Recent development in random planar maps: exercises for lecture I

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Abstract

The purpose of this exercise session is to enumerate rooted planar triangulations with a simple boundary, by solving Tutte's equation. The results will be useful for lecture IV. For some questions, a computer algebra system (Mathematica, Maple, Sage...) might be helpful.

1 Preliminaries

A *planar triangulation* is a planar map whose all faces have degree 3. We may actually distinguish different classes of triangulations, depending on their possible "singularities": general triangulations may have loops and multiple edges, and we refer to them as *type I* triangulations, *type II* triangulations may have multiple edges but no loops, finally *type III* triangulations have neither loops nor multiple edges.

Question 1 (Optional)**.** Show that a triangulation is of type II if and only if it is 2-connected, and of type III if and only if it is 3-connected. (Recall that a map or a graph is said *k-connected* if it remains connected whenever one removes at most $(k-1)$ of its vertices. Note that being 1-connected is the same as being connected, and this is the case for any map by definition.)

Solution : Note first that any triangulation containing a loop is not 2-connected, since removing the endpoint of the loop disconnects the map (there are necessarily vertices in both regions delimited by the loop since the map is a triangulation). Conversely, a separating vertex is necessarily incident twice to a same triangle, and then clearly one side of the triangle must be a loop. This shows that a triangulation is of type II if and only if it is 2-connected. The reasoning for type III is similar: in a triangulation, pairs of separating vertices are always connected by multiple edges, and vice-versa. Note that these equivalences fail as soon as we allow faces of degree other than 3. △

More generally, a *planar triangulation with a simple boundary* is a rooted planar map such that every non-root face has degree 3, and such that the root face (whose degree k is arbitrary) is simple (i.e. is incident to k distinct vertices, in other words there are no "pinch points"). Type I, II and III triangulations with a simple boundary are defined as above (no loops for type II, neither loops nor multiple edges for type III). The main purpose of this exercise is to enumerate planar triangulations of type II, by solving Tutte's equation.

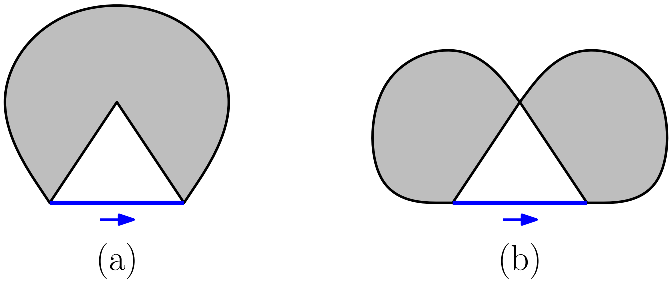

Figure 1: Possible outcomes when removing the root edge of a type II planar triangulation with a simple boundary: we may either obtain (a) another triangulation with a simple boundary or (b) a pair of triangulations with simple boundaries sharing a common vertex.

2 Tutte's equation

For n, m nonnegative integers, we denote by $F_{n,m}$ the number of type II planar triangulations with a simple boundary of length $m+2$ and n internal vertices. It is convenient to view the *link map* (map reduced to a single edge, two vertices and one face) as such a triangulation with $n = m = 0$, hence by convention we set $F_{0,0} = 1$.

Question 2 (Optional)**.** For $n \geq 3$, let T_n be the number of rooted type II planar triangulations (without boundary). Explain why

$$T_n = F_{n-2,0} = F_{n-3,1}.$$

Would these equalities hold in type I or III?

Solution : The equality $T_n = F_{n-3,1}$ is immediate by treating the root face as a boundary (this equality would however fail in type I as the root face might be non simple).

The equality $T_n = F_{n-2,0}$ is obtained by "splitting" the root edge of a rooted triangulation into a pair of edges delimiting a 2-gon. (This would again fail in type I as the 2-gon has a non simple boundary when the root edge is a loop, and also in type III since multiple edges are forbidden.) $\triangle$

2.1 Derivation

We introduce the generating functions

$$F \equiv F(t,z) = \sum_{n\geq 0}\sum_{m\geq 0} F_{n,m} t^n z^m, \qquad F_0 \equiv F_0(t) = \sum_{n \geq 0} F_{n,0} t^n.$$

Question 3. Derive Tutte's equation

$$F(t,z) = 1 + t\frac{F(t,z) - F_0(t)}{z} + zF(t,z)^2. \tag{1}$$

Solution : Starting with a type II planar triangulation with a simple boundary of length $m+2$ and n internal vertices, we remove the root edge and consider the possible outcomes. For $m = n = 0$ the map is the link map hence nothing remains. Otherwise, the face on the left of the root edge is necessarily a triangle distinct from the root face, and we are in one of the two possible situations illustrated on Figure 1:

(a) when the third vertex of the triangle is an internal vertex, we end up with a triangulation with a simple boundary of length $m+3$ and $n-1$ internal vertices,

(b) when the third vertex of the triangle is a boundary vertex, we end up with a pair of triangulations with simple boundaries sharing a common vertex: their outer lengths sum up to $m+3$ and they have n internal vertices in total.

This decomposition is clearly bijective (each resulting map can be canonically rooted), hence we end up with the relation

$$F_{n,m} = \delta_{n,0}\delta_{m,0} + F_{n-1,m+1} + \sum_{k=0}^{m-1} \sum_{\ell=0}^{n} F_{\ell,k} F_{n-\ell,m-1-k}.$$

Multiplying this equation by $t^n z^m$ and summing over $n, m \geq 0$ we end up with the desired equation. $\triangle$

2.2 Solution I

We first derive a closed form expression for the number of triangulations without a boundary. We set $P(F, F_0, t, z) = F - 1 - t(F - F_0)/z - zF^2$ so that Tutte's equation amounts to

$$P(F(t,z), F_0(t), t, z) = 0. \tag{2}$$

Question 4. Show that there is a unique power series $U \equiv U(t) = 1 + o(t)$ such that

$$\frac{\partial P}{\partial F}(F(t, tU(t)), F_0(t), t, tU(t)) = 0. \tag{3}$$

Show that it also satisfies

$$\frac{\partial P}{\partial z}(F(t, tU(t)), F_0(t), t, tU(t)) = 0. \tag{4}$$

Solution : We have $\frac{\partial P}{\partial F} = 1 - t/z - 2zF^2$ hence U must satisfy

$$U = 1 + 2tU^2 F(t, tU(t)).$$

This equation clearly determines, order by order, a unique power series $U(t) = 1 + o(t)$: observe that the coefficient of t^n in $U(t)$ is uniquely determined in terms of coefficients of lower order and the (yet unknown, but well-defined) coefficients of F.

Now, by differentiating (2) with respect to z, we find

$$\frac{\partial P}{\partial F}(F(t,z), F_0(t), t, z)\frac{\partial F}{\partial z}(t,z) + \frac{\partial P}{\partial z}(F(t,z), F_0(t), t, z) = 0.$$

Substituting $z = tU(t)$ we obtain (4). $\triangle$

Question 5. By elimination, derive the algebraic equation satisfied by U. Express F_0 in terms of U. (In a more educated language, this is a rational parametrization of the spectral curve.)

Solution : We may for instance eliminate $F(t, tU(t))$ using (3), then $F_0(t)$ using (4). We find that U satisfies

$$U = 1 + 2tU^3$$

and then $F_0(t)$ is given by

$$F_0 = \frac{U(3-U)}{2}.$$

Interestingly, U can be interpreted as the generating function of plane ternary trees with an extra weight 2 per node. This suggests a possible correspondence between such trees and type II planar triangulations, which was indeed given by Schaeffer [2, Théorème 2.14]. $\triangle$

We now recall the Lagrange inversion formula, see e.g. [1, Section A.6]: if $\phi \equiv \phi(y)$ is a power series whose constant coefficient is nonzero, then there is a unique power series $u \equiv u(t)$ satisfying $u = t\phi(u)$, and its coefficients read explicitly

$$[t^n]u(t) = \frac{1}{n}[y^{n-1}]\phi(y)^n, \qquad n \geq 1.$$

(Here $[t^n]u(t)$ denotes the coefficient of t^n in $u(t)$, etc.) Furthermore, for an arbitrary function H, we have

$$[t^n]H(u(t)) = \frac{1}{n}[y^{n-1}]\left(H'(y)\phi(y)^n\right), \qquad n \geq 1.$$

Question 6. Apply the Lagrange inversion formula to compute $[t^n]U(t)$ and $[t^n]F_0(t)$. (Hint: take $u(t) = U(t) - 1$.)

Solution : Taking $u = U - 1$, we find that $u = t\phi(u)$ with $\phi(u) = 2(u+1)^3$. By the Lagrange inversion formula we find that, for $n \geq 1$,

$$[t^n]U(t) = [t^n]u(t) = \frac{2^n}{n}[y^{n-1}](y+1)^{3n} = \frac{2^n(3n)!}{n!(2n+1)!}.$$

More generally we have, for $k \geq 1$, the nice expression

$$[t^n]U^k = \frac{k\,2^n(3n+k-1)!}{n!(2n+k)!}$$

(note that this expression remains correct for $n = 0$). Using now $F_0 = U(3-U)/2$ we obtain, after simplification, the number of rooted type II planar triangulations with $n+2$ vertices:

$$T_{n+2} = [t^n]F_0(t) = \frac{2^{n+1}(3n)!}{n!(2n+2)!}.$$

$\triangle$

2.3 Solution II

With a bit more work it is possible to derive a bivariate closed form expression for $F_{n,m}$.

Question 7. Replace t and $F_0(t)$ by their expressions in terms of U in Tutte's equation, to obtain an algebraic equation relating F, U and z. What do you observe about its discriminant with respect to F? Show that

$$F = \frac{1 - U + 2zU^3 - (1 - U + 2zU^2)\sqrt{1 - 4zU^2}}{4z^2U^3}. \tag{5}$$

Solution : We find a quadratic equation for F whose coefficients are polynomials in U and z. We observe that its discriminant has the nice factorization

$$\Delta = 4(1 - U + 2zU^2)^2(1 - 4zU^2)$$

which is no miracle, see e.g. [3]. Solving the quadratic equation in the usual way, we obtain the desired expression (5) (the correct sign is determined by the requirement that the expansion of F in powers of z contains only nonnegative powers). △

Question 8. Deduce a closed form expression for $[z^m]F(t,z)$ in terms of U. (Hint: $\sqrt{1-4x} = 1 - 2xC(x)$ where $C(x)$ is the generating function of Catalan numbers, $C(x) = \sum_{k\geq 0} \frac{(2k)!}{k!(k+1)!}x^k$.)

Solution : Using the hint we get that

$$F = UC(zU^2) - \frac{U-1}{2U}\frac{C(zU^2)-1}{z}$$

which immediately yields

$$[z^m]F(t,z) = \frac{(2m)!}{m!(m+1)!}U^{2m+1} - \frac{(2m+2)!}{(m+1)!(m+2)!}\frac{U^{2m+2} - U^{2m+1}}{2}.$$

△

Question 9. By the Lagrange inversion formula, obtain a closed form expression for $F_{n,m} = [t^n z^m]F(z,t)$.

Solution : Using the above expression for $[t^n]U^k$ we obtain, after simplification, the nice expression

$$[t^n z^m]F(t,z) = \frac{2^{n+1}(2m+1)!(3n+2m)!}{(m!)^2 n!(2n+2m+2)!}.$$

△

3 Asymptotics

Question 10. Show that the radius of convergence of $F_0(t)$ and, more generally, $[z^m]F(t,z)$ for any m, is

$$t_c = \frac{2}{27}.$$

Solution : It is easily seen from the algebraic equation for U, or from the explicit expression of its coefficients, that its radius of convergence is $t_c = 2/27$. Note that $U(t_c) = 3/2$. As $[z^m]F(t,z)$ is a polynomial in U, it has the same radius of convergence. △

Question 11. Compute $[z^m]F(t_c,z)$. (This is the partition function of the "free distribution" on rooted triangulations of the $(m+2)$-gon, useful when studying local limits.)

Solution : From the expression of $[z^m]F(t,z)$ in terms of U, and the value $U(t_c) = 3/2$, we readily get

$$[z^m]F(t_c, z) = \frac{(2m)!}{m!(m+2)!}\left(\frac{3}{2}\right)^{2m+2}.$$

△

Question 12. Show that, for any fixed $m \geq 0$, we have as $n \to \infty$

$$F_{n,m} \sim C_m t_c^{-n} n^{-5/2}$$

and compute C_m as well as its asymptotic behaviour as $m \to \infty$.

Solution : This easily follows from the Stirling formula, with the explicit value

$$C_m = \frac{\sqrt{3}(2m+1)!}{4\sqrt{\pi}(m!)^2}\left(\frac{3}{2}\right)^{2m} \sim C 9^m \sqrt{m}.$$

△

References

[1] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[2] G. Schaeffer, *Conjugaison d'arbres et cartes combinatoires aléatoires*, PhD thesis, Université Bordeaux I (1998).

[3] M. Bousquet-Mélou and A. Jehanne, *Polynomial equations with one catalytic variable, algebraic series and map enumeration*, Journal of Combinatorial Theory, Series B 96 (2006) 623–672