Recent development in random planar maps: exercises for lecture III

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1 Distance statistics for generalized CDT

We consider pointed rooted planar quadrangulations endowed with their canonical labeling: for any vertex v, we take

$$\ell(v) = d(v_0, v)$$

where v_0 is the origin (pointed vertex) and d is the graph distance in the quadrangulation. We denote by v_1 the endpoint of the root edge farther from the origin. A quadrangulation is said *causal* if ℓ admits a unique local maximum. In an arbitrary quadrangulation, there may be (and typically there are) many local maxima of ℓ.

The model of generalized CDT can be seen as an interpolation between causal and arbitrary quadrangulations, see [1] and references therein. It consists in "penalizing" local maxima by attaching them a weight $h < 1$, in addition to the usual weight g per face. We wish to study the distribution of the root-origin distance, namely of $d(v_0, v_1)$, in this model. In combinatorial terms, we wish to compute the generating function

$$T_i = \sum g^{\#\text{faces}} h^{\#\text{local maxima}}$$

where the sum is over all pointed rooted planar quadrangulations such that $d(v_0, v_1) \leq i$.

Question 1. By the CVS bijection, show that T_i is equal to the generating function of well-labeled trees with positive labels, root label i, counted with a weight g per edge and h per local maximum.

Question 2. Using recursive decomposition of trees, show that T_i is determined by the system of recurrence equations

$$\begin{aligned} T_i &= h + g(T_i U_{i-1} + T_i^2 + U_i T_{i+1}) \\ U_i &= 1 + g(U_i U_{i-1} + U_i T_i + U_i T_{i+1}) \end{aligned} \tag{1}$$

where U_i is an auxiliary variable.

Miraculously, it is again possible to solve explicitly this system [1]: it takes the form

$$T_i = T\frac{(1-y^i)(1-\alpha^2 y^{i+3})}{(1-\alpha y^{i+1})(1-\alpha y^{i+2})}, \qquad U_i = U\frac{(1-y^i)(1-\alpha y^{i+3})}{(1-y^{i+1})(1-\alpha y^{i+2})} \tag{2}$$

where T, U, y and α are power series in g, h determined by some algebraic equations.

Question 3 (Optional, requires a computer algebra system)**.** From the knowledge of (2), find these algebraic equations. Hint: satisfying (1) amounts to satisfying two equations

$$P_s(g, h, T, U, y, \alpha, y^i) = 0, \qquad s = 1, 2$$

where the P_s are polynomials that *do not depend* on i. Thus, in their expansion with respect to the last variable, we want all coefficients to vanish identically. This yields a system of algebraic equations relating g, h, T, U, y, α: show that it defines a two-dimensional variety. More difficult, prove that it uniquely determines T, U, y, α as power series in g and h, and compute their first coefficients.

2 The expected volume of a ball in the UIPQ

Recall that the "two-point function" of planar quadrangulations is given by

$$R_i = R\frac{(1-x^i)(1-x^{i+1})}{(1-x^{i+1})(1-x^{i+2})} \tag{3}$$

where $R = (1-\sqrt{1-12t})/(6t)$ and x is determined by

$$x + \frac{1}{x} = \frac{1-4tR}{tR}. \tag{4}$$

Question 4. Provide a probabilistic interpretation for the ratio $\frac{[t^n]R_i}{[t^n]R_1}$.

Question 5 (Requires computations)**.** Determine the limit of this ratio as $n \to \infty$, i being fixed. Hint: use a suitable "transfer theorem" [2, Section VI.3].

References

[1] J. Ambjørn and T. Budd, *Trees and spatial topology change in generalized CDT*, J. Phys. A: Math. Theor. 46 (2013), 315201, arXiv:1302.1763 [hep-th].

[2] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009. Available online at `http://algo.inria.fr/flajolet/Publications/books.html`.