1 Distance statistics for generalized CDT

We consider pointed rooted planar quadrangulations endowed with their canonical labeling: for any vertex \( v \), we take
\[
\ell(v) = d(v_0, v)
\]
where \( v_0 \) is the origin (pointed vertex) and \( d \) is the graph distance in the quadrangulation. We denote by \( v_1 \) the endpoint of the root edge farther from the origin. A quadrangulation is said causal if \( \ell \) admits a unique local maximum. In an arbitrary quadrangulation, there may be (and typically there are) many local maxima of \( \ell \).

The model of generalized CDT can be seen as an interpolation between causal and arbitrary quadrangulations, see [1] and references therein. It consists in penalizing local maxima by attaching them a weight \( h < 1 \), in addition to the usual weight \( g \) per face. We wish to study the distribution of the root-origin distance, namely of \( d(v_0, v_1) \), in this model. In combinatorial terms, we wish to compute the generating function
\[
T_i = \sum g^{\# \text{faces}} h^{\# \text{local maxima}}
\]
where the sum is over all pointed rooted planar quadrangulations such that \( d(v_0, v_1) \leq i \).

**Question 1.** By the CVS bijection, show that \( T_i \) is equal to the generating function of well-labeled trees with positive labels, root label \( i \), counted with a weight \( g \) per edge and \( h \) per local maximum.

**Question 2.** Using recursive decomposition of trees, show that \( T_i \) is determined by the system of recurrence equations
\[
T_i = h + g(T_i U_{i-1} + T_i^2 + U_i T_{i+1}) \quad U_i = 1 + g(U_i U_{i-1} + U_i T_i + U_i T_{i+1})
\]
where \( U_i \) is an auxiliary variable.

Miraculously, it is again possible to solve explicitly this system [1]: it takes the form
\[
T_i = T \frac{(1 - y^i)(1 - \alpha y^{i+3})}{(1 - \alpha y^{i+1})(1 - \alpha y^{i+2})}, \quad U_i = U \frac{(1 - y^i)(1 - \alpha y^{i+3})}{(1 - y^{i+1})(1 - \alpha y^{i+2})}
\]
where \( T, U, y \) and \( \alpha \) are power series in \( g, h \) determined by some algebraic equations.
Question 3 (Optional, requires a computer algebra system). From the knowledge of (2), find these algebraic equations. Hint: satisfying (1) amounts to satisfying two equations

\[ P_s(g, h, T, U, y, \alpha, y^i) = 0, \quad s = 1, 2 \]

where the \( P_s \) are polynomials that do not depend on \( i \). Thus, in their expansion with respect to the last variable, we want all coefficients to vanish identically. This yields a system of algebraic equations relating \( g, h, T, U, y, \alpha \): show that it defines a two-dimensional variety. More difficult, prove that it uniquely determines \( T, U, y, \alpha \) as power series in \( g \) and \( h \), and compute their first coefficients.

2 The expected volume of a ball in the UIPQ

Recall that the “two-point function” of planar quadrangulations is given by

\[ R_i = R \frac{(1 - x^i)(1 - x^{i+1})}{(1 - x^{i+1})(1 - x^{i+2})} \]

where \( R = (1 - \sqrt{1 - 12t}) / (6t) \) and \( x \) is determined by

\[ x + \frac{1}{x} = 1 - 4tR \]

Question 4. Provide a probabilistic interpretation for the ratio \( \frac{[x^n]R_i}{[x^n]R_1} \).

Question 5 (Requires computations). Determine the limit of this ratio as \( n \to \infty \), \( i \) being fixed. Hint: use a suitable “transfer theorem” [2, Section VI.3].

References
