Recent development in random planar maps: exercises for lecture IV

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Abstract

In this exercise session, we study the site percolation on the UIHPT (Uniform Infinite Half-Plane Triangulation). More precisely, we use the domain Markov property of the UIHPT to construct the so-called peeling processes and we encode information about percolation clusters with these processes.

1 Definitions and enumerative background

Recall that *a planar triangulation with simple boundary* is a rooted planar map such that every non-root face has degree 3, and such that the root face has a simple boundary. When the root face is of degree m, it is also called a triangulation of the m-gon. By taking the local limit of finite triangulations of the m-gon as $m \to \infty$, we can generalize the notion of "planar triangulation with simple boundary" to include triangulations of the half-plane. Note that the boundary vertices of a triangulation of the half-plane are naturally indexed by $\mathbb{Z}$, with the root vertex being of index 0. By convention, all triangulations considered below are of type II (i.e. no self-loops but may have multiple edges, c.f. exercises for lecture I).

Let $\mathcal{M}_{n,m}$ be the set of all triangulations of the m-gon with n internal vertices. Let $\phi_{n,m}$ be the cardinal of $\mathcal{M}_{n,m}$. Recall the following enumeration results from the session I:

$$\phi_{n,m} \sim_{n\to\infty} C_m n^{-5/2} \rho^n \qquad \text{where } \rho = \frac{27}{2} \tag{1}$$

$$C_m \sim_{m\to\infty} \kappa m^{1/2} \alpha^m \qquad \text{where } \alpha = 9 \tag{2}$$

$$Z_m \overset{\text{def}}{=} \sum_{n=0}^{\infty} \phi_{n,m} \rho^{-n} = \frac{(2m-4)!}{m!(m-2)!} \left(\frac{9}{4}\right)^{m-1} \tag{3}$$

Z_m can be viewed as the partition function of the following Boltzmann distribution μ_m on $\bigcup_{n\geq 0} \mathcal{M}_{n,m}$:

$$\mu_m(T) = \frac{1}{Z_m} \rho^{-n} \tag{4}$$

for all $T \in \mathcal{M}_{n,m}$. A sample from the distribution μ_m is called the *free triangulation of the m-gon*. Also, denote by $\theta_{n,m}$ the law of the *uniform triangulation of the m-gon with n internal vertices*, that is,

$$\theta_{n,m}(T) = \phi_{n,m}^{-1} \tag{5}$$

for all $T \in \mathcal{M}_{n,m}$.

We call *half-plane triangulation with boundary* a finite triangulation of a compact domain in the upper half-plane which has only a finite number of holes. We assume that the compact domain contains the root vertex. And we include the indices of vertices which are both on the boundary of the compact domain and the boundary of the half-plane as part of the specification of the half-plane triangulation with boundary.

In other words, a half-plane triangulation with boundary is the planar map that one obtains by removing all but a finite number of faces from a triangulation of the half-plane, while retaining the boundary of the half-plane with the indices of all vertices on it. Note that, since a half-plane triangulation with boundary is finite, it can also be obtained by removing faces from a finite triangulation of the m-gon for m large enough, then embedding appropriately the boundary of the m-gon into a straight line.

A half-plane triangulation with boundary V is *rigid* if it is connected and no triangulation with simple boundary (that is, triangulation of an m-gon or of the half-plane) contains two distinct copies of V.

Question 1 (Optional)**.** Show that a half-plane triangulation with boundary V is rigid if and only if the union of all faces of V with the lower half-plane is 3-connected. (Recall that a topological set is k-connected if it remains connected whenever one removes at most $k-1$ points from it.)
Note: the "if" part is much harder than the "only if" part.

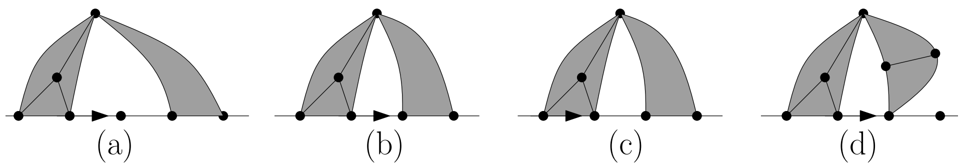

Figure 1: **Examples of half-plane triangulation with boundary**: (a), (b) and (c) are three half-plane triangulations with boundary which differs from each other by the index of the vertices on the boundary of the half-plane. They are all rigid. (d) is a non-rigid half-plane triangulations with boundary. All of the four half-plane triangulations with boundary above have only one hole ($k = 1$) of size $m_1 = 3$.

2 The UIHPT and its domain Markov property

Let V be a rigid half-plane triangulation with boundary having k holes. We denote by $m_1, \dots, m_k$ the perimeters of the holes, and by δ the net number of vertices added to the boundary of the half-plane when one removes V and its hole components from the half-plane. Also, let N be the number of vertices in V which are not on the boundary of the half-plane.

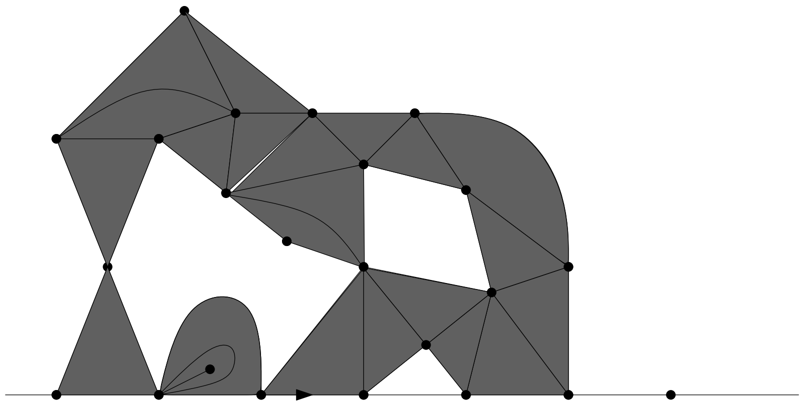

Figure 2: A half-plane triangulation with boundary which has $k = 3$ holes of respective sizes $m_1 = 7$, $m_2 = 4$ and $m_3 = 3$. Its outer boundary has length 7 and it covers 5 boundary edges of the half-plane, so $\delta = 7 - 5 = 2$.

We denote by $\{V \subset T\}$ the event that a random planar triangulation T with simple boundary T contains V as a sub-triangulation. For nonnegative integers $n_1, \dots, n_k$, let $E(V; n_1, \dots, n_k)$ be the event that $V \subset T$ *and* for all $i \in \{1, \dots, k\}$, the i-th hole of V contains n_i vertices of T which are not in V.

Question 2. For large enough n and m, express $\mu_m(E(V; n_1, \dots, n_k))$ and $\theta_{n,m}(E(V; n_1, \dots, n_k))$ with the help of $\phi_{\cdot,\cdot}$, $C_\cdot$ and $Z_\cdot$. Show that they have the same limit as $m \to \infty$ for μ_m and as $n \to \infty$ then $m \to \infty$ for $\theta_{n,m}$.

Modulo some tightness arguments that we admit (see [1] for similar arguments written in the case of the UIPT), the above convergence implies that μ_m and $\theta_{n,m}$ converge weakly with respect to the local distance to the same distribution ν, which is characterized by

$$\nu(E(V; n_1, \dots, n_k)) = \lim_{m \to \infty} \mu_m(E(V; n_1, \dots, n_k)) \quad (6)$$

$$= \lim_{m \to \infty} \lim_{n \to \infty} \theta_{n,m}(E(V; n_1, \dots, n_k)) \quad (7)$$

for all rigid half-plane triangulation with boundary V and all $n_1, \dots, n_k \in \mathbb{N}$. We call a sample of ν a Uniform Infinite Half-Plane Triangulation (UIHPT).

Using estimates on the probability that one of the n_i ($i \in \{1, \dots, k\}$) is large, it can be shown that with probability 1 the UIHPT is *one-ended*, that is, the complement of any finite set of vertices in the UIHPT contains only one infinite connected component.

Question 3. For fixed V, compute $\nu(V \subset T)$.

Question 4. Show that, conditionally to the event $\{V \subset T\}$, the holes of V contains free triangulations of the m_i-gon ($i = 1, \dots, k$) which are mutually independent and independent from the infinite component of $T \setminus V$.

Question 5. Show that, conditionally to $\{V \subset T\}$, the infinite component has the law of a UIHPT.

3 Peeling process and site percolation on the UIHPT

One-step peeling Let T be a UIHPT. The *peeling of T at the root edge* consists of revealing in T the face on the left of the root edge. (By convention, the outer face of T is on the right of the root edge.) The third vertex of the revealed triangle may be an internal vertex, or it may lie on the boundary. In the latter case, the revealed triangle separates the remaining part of T into two regions, one infinite and one finite. We say that this finite region is *swallowed* by the revealed triangle.

Question 6. Write down the probabilities of the following events

1. the third vertex of the revealed triangle is an interval vertex of the half-plane triangulation. (No swallowed boundary edge.)
2. the third vertex of the revealed triangle is on the boundary at a distance k to the left (or right, which is symmetric) of the root edge. (k boundary edges are swallowed.)

Let $\mathcal{R}$ be the number of swallowed boundary edges on *right* of the root edge. That is, $\mathcal{R} = 0$ when there is no swallowed boundary or when the swallowed boundary is on the left of the root edge.

Question 7. Compute $\mathbb{E}[\mathcal{R}]$, and find an asymptotic of $\mathbb{P}(\mathcal{R} = k)$ when $k \to \infty$.

Peeling process For a half-plane triangulation T and $a \in \mathbb{Z}$, we denote by $\mathrm{Peel}(T, a)$ the half-plane triangulation obtained by peeling T at the edge $a \to a+1$ and removing the revealed triangle together with the swallowed region (if exists). A *peeling process* is a randomized algorithm which explores T by revealing one triangle at each step. More precisely, it can be defined as a sequence of half-plane triangulations $T = T_0 \supset T_1 \supset \cdots$ such that for all $i \in \mathbb{N}$,

$$T_{i+1} = \mathrm{Peel}(T_i, a_i) \tag{8}$$

where a_i is chosen by an algorithm independently from T_i. Basically, a_i is a function of P_i, the complement of T_i in T, and possibly another source of randomness which is independent from T_i.

Question 8. Show that for all $i \geq 0$, T_i is an UIHPT independent from P_i.

Let $\mathcal{R}_i$ be the number of swallowed boundary edges on the right in the peeling step $\mathrm{Peel}(T_i, a_i)$.

Question 9. Show that $(\mathcal{R}_i)_{i\geq 0}$ is an i.i.d. sequence.

We consider the site percolation model on the UIHPT with the following boundary condition: we color the root vertex white and all the other boundary vertices black. Conditionally on the triangulation, all the other vertices are colored independently in white with probability $p \in (0,1)$, and in black with probablity $1-p$. We are interested in the white cluster containing the root. To explore this cluster, we consider the following algorithm for choosing edges to peel:

> ALGORITHM: whenever the boundary of T_i is colored in the form "black-white-black" (that is, all whites vertices on the boundary form a finite interval like in), let a_i be the edge on the left of the left-most white vertex. If the boundary of T_i is not colored this way, then a_i is undefined.

Question 10. Show that this defines a peeling process as long as the boundary vertices of T_i is not all black. Explain how this peeling process explores the white cluster containing the root.

Question 11. Show that as long as the peeling process is defined, the number of white boundary vertices of T_i evolves as a random walk with i.i.d. steps.

Let $\mathcal{E}_i$ be the event that the peeling step $\mathrm{Peel}(T_i, a_i)$ reveals a white interval vertex of T_i.

Question 12. Express the steps of the above random walk with $\mathcal{R}_i$ and $\mathcal{E}_i$.

Question 13. Show that the white cluster containing the root is finite with probability 1 if and only if $p \leq p_c = \frac{1}{2}$.

References

[1] O. Angel and O. Schramm, *Uniform Infinite Planar Triangulations*, Comm. Math. Phys. 241 (2003) 191–213, arXiv:math/0207153 [math.PR]

[2] O. Angel and N. Curien, *Percolations on random maps I: half-plane models*, arXiv:1301.5311 [math.PR]