Solutions for the exercises session II.

— Preliminary — : Geodesics on a graph

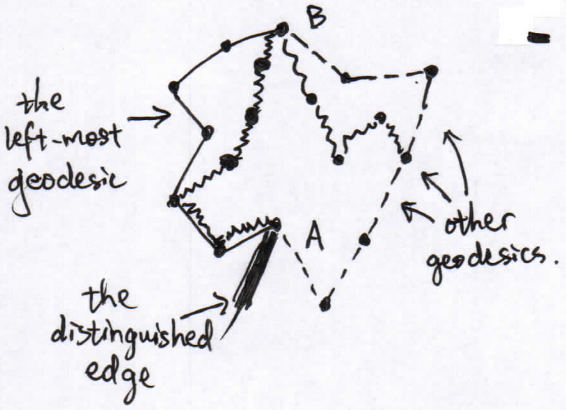
Let A and B be two vertices on the graph G,

γ is a path from A to B in G, we denote by $|\gamma|$ the number of edges it contains.

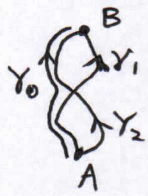
Def: γ' denotes a generic path from A to B.

- γ is a geodesic from A to B if $\forall \gamma'$, $|\gamma| \leq |\gamma'|$
- γ is the unique geodesic $A \to B$ if $\forall \gamma' \neq \gamma$, $|\gamma| < |\gamma'|$

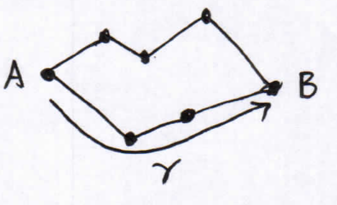
When G is the graph of a planar map, and when one of the edges incident to A is distinguished, then we can define the left-most geodesic from A to B.



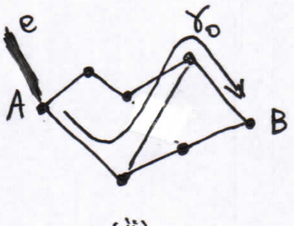
- The distinguished edge prevents us to "turn around" the vertex A, so that there is a well-define left $\to$ right order.
- It is possible that two geodesics γ_1, γ_2 intersect, so that neither of them is on the left of the other. But in this case, there is another geodesic γ_0 which is on the left of both γ_1 and γ_2 :



Examples :



(i)



(ii)

In (i), γ is the unique geodesic $A \to B$.

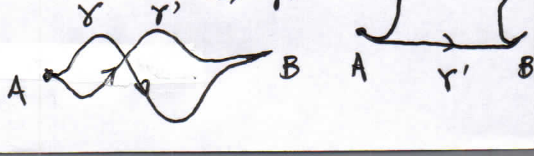
In (ii), it is no longer the case because of the existence of γ_0.

In (ii), γ_0 is the left-most geodesic if we add a distinguished edge e.

From the example (ii) we see that geodesics on graphs may do things that one does not expect from the Euclidean geometry.

Especially, these discrete geodesics may share one or more segments with each other, yet remaining distinct :

Possible configurations :

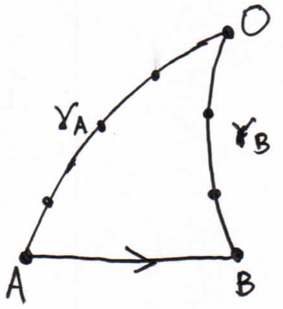


Definition : slices.

A *slice* is a rooted quadrangulation with simple boundary, with a marked point O on the boundary such that, if we call $A \to B$ the root edge (which is on the boundary) call γ_A (resp. γ_B) the part of the boundary that goes from A (resp. B) to O without visiting B (resp. A).

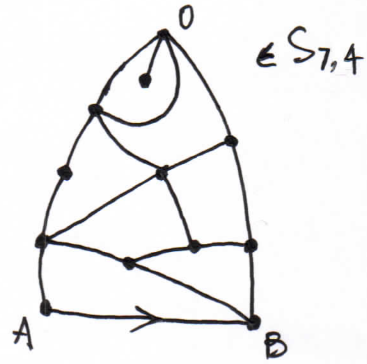
then ① γ_A is a geodesic

② γ_B is the unique geodesic from B to O.

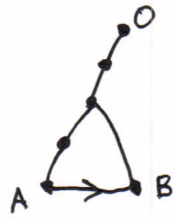


← scheme of a slice

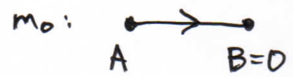
An example of a slice of length 4 (see Q1) having 7 internal faces →



Since the boundary must be simple, this is not a slice →



But, by convention, the map m_0 consisting of a single edge is a slice :



Answer 1

Step 1 : use ① & ② in the definition to show that

$$d(A,O) - d(B,O) \; (= |\gamma_A| - |\gamma_B|) \in \{0,1\}$$

↳ Note that (the root edge) $\cup \, \gamma_B$ is a path from A to O

so by ①, $|\gamma_A| \leq 1 + |\gamma_B|$

↳ (the root edge) $\cup \, \gamma_A$ is a path from B to O,

so by ②, $|\gamma_B| < 1 + |\gamma_A|$ $\Rightarrow$ $|\gamma_A| - |\gamma_B| \in \{0,1\}$.

Step 2 : since a quadrangulation with simple boundary is always bipartite, the distances from two neighboring points to O always differs by 1 in absolute value.

So $\boxed{|\gamma_A| - |\gamma_B| = 1}$

Notations: Q_n : the set of all rooted and pointed planar quadrangulations (without a boundary) having n faces.

S_n : the set of all slices having n internal faces

$S_{n,i}$: the set of all slices in S_n which are of length $\leq i$.

so that : $R(g) = \sum_{n\geq 0} \#(S_n)\, g^n = \sum_{Q\in S_n} g^{\#(\text{internal faces of } Q)}$

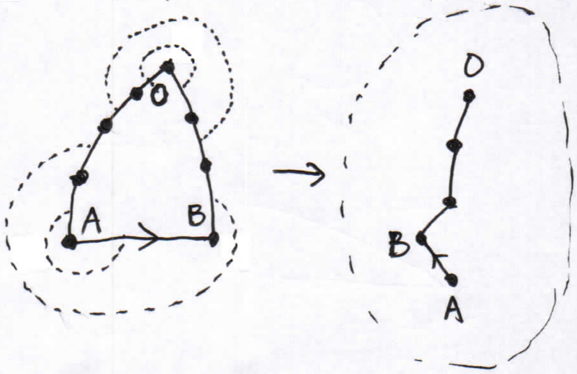
$\forall i \geq 0$ $R_i(g) = \sum_{n\geq 0} \#(S_{n,i})\, g^n$ Note that $S_{n,0} = \emptyset$ (there is no slice of length zero) so $R_0(g) = 0$.

— Answer 2 —

We want to show that $R(g) = \sum_{n\geq 0} (\#Q_n)\, g^n$, or $\forall n \geq 0$ $\#S_n = \#Q_n$, that is, there is a bijection between S_n and Q_n.

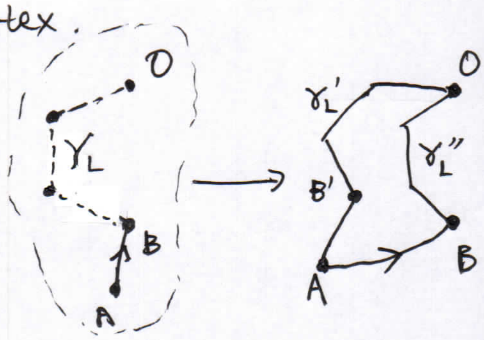
- Construction :

$S_n \to Q_n$: we glue the two boundary components between A and O (By Answer 1, they are of the same length)



This gives a quadrangulation in Q_n with $A \to B$ as the root and O as the distinguished vertex.

$Q_n \to S_n$: Let $A\to B$ be the root and O be the distinguished vertex in the quadrangulation $Q \in Q_n$. We draw the left-most geodesic γ_L from B to O, then cut along the path (root edge) $\cup\ \gamma_L$.



The fact that γ_L is left-most in the quadrangulation ensures ② in the definition of a slice.

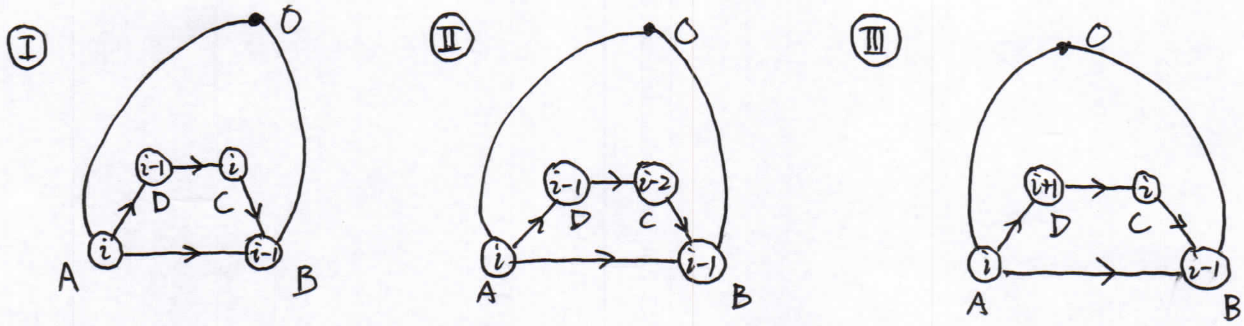
- In this bijection, the length of the slice corresponds to $1 +$ (the distance from B (end of the root edge) to O) in the quadrangulation.

So r_i is the generating function of the number of all rooted and pointed quadrangulation having n faces, such that the end of the root edge is at distance $i-1$ from the distinguished vertex.

Therefore, knowledge about the r_i will allow us to obtain the distance statistics between the root and the marked vertex.

— Answer 3 —

We label the vertices of a slice by their distance to O.
There are 3 cases for the labels of C and D

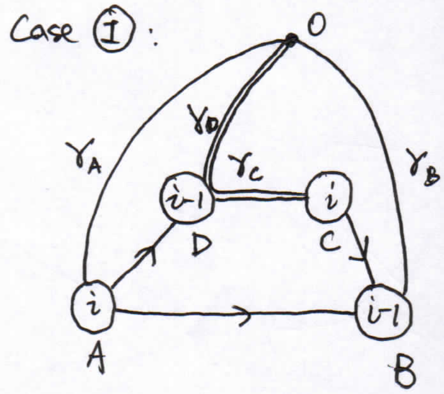


In each case, let's call L, M and R the 3 small slices (from Left to Right) delimited by the left-most geodesics from D and C to O.

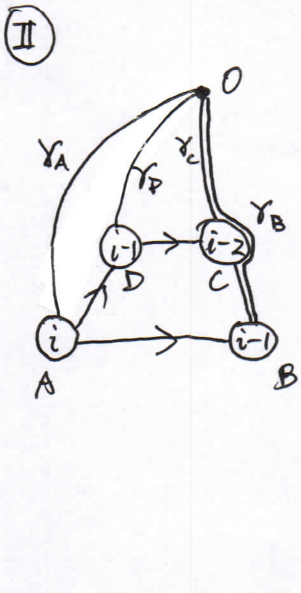
↳ We concentrate on the case I.
First remark that the middle slice M has an abnormal labeling on its root edge: the relation $|\gamma_D| - |\gamma_C| = 1$ (c.f. Answer 1) is not respected.
This is a sign that M is degenerated.

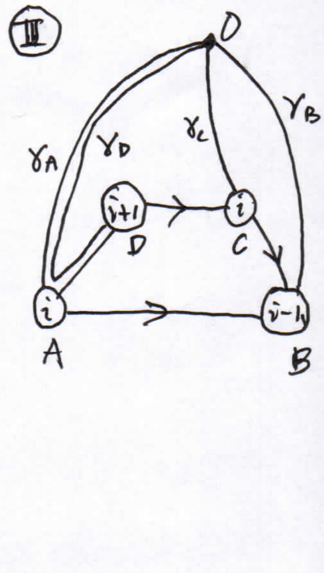
Indeed, one can check that in case I,
the concatenation $(D \to C) \cup \gamma_C$ is the left-most geodesic from C to O,
i.e. $\gamma_D = (D \to C) \cup \gamma_C$.

So M is reduced to m_0 (•→•)



↳ Similarly, in the two other cases, exactly one of the 3 slices L, M, R is reduced to m_0 in the generic case.
(Because to get from $\overset{A}{(i)}$ to $\overset{B}{(i-1)}$ in 3 steps of ± 1, one needs to take exactly one step $+1$.)





— The rest of Answer 3 —

We decompose the set S_n according to the 3 cases seen above.

$$S_n = \begin{cases} \{m_0\} & \text{if } n=0 \\ \{\mathrm{I}\} \sqcup \{\mathrm{II}\} \sqcup \{\mathrm{III}\} & \text{if } n \geq 1 \end{cases}$$

$$\{\mathrm{I}\},\ \{\mathrm{II}\},\ \{\mathrm{III}\} \ \leftrightarrows\ \bigsqcup_{n_1+n_2=n-1} S_{n_1} \times S_{n_2}$$

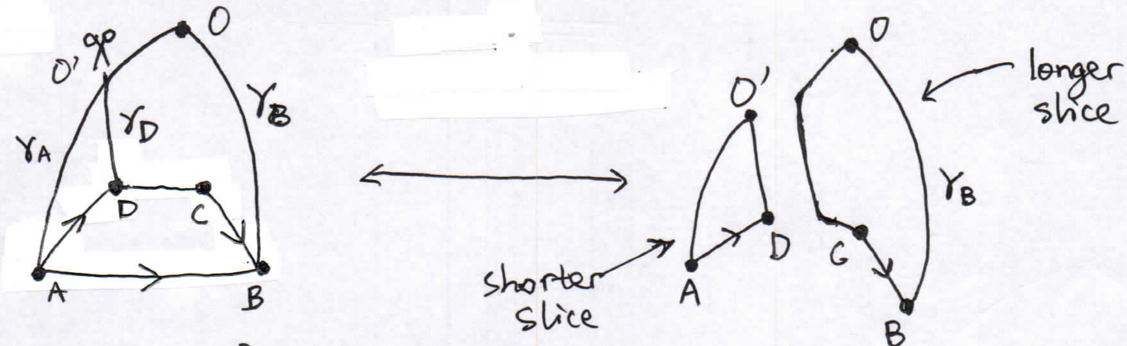
The "$\leftrightarrows$" means there are bijections from the 3 subsets to $\bigsqcup_{n_1+n_2=n-1} S_{n_1} \times S_{n_2}$.

This means that, if s_1 and s_2 are two arbitrary slices, then regardless of their lengths, we can always glue them together according to the scheme I (resp. II, III) to get a larger slice.

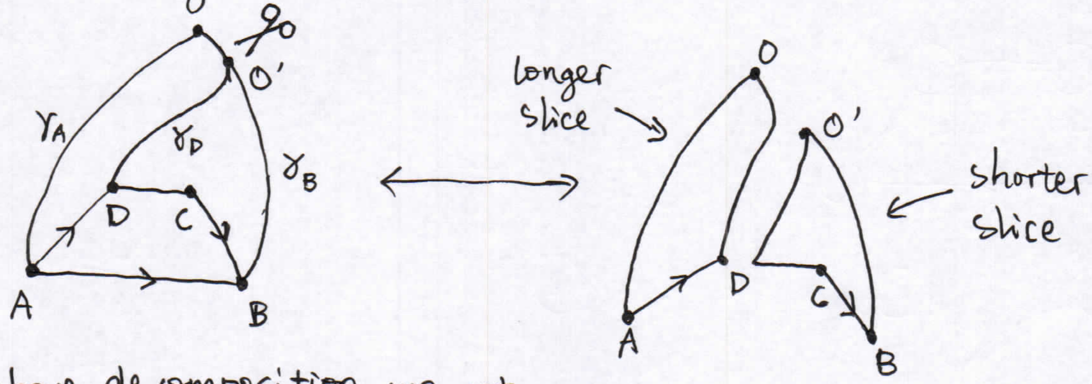
This is true thanks to the fact that, in case I for example, the geodesic γ_D may merge with γ_A or γ_B before it reaches O.

Case I

γ_D merges with γ_A.



γ_D merges with γ_B



Using the above decomposition we get

$$\# S_n = \delta_{n,0} + 3 \sum_{n_1+n_2=n-1} (\# S_{n_1}) \times (\# S_{n_2}) \qquad \left(\text{By convention } S_n = \emptyset \text{ for } n<0 \right)$$

$$\sum_n g^n \times \quad R(g) = 1 + 3g\, R(g)^2 .$$

— Answer 4 —

We want to show the following decomposition

$$S_{n,i} = \{\mathrm{I}\} \sqcup \{\mathrm{II}\} \sqcup \{\mathrm{III}\}$$

$$\{\mathrm{I}\} \leftrightarrows \bigsqcup_{n_1+n_2=n-1} S_{n_1,i} \times S_{n_2,i}, \qquad \{\mathrm{II}\} \leftrightarrows \bigsqcup_{n_1+n_2=n-1} S_{n_1,i} \times S_{n_2,i-1}, \qquad \{\mathrm{III}\} \leftrightarrows \bigsqcup_{n_1+n_2=n-1} S_{n_1,i+1} \times S_{n_2,i}$$

For example, the bijection $\{\mathrm{I}\} \rightleftharpoons \bigsqcup_{n_1+n_2=n-1} S_{n_1,i}\times S_{n_2,i}$ means that, whenever we take tw slices L and R of respective length $j_1\le i$ and $j_2\le i$, and that we glue them together according to scheme I, we get a larger slice of length $j\le i$.

Indeed, if $j_1\le j_2$, then we fall into the situation "γ_D merges with γ_A" shown on the last page. In this case we have $j=j_2$.

If $j_1>j_2$, we fall into the situation "γ_D merges with γ_B", then $j=j_1$

therefore, in Case I, $\boxed{j=\max(j_1,j_2)}$,

$$\text{so}\quad \{\mathrm{I}\} \rightleftharpoons \bigsqcup_{n_1+n_2=n-1} S_{n_1,i}\times S_{n_2,i}\quad \text{is bijective.}$$

We can treat $\{\mathrm{II}\}$ (where $j=\max(j_1,j_2+1)$)

and $\{\mathrm{III}\}$ (where $j=\max(j_1-1,j_2)$) similarly.

We conclude by the set decomposition that

$$\#S_{n,i}=\delta_{n,0}\,\delta_{i,1}+\sum_{n_1+n_2=n-1}(\#S_{n_1,i})(\#S_{n_2,i})+(\#S_{n_1,i})(\#S_{n_2,i-1})+(\#S_{n_1,i+1})(\#S_{n_2,i})$$

$$\sum_n g^n\times \quad R_i(g)=\delta_{i,1}+g\,R_i(g)\cdot\big(R_i(g)+R_{i-1}(g)+R_{i+1}(g)\big)$$

— Answer 5 —

We construct a bijection between

- $Q_{n,p}$: the set of rooted quadrangulations with a (connected) boundary of length $2p$, having n internal faces and a distinguished vertex.

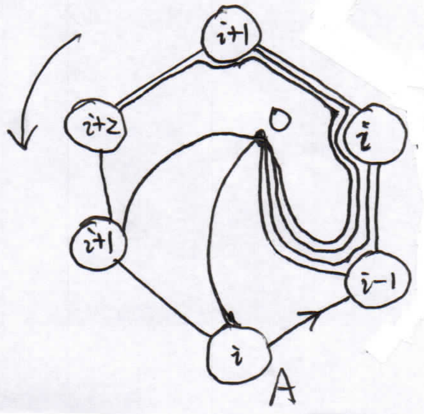
and

- $W_p\times\left(\bigsqcup_{n_1+\dots+n_p=n} S_{n_1}\times\dots\times S_{n_p}\right)$: W_p is the set of all walks on $\mathbb{Z}$ of steps ±1 (Dyck's paths) which start at $t=0$ from $x=0$ and ends at $t=2p$ and $x=0$.

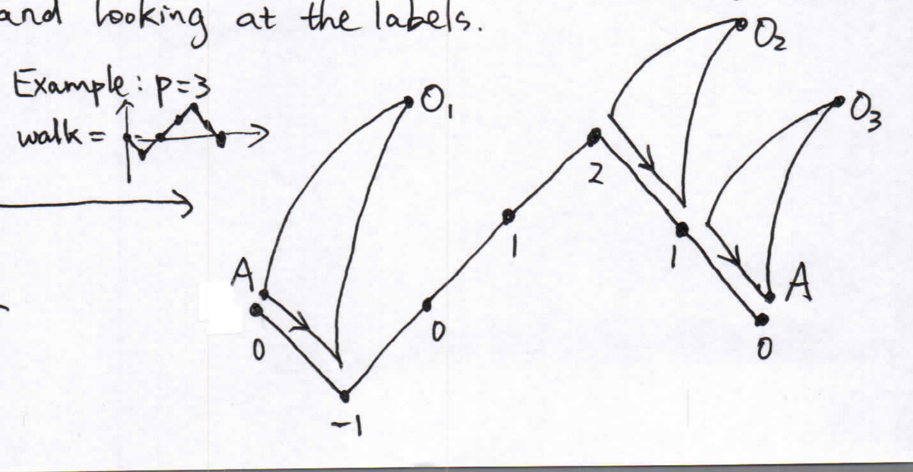
Construction:

$Q_{n,p}\longrightarrow W_p\times\bigsqcup_{n_1+\dots+n_p=n}\cdots$: Let A be the root vertex and O the distinguished vertex in a quadrangulation $Q\in Q_{n,p}$.

— Label the boundary vertices of Q by their distances to O, and subtract the label of A. Then we get a walk in W_p by turning around the boundary and looking at the labels.



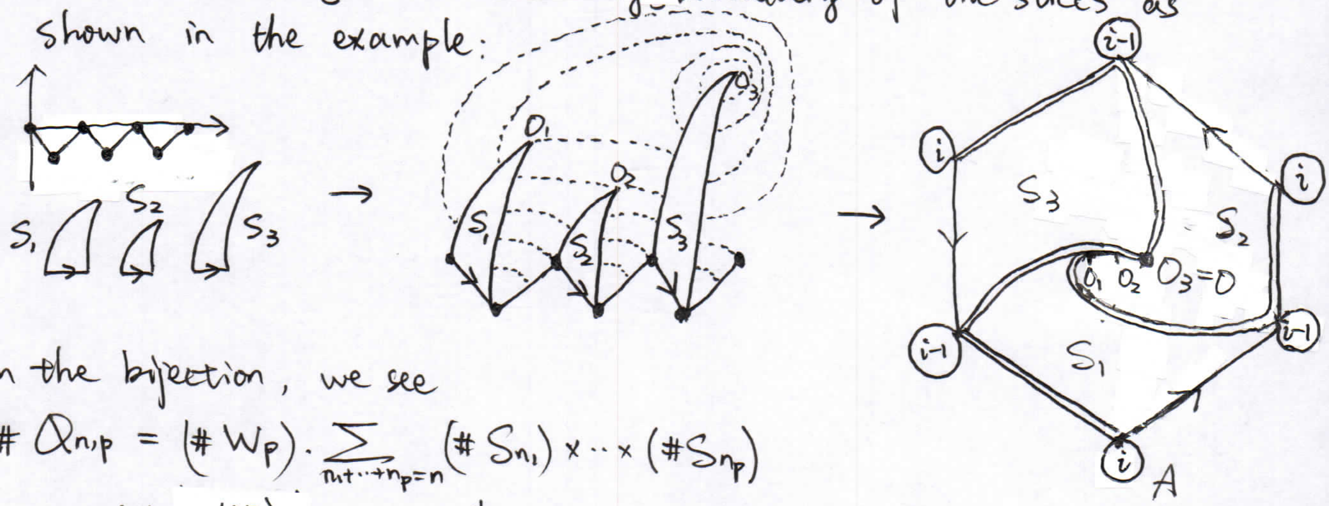
Example: p=3
walk =



- From each boundary vertex, draw the left-most geodesic to O.
  These geodesics divide Q into $2p$ slices, and generically, p slices among them are reduced to m_0 ($\bullet\!\!\longrightarrow\!\!\bullet$).
  These reduced slices corresponds to the increasing steps in the walk.
  The remaining slices contain $n_1+n_2+\cdots+n_p=n$ faces.

$W_p \times \left(\bigsqcup_{n_1+\cdots+n_p=n} S_{n_1}\times\cdots\times S_{n_p}\right) \longrightarrow Q_{n,p}$:

Given a walk in W_p and p slices $(S_1,\cdots S_p)\in S_{n_1}\times\cdots\times S_{n_p}$, first "put" the roots of the slices on the p decreasing steps of the walk. Then, glue the remaing boundary of the slices as shown in the example:



From the bijection, we see

$$\# Q_{n,p} = (\# W_p)\cdot \sum_{n_1+\cdots+n_p=n} (\# S_{n_1})\times\cdots\times(\# S_{n_p})$$

Since $\# W_p = \binom{2p}{p}$ (choose the p decreasing steps among $2p$ steps)

$\sum_n g^n$ gives $\boxed{F_p^\bullet = \binom{2p}{p} R^p}$

Since $R = 1+3gR^2$, we apply Lagrange inversion formula (c.f. exercise sheet I) with $u=R-1$ and $\phi(y)=3(y+1)^2$:

$u := R-1 = g\cdot 3(R-1+1)^2$

So $[g^n]\, R^p(g) = [g^n]\,(u+1)^p(g) = \frac{1}{n}[y^{n-1}]\left(\frac{d}{dy}(y+1)^p\right)\cdot\left(3(y+1)^2\right)^n$

$= \frac{1}{n}[y^{n-1}]\; p\cdot 3^n\cdot (y+1)^{2n+p-1} = \frac{p\cdot 3^n}{n}\cdot\frac{(2n+p-1)!}{(n-1)!\,(n+p)!}$

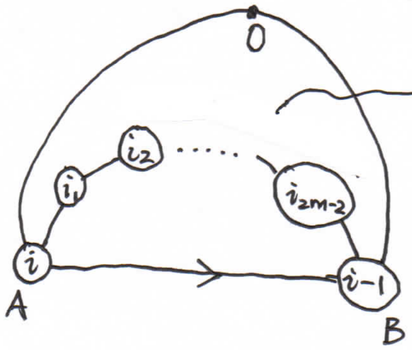
$\Rightarrow \boxed{[g^n]\,F_p^\bullet(g) = 3^n\cdot\frac{(2p)!\,(2n+p-1)!}{(p-1)!\,p!\,n!\,(n+p)!}}$

— Answer 6 —

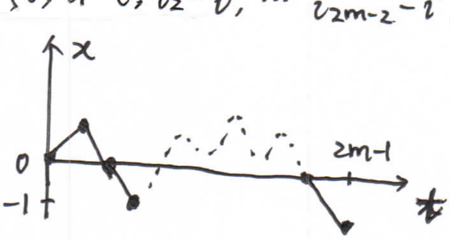
Now, a slice is <u>redefined</u> as a ~~rooted~~ rooted and pointed <u>$2m$</u>-angulation with simple boundary, satisfying similar conditions as before.

Since a $2m$-angulation is also bipartite, the result from Question 1 remains valid, and we can define the length of a slice as before.

Then we can repeat the recursive decomposition shown in Answer 3.
The only difference is that now we have, instead of 3, $\binom{2m-1}{m}$ cases for the labels of the vertices on the face adjacent to the root edge.
These cases correspond to all the walks of step ± 1 from $t=0$, $x=0$ to $t=2m-1$, $x=-1$:



walk $= (0, i_1 - i, i_2 - i, \cdots i_{2m-2} - i, -1)$



As before, in each of these $\binom{2m-1}{m}$ cases, one draw the left-most geodesics from each vertex of the revealed face to 0.
These geodesics cut the slice into m (not $2m-1$) smaller slices.
(The parts above the edges of the form i_k — i_k+1 are reduced to m_0).
Let $S_n^{(m)}$ be the set S_n in the case of $2m$-angulation, then we have a bijection (for $n \geq 1$):

$$S_n^{(m)} \rightleftharpoons \widetilde{W}_m \times \left(\bigsqcup_{n_1 + \cdots + n_m = n-1} S_{n_1}^{(m)} \times \cdots \times S_{n_m}^{(m)} \right)$$

$\widetilde{W}_m$: the set of walks from $(0,0)$ to $(2m-1, -1)$.

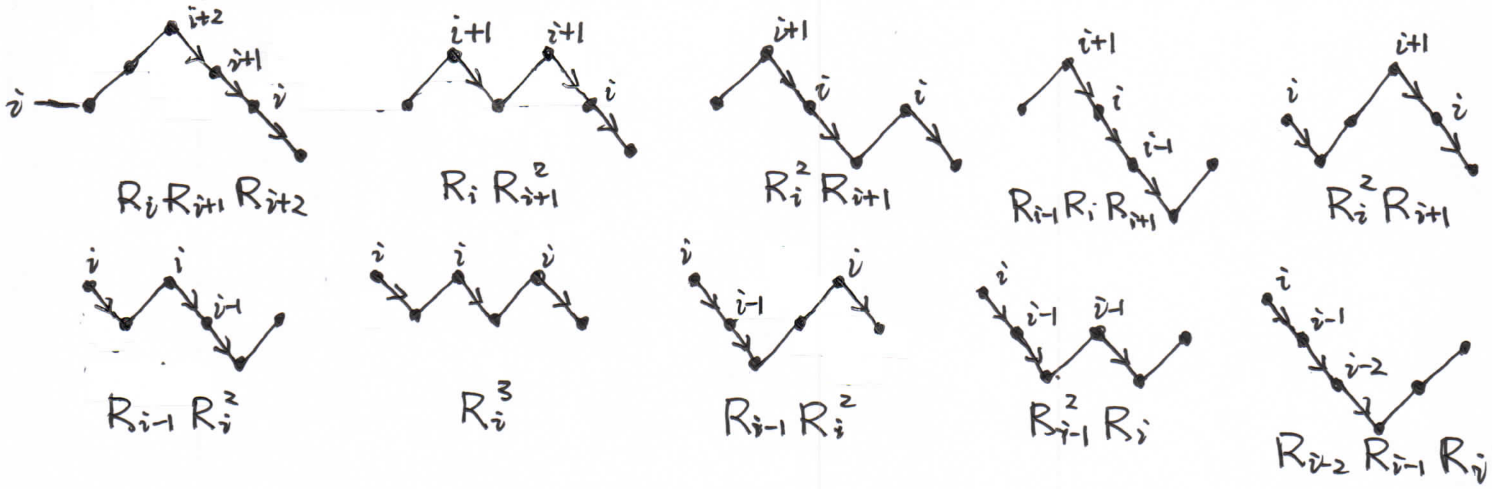
Therefore

$$\# S_n^{(m)} = \delta_{n,0} + \binom{2m-1}{m} \cdot \sum_{n_1 + \cdots + n_m = n-1} (\# S_{n_1}^{(m)}) \times \cdots \times (\# S_{n_m}^{(m)})$$

$$\xRightarrow{\sum_n g^n \times} \boxed{R(g) = 1 + g \binom{2m-1}{m} R^m(g)}$$

When $m=3$, we can work out the relation between the length of the original slice and the length of the small slices obtained from the cut.
In each of the $\binom{2m-1}{m} = \binom{5}{3} = 10$ cases, we should replace the R^m in (5) by $\prod_{k=1}^{m} R_{i_k}$, where $(i_1, \ldots, i_m)$ are the positions (y-coordinate) from where the walk in $\widetilde{W}_3$ made a step -1.

Explicitly:



So the equation for R_i in the case $m=3$ is

$$R_i = 1 + gR_i\left[R_{i+1}(R_{i+2} + R_{i+1} + 2R_i) + R_{i-1}R_{i+1} + R_i^2 + R_{i-1}(R_{i-2} + R_{i-1} + 2R_i)\right].$$

(4) holds unchanged because the bijection constructed in Answer 5 remains.

— Answer 7 —

Now redefine slices as triangulations with simple boundary.

It always follows from the definition that $|\gamma_A| - |\gamma_B| \in \{0,1\}$

But since a triangulation is not bipartite, we can not rule out the case 0.

↳ So there are 2 types of slices now. We enumerate the slices in which $|\gamma_A| - |\gamma_B| = 1$ by R_i and R as before, and enumerate the slices with $|\gamma_A| - |\gamma_B| = 0$ by S_i and S.

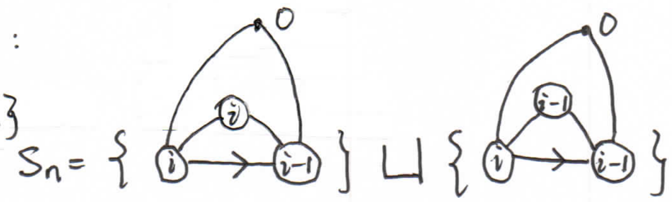
— Answer 8 —

slice decomposition:

→ when $|\gamma_A| - |\gamma_B| = 1$:

$n=0$: $S_0 = \{m_0\}$

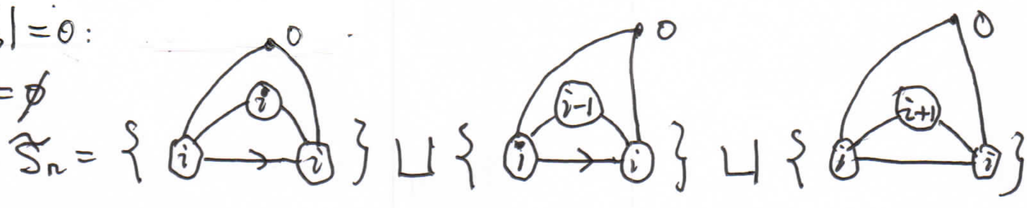
$n \geq 1$: $S_n =$



→ when $|\gamma_A| - |\gamma_B| = 0$:

$n=0$: $\tilde{S}_0 = \emptyset$

$n \geq 1$: $\tilde{S}_n =$



We get (7) and (8) in a similar way as in Question 3 and 4.

Solutions for the exercises session II.

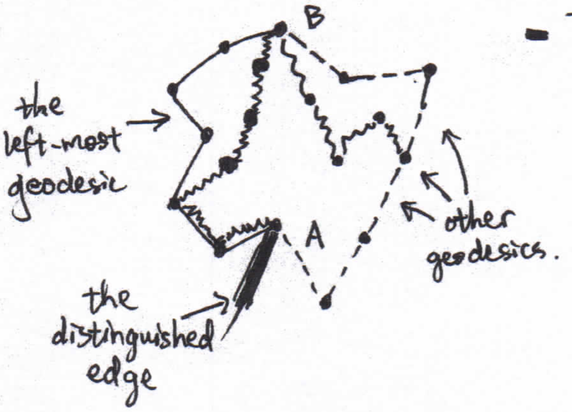
— Preliminary —: Geodesics on a graph

Let A and B be two vertices on the graph G,
γ is a path from A to B in G, we denote by $|\gamma|$ the number of edges it contains.

Def: γ' denotes a generic path from A to B.

- γ is a geodesic from A to B if $\forall \gamma'$, $|\gamma| \leq |\gamma'|$
- γ is the <u>unique</u> geodesic $A \to B$ if $\forall \gamma' \neq \gamma$, $|\gamma| < |\gamma'|$

When G is the graph of a planar map, and when one of the edges incident to A is distinguished, then we can define the <u>left-most</u> geodesic from A to B.



- The distinguished edge prevents us to "turn around" the vertex A, so that there is a well-define left $\to$ right order.
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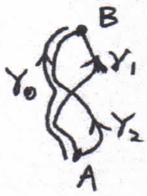
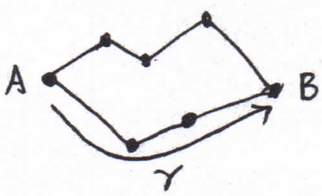
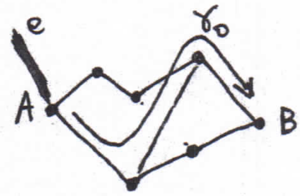


Examples :



(i)



(ii)

In (i), γ is the unique geodesic $A \to B$.

In (ii), it is no longer the case because of the existence of γ_0.

In (ii), γ_0 is the left-most geodesic if we add a distinguished edge e.

From the example (ii) we see that geodesics on graphs may do things that one does not expect from the Euclidean geometry.

Especially, these discrete geodesics may share one or more segments with each other, yet remaining distinct:

Possible configurations :