Coherent Network Analysis

- Inverse problem for bursts
- Likelihood analysis
- Detection statistics
- Astrophysical and network constraints

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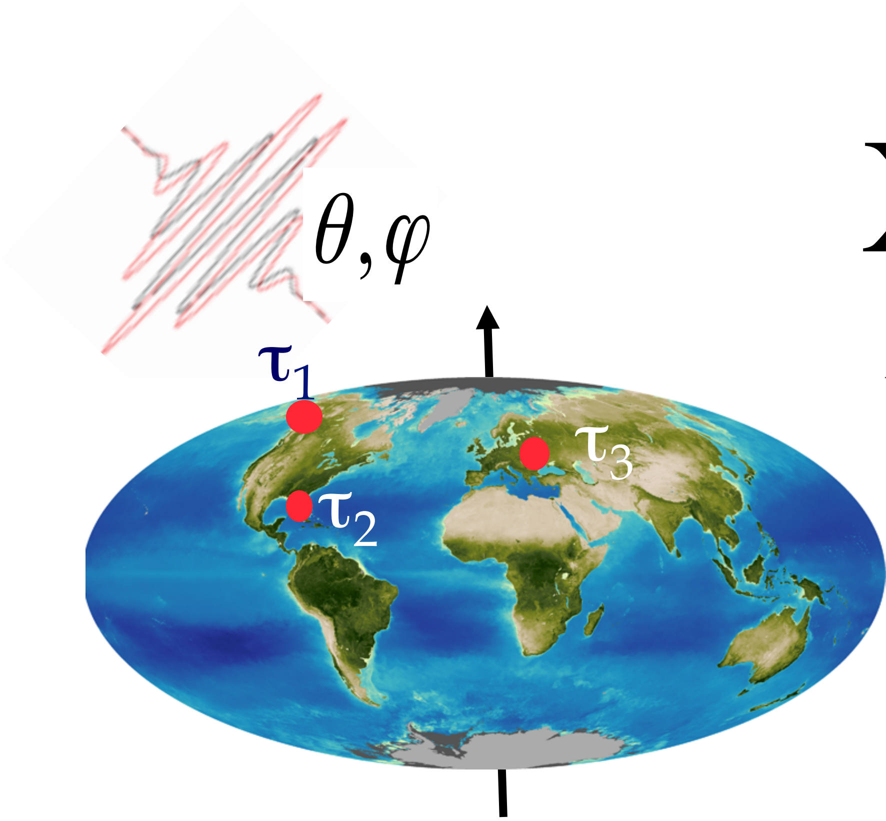
Recap

- Lecture 2 describes networks of detectors
  - detector & network response
  - fundamental network parameters and how they affect detection and reconstruction
  - polarization patterns
  - sky localization
- Lecture 3 describes analysis of a single detector
  - data conditioning & regression
  - time-frequency transformations
  - multi-resolution analysis
  - selection of excess power samples & clustering
- In this lecture we combine all this together in the framework of the coherent network analysis

Inverse Problem for GW transients



$$X = F \times H + N$$

data = network x wave + noise

Data analysis questions:

1. Detection: Is GW signal present in X?
2. Reconstruction: What can we learn about H from X?

DA scenarios:	known	unknown
arrival time τ	ExtTrig	all-time
arrival direction (θ,ϕ)	ExtTrig	all-sky
GW waveforms	template	unmodeled

Likelihood Method

Guersel&Tinto, 1998
lanagan & Hughes, 199
Finn, 2001

- **Likelihood ratio** (global fit to GW data):
- **Noise model:** usually multivariate Gaussian noise

$$\Lambda = \frac{p(X \mid h)}{p(X \mid 0)}$$

- **signal model (defined by detector response)**

$$p(X \mid 0) \propto \exp[-X\Sigma^{-1}X^{T}] \qquad \Sigma\text{-noise covariance matrix}$$

$$\vec{\xi}[i] = h_{+}[i]\vec{F}_{+} + h_{\times}[i]\vec{F}_{\times}, \quad h_{+}(\Omega), h_{\times}(\Omega), \quad \Omega - \text{signal model}$$

$$p(X \mid h) \propto \exp[-\left(X-\xi\right)\Sigma^{-1}\left(X-\xi\right)^{T}]$$

- **find GW polarizations (*h_+*,*h_x*) at maximum of Λ**
- **find source sky location by variation of Λ over θ and ϕ**
- **Ambiguity due to a large number of free parameters**

Matched Filter

$$L = 2\ln\Lambda = 2\sum_i \left(\vec{X}[i]\cdot\vec{\xi}[i,h]\right) - \sum_i \left(\vec{\xi}[i,h]\cdot\vec{\xi}[i,h]\right)$$

modeled(Inspiral)

- ξ is calculated from theoretical waveforms $h_+ h_x$ described by source parameters Ω
- Parameter space Ω is constrained by the model
- Sample Ω with templates (explicit template banks)
- Find τ, θ, ϕ, Ω (thus ξ) from best matching template
- Increase Ω by expanding models: spin, eccentricity, etc

un-modeled(burst)

- Amplitudes $h_+[i]$, $h_x[i]$ are free source parameters
- Parameter space is constrained by signal duration and bandwidth
- Search through parameter space analytically.
- Find τ, θ ,ϕ, ξ at maximum of L
- Decrease parameter space by adding astrophysical constraints

conceptually the same method, but approaches is radically different

Standard likelihood solution for inspirals

"forward" approach

- Select source model
  - for example, non-spinning, non-eccentric BHs
- Select parameter space
  - range of total masses
  - range of mass ratios
  - … other parameters for more complex models
- Construct template bank of detector responses covering the source parameter space, inclination angles and sky locations. Make sure there are no cracks in the coverage – overlap > 0.98 between nearby templates
- Find matching template (and thus source parameters) at max likelihood
  - Find nearby templates to estimate errors
- Practical inspiral algorithms do not really work this way
  - detection and reconstruction algorithms are quite different
  - optimal placement of templates is very non-trivial
  - To make sure that astrophysical sources (NSNS,NSBH,BHBH) are not missed template bank should be expanded to cover the whole parameter space (17par)

Network response to a GW event

- Consider a GW event consisting of I TF samples

$$\begin{bmatrix} \xi[1] \\ \xi[2] \\ \dots \\ \xi[I] \end{bmatrix} = \begin{bmatrix} f[1] & 0 & .. & 0 \\ 0 & f[2] & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & f[I] \end{bmatrix} \begin{bmatrix} h[1] \\ h[2] \\ \dots \\ h[I] \end{bmatrix}$$

$$\Xi = F\,H$$

  - Ξ – network response to a GW event
  - F – network matrix
  - H – GW amplitudes

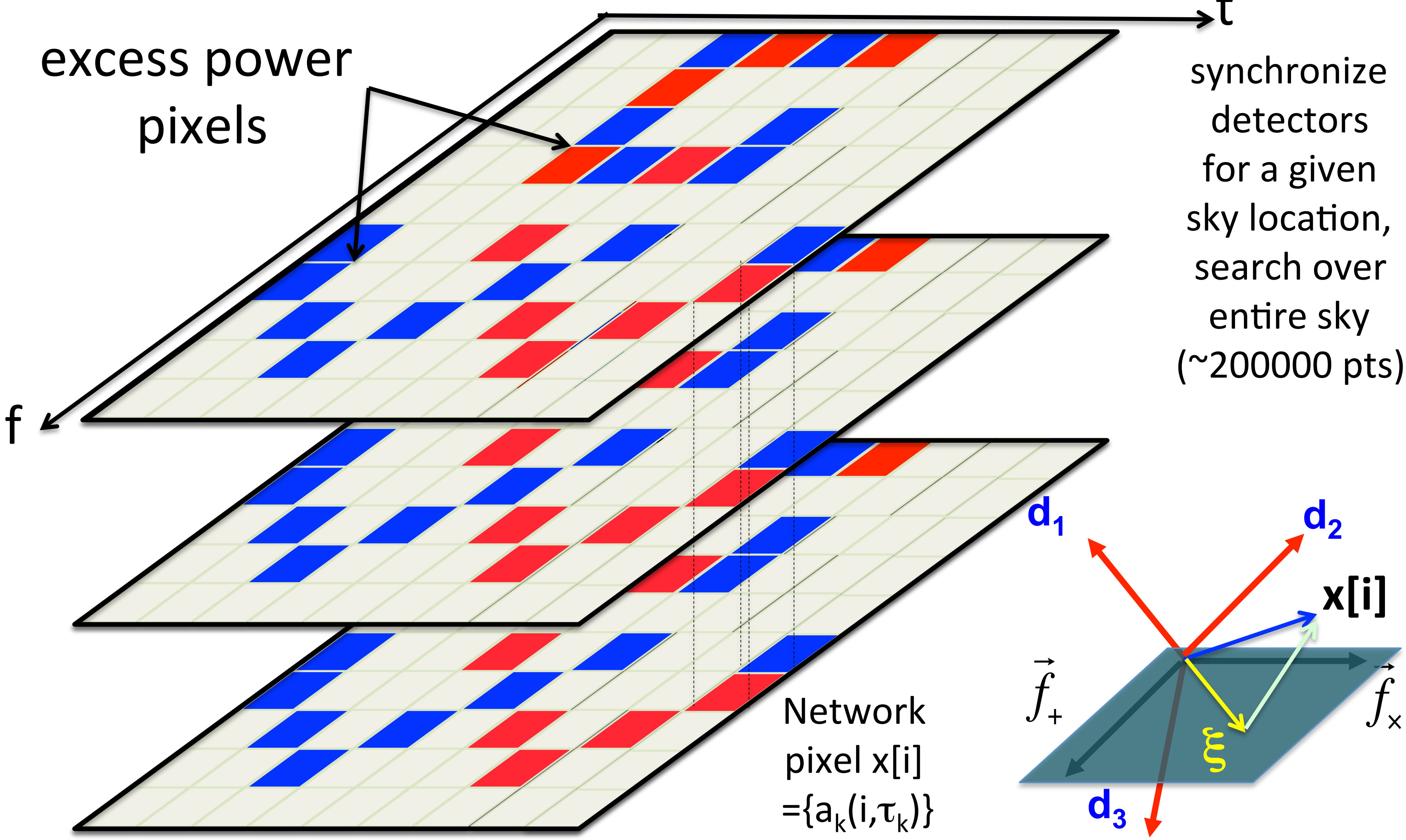
- Network event X, where N is network noise.

$$X = F\,H + N$$

- Template search: events are matching waveforms in the bank
- Burst search: How do we define a network event?

Burst Network Event

- after conditioning detector data is transformed into WDM domain, whitened, excess power (above Gaussian noise) data samples are selected.



Standard likelihood solution for bursts

"inverse" approach

$$F = \begin{bmatrix} f[1] & 0 & .. & 0 \\ 0 & f[2] & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & f[I] \end{bmatrix}$$

- Select sky location (θ,ϕ)
  - calculate network matrix *F* for TF "event" {1,..,I}
  - Calculate data vector *X* by time-shifting data streams to synchronize detectors: $X = \{\vec{\mathrm{x}}[1],\ldots,\vec{\mathrm{x}}[\mathrm{I}]\}$
- Parameterize GW signal: $H = \{\vec{h}[1],\ldots,\vec{h}[[\mathrm{I}]\},\ h[i] = \left(h_{+}[i], h_{\times}[i]\right)$
- Find likelihood and its derivatives

$$L = 2\ln\Lambda = X^T\left(FH\right) + \left(FH\right)^T X - \left(FH\right)^T\left(FH\right) \qquad \frac{\partial L}{\partial h} = 0$$

- Solution for H is coherent combination of X
- Repeat for all-sky locations maximizing $L(H_s)$
- Find waveforms H_m and (θ_m,ϕ_m) at max{L}
- Confront waveforms with source models

$$H_s = \underbrace{\left(F^T F\right)^{-1} F^T}_{\text{Moore-Penrose inverse}} X$$

does not work for practical networks – MP inverse may not exist

Rank Deficiency of Network Matrix

$$F = \begin{bmatrix} f[1] & 0 & .. & 0 \\ 0 & f[2] & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & f[I] \end{bmatrix} \qquad \vec{\xi}[i] = [\vec{f}_+[i], \vec{f}_\times[i]]^T \begin{bmatrix} h_+[i] \\ h_\times[i] \end{bmatrix} = f[i]\cdot \mathrm{h[i]}$$

i – is a single sample of network response

- Multiply data $\vec{x} = \vec{\xi} + \vec{n}$ by the network pattern vectors (i is omitted)
  - DPF is assumed $\left(\vec{f}_+ \cdot \vec{f}_\times\right) = 0$ - diagonalize network matrix

$$\begin{matrix} \left(\vec{\xi} + \vec{n}\right)\cdot \vec{f}_+ \\ \left(\vec{\xi} + \vec{n}\right)\cdot \vec{f}_\times \end{matrix} \longrightarrow \underbrace{\begin{bmatrix} \vec{x}\cdot\vec{f}_+ \\ \vec{x}\cdot\vec{f}_\times \end{bmatrix} = \begin{bmatrix} \left|\vec{f}_+\right|^2 & 0 \\ 0 & \left|\vec{f}_\times\right|^2 \end{bmatrix}\begin{bmatrix} h_+ \\ h_\times \end{bmatrix}}_{\partial L/\partial h_+ = 0,\ \partial L/\partial h_\times = 0} + \begin{bmatrix} \vec{n}\cdot\vec{f}_+ \\ \vec{n}\cdot\vec{f}_\times \end{bmatrix}$$

- $|f_\times| << |f_+|$ (A<<1) - hx can not be reconstructed from noisy data
- **need regulators – un-modeled constraints**

Klimenko, et al (2005)
Rakhmanov (2006)

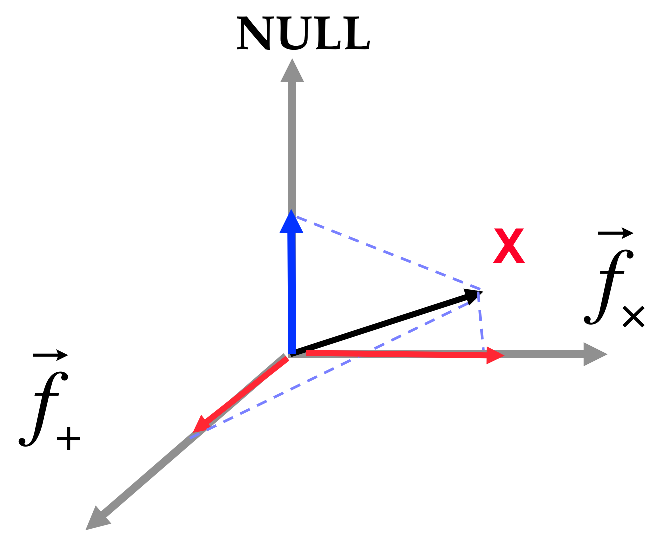
Network projections

- **To find detection statistic L_{max} we do not need explicit h_+ & h_x**
- **$L_{max} = L_+ + L_x$**

$$\begin{bmatrix} \vec{x}\cdot\vec{f}_+ \\ \vec{x}\cdot\vec{f}_\times \end{bmatrix} = \begin{bmatrix} \left|\vec{f}_+\right|^2 & 0 \\ 0 & \left|\vec{f}_\times\right|^2 \end{bmatrix} \begin{bmatrix} h_+ \\ h_\times \end{bmatrix}$$

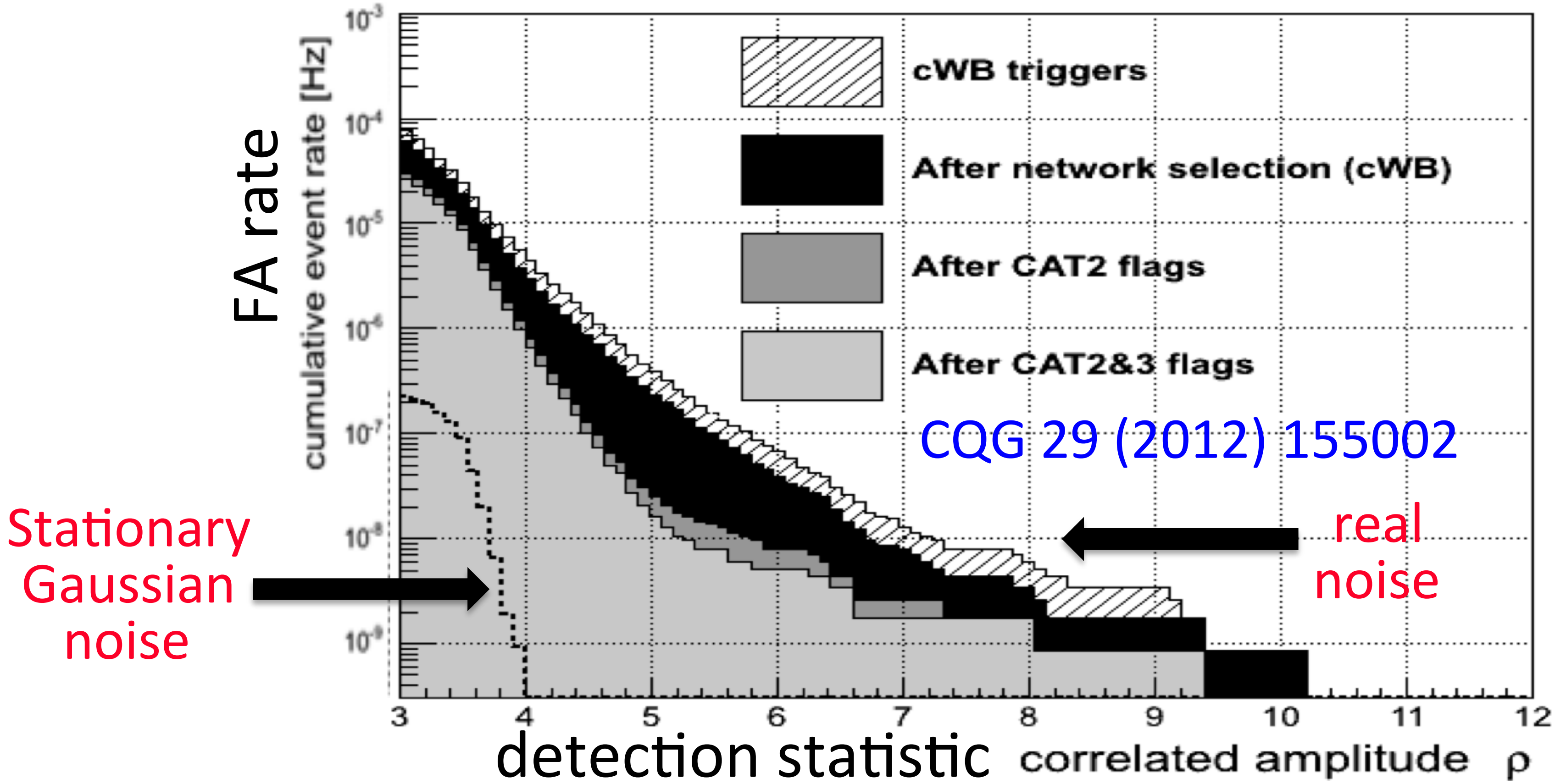
$$L_+ = \frac{\left(\vec{x}\cdot\vec{f}_+\right)^2}{|\vec{f}_+|^2} = X^T P_+ X, \qquad P_{+ij} = \frac{f_{+i} f_{+j}}{|f_+|^2} = \mathrm{e}_{+\mathrm{i}}\mathrm{e}_{+\mathrm{j}}$$

$$L_+ = \frac{\left(\vec{x}\cdot\vec{f}_\times\right)^2}{|\vec{f}_\times|^2} = X^T P_\times X, \qquad P_{\times ij} = \frac{f_{\times i} f_{\times j}}{|f_\times|^2} = \mathrm{e}_{\times\mathrm{i}}\mathrm{e}_{\times\mathrm{j}}$$



- **Textbook detection: given L_{max} calculate probability to mimic it by noise (significance), declare discovery of GWs if significance >5σ**

Real-life Detection



- **Data is non-stationary, non-gaussian and affected by artifacts**
- **Empirical background sample for estimation of FA probability**
  - **constructed by time-shifting data → may be biased wrt true background**
  - **need a massive background set (T observation x 10^6)**

Coherent Statistics

- **True GW signal should be in the f_+,f_x plane**

$$\mathbf{L} = \mathbf{E} - \mathbf{N}$$

detected (signal) energy — total energy — noise (null) energy

- **Likelihood quadratic form**

$$L_{\max} = X^T P\, X, \qquad P_{nm} = \mathrm{e}_{+n}\mathrm{e}_{+m} + \mathrm{e}_{\times n}\mathrm{e}_{\times m}$$

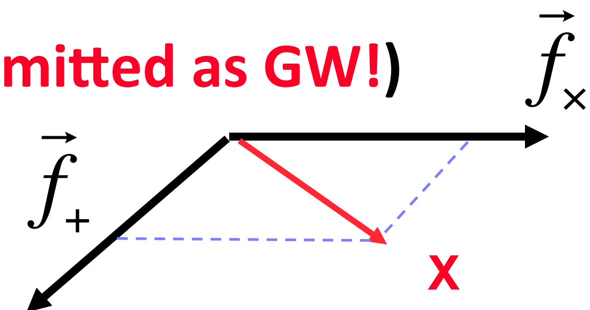
$$L = \sum_i \sum_{n,m} x_n[i] x_m[i] P_{nm}[i] = L_{i=j} + L_{i\neq j}$$

L matrix — incoherent — coherent

- **Detection statistics**
  - event ranking: characterize event strength, preferable if ~SNR
  - event consistency: significant null stream can be indication of a noise artifact

Two detector case

- no null space (any unconstrained event is admitted as GW!)
- A << 1 for significant fraction of the sky
- L=const(θ,φ)



$$\xi_1 = x_1, \quad \xi_2 = x_2 \qquad L_+ + L_\times = \langle x_1 x_1 \rangle + \langle x_2 x_2 \rangle$$

- **Two detector paradox** (Mohanty et al, CQG 21 S1831 (2004))
  - no x-correlation term in the likelihood matrix! $P_{12} = 0!$
  - contradict to the case of two co-aligned detectors where

$$\xi_1 = \xi_2 = \frac{x_1 + x_2}{2}, \qquad L_+ + L_\times = \tfrac{1}{2}\left[\langle x_1 x_1 \rangle + \langle x_2 x_2 \rangle + 2\langle x_1 x_2 \rangle\right]$$

power

cross-correlation

- What is meaning of coherent energy?

In-coherent/Coherent Energy

$$L_+ = \sum_{i,j} x_i x_j P_{ij,+} = E_{+(i=j)} + C_{+(i\neq j)}$$

$$L_\times = \sum_{i,j} x_i x_j P_{ij,\times} = E_{\times(i=j)} + C_{\times(i\neq j)}$$

- **quadratic forms C_+ & C_x depend on time delays between detectors and carry information about θ,ϕ – sensitive to source coordinates**
- **properties of the likelihood quadratic forms**

arbitrary network

$$\mathrm{cov}(L_+ L_\times) = 0$$

$$\mathrm{cov}(C_+ C_\times) = -\sum \mathrm{e}_{+\mathrm{i}}^2 \mathrm{e}_{\times\mathrm{i}}^2$$

$$\mathrm{cov}(E_+ E_\times) = \sum \mathrm{e}_{+\mathrm{i}}^2 \mathrm{e}_{\times\mathrm{i}}^2$$

2 detector network

$$C_+ + C_\times = 0$$

$$E_+ + E_\times = \mathrm{x}_1^2 + \mathrm{x}_2^2$$

- E+, Ex, C+, Cx are dependent
- **How should we calculate "generalized" network x-correlation?**

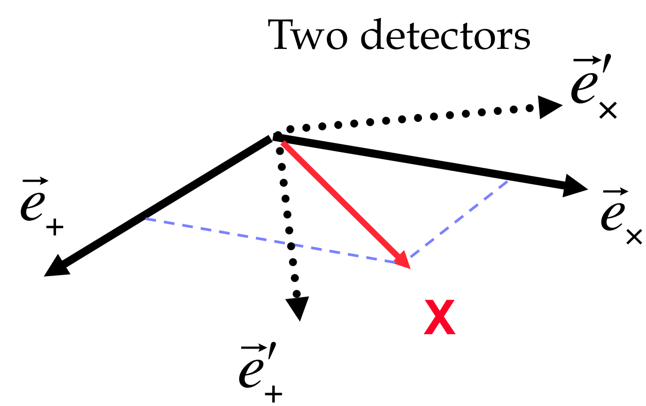
THE Projection Operator

- **Construction of the projection operator**

$$P_{nm} = \mathrm{e}_{+n}\mathrm{e}_{+m} + \mathrm{e}_{\times n}\mathrm{e}_{\times m}$$

is ambiguous: $e_+ e_+ \rightarrow rotation \rightarrow e'_+ e'_+$

$$L_{\max} = X^T P\ X = X^T P'\ X$$

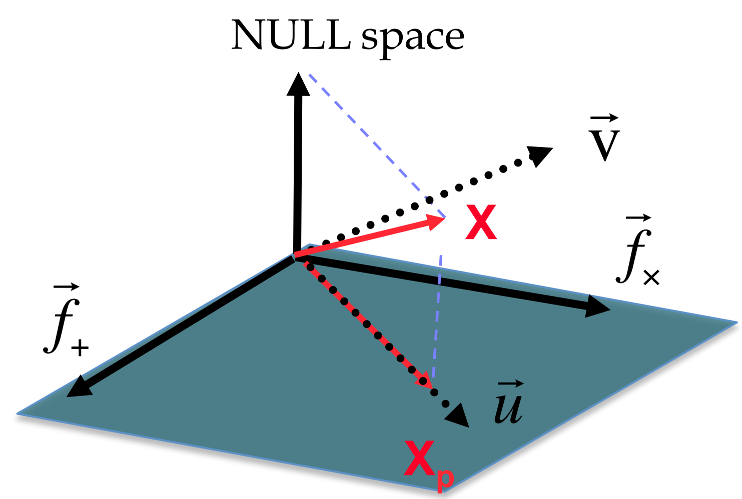


- **incoherent & coherent terms are not invariant**

- **Select the projection operator as**

$$P_{nm} = u_n u_m$$

(*solves two-detector paradox*)



$$\vec{u} \cdot \vec{\mathrm{v}} = 0$$

- **coherent/incoherent energies**

$$C = X^T P_u(n \neq m) X \qquad E_I = X^T P_u(n = m) X$$

Meaning of Coherent Energy

- **Reconstructed network response**

$$\vec{\xi} = (\vec{x}\cdot\vec{u})\ \vec{u}$$

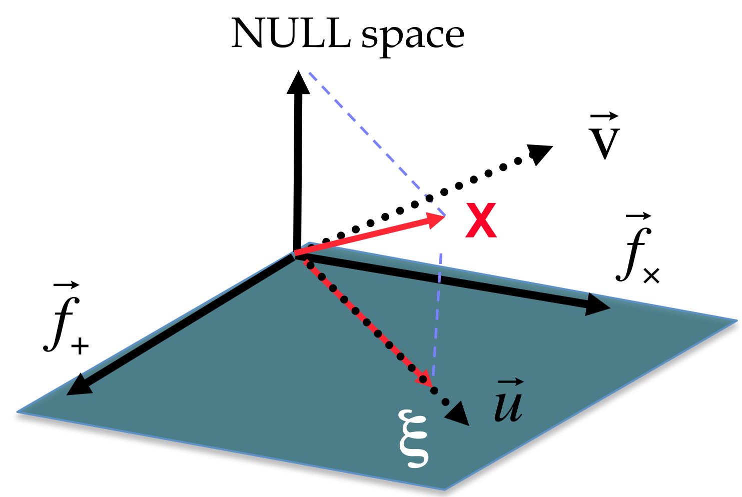
- **Total signal energy**

$$L_{\max} = (x\,|\,\xi) = \sum_i (\vec{x}[i]\cdot\vec{u}[i])^2$$

- **Lets consider the case when x=ξ**

$$L_{\max} = \sum_i |\vec{\xi}[i]|^2 \cdot (\vec{u}[i]\cdot\vec{u}[i])^2$$

$$= \sum_i |\vec{\xi}[i]|^2 \sum_{nm} u_n^2[i]u_m^2[i]$$



$$\vec{u}\cdot\vec{v} = 0$$

- **Coherent energy**

$$C = \sum_i |\vec{\xi}[i]|^2 \left[1 - \sum_n u_n^4[i]\right]$$

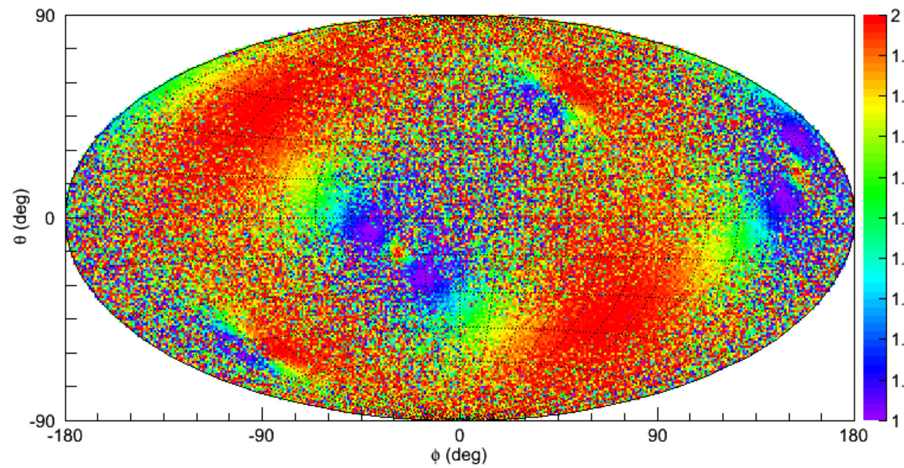
Response Index

$$I_r = \sum_k u^4[k]$$

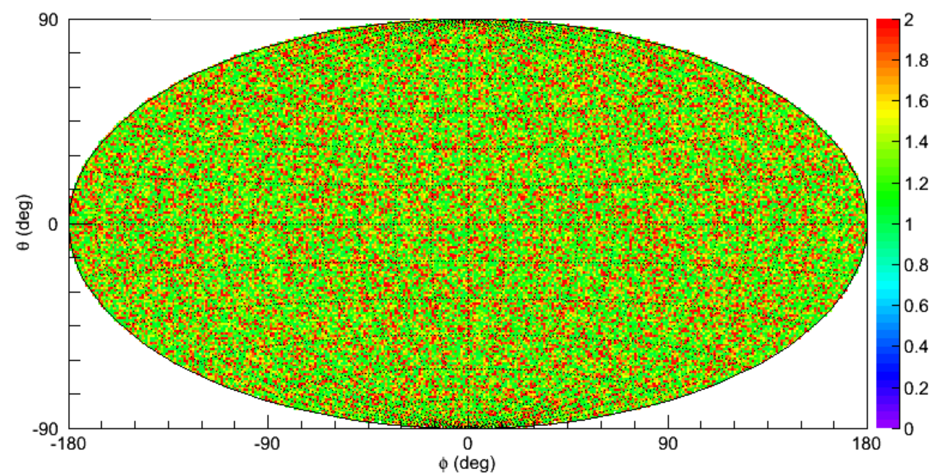
$$\vec{u} = \vec{\xi} / |\vec{\xi}|$$

- $1/I_r$ - effective number of detectors contributing to total network SNR: distributed between 1 and K
- For GW signals response index correlates with network index
- For noise and glitches there is no correlation
- Describes how similar (coherent) are responses in individual detectors
- Great tool to distinguish signal from glitches

HL: signal with random polarization



HL: random noise

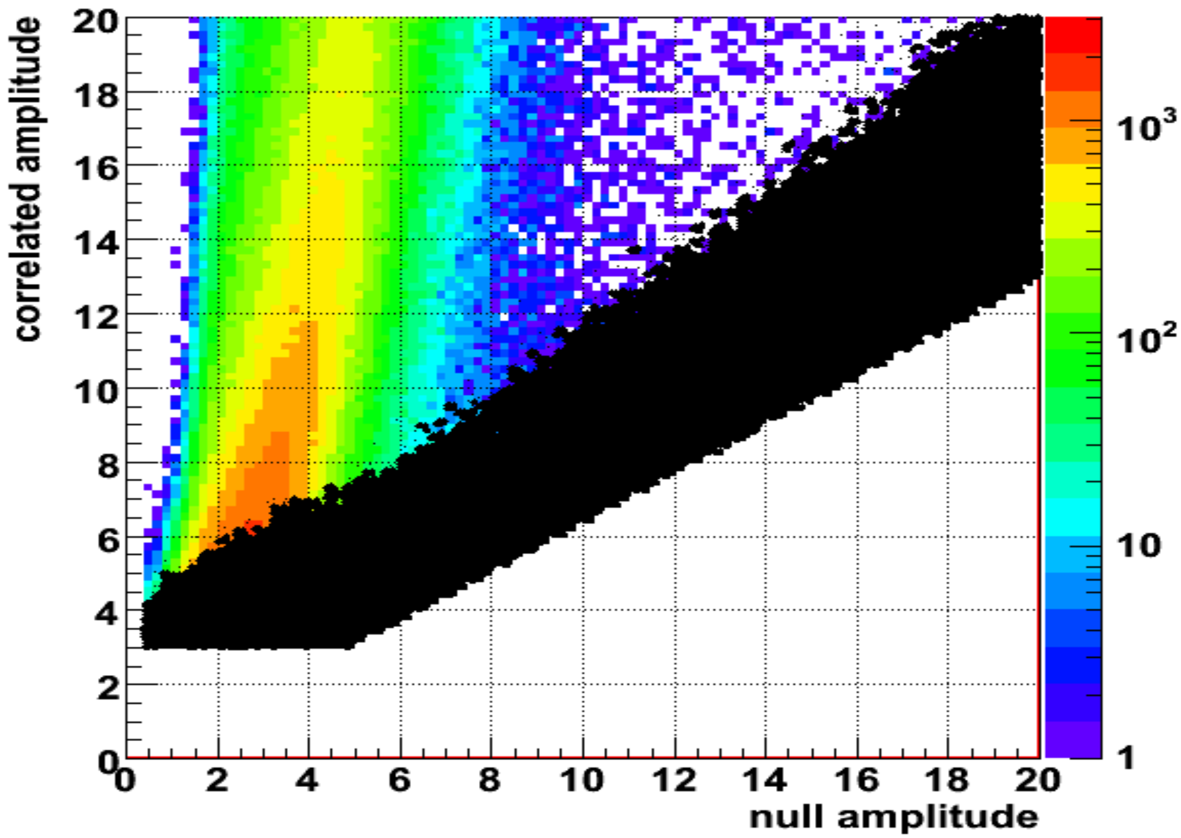


Detection statistics: coherent – null energy

- **coherent energy: sum of the off-diagonal elements of L matrix**

$$E_{coherent} = \sum_{i \neq j} L_{ij}$$

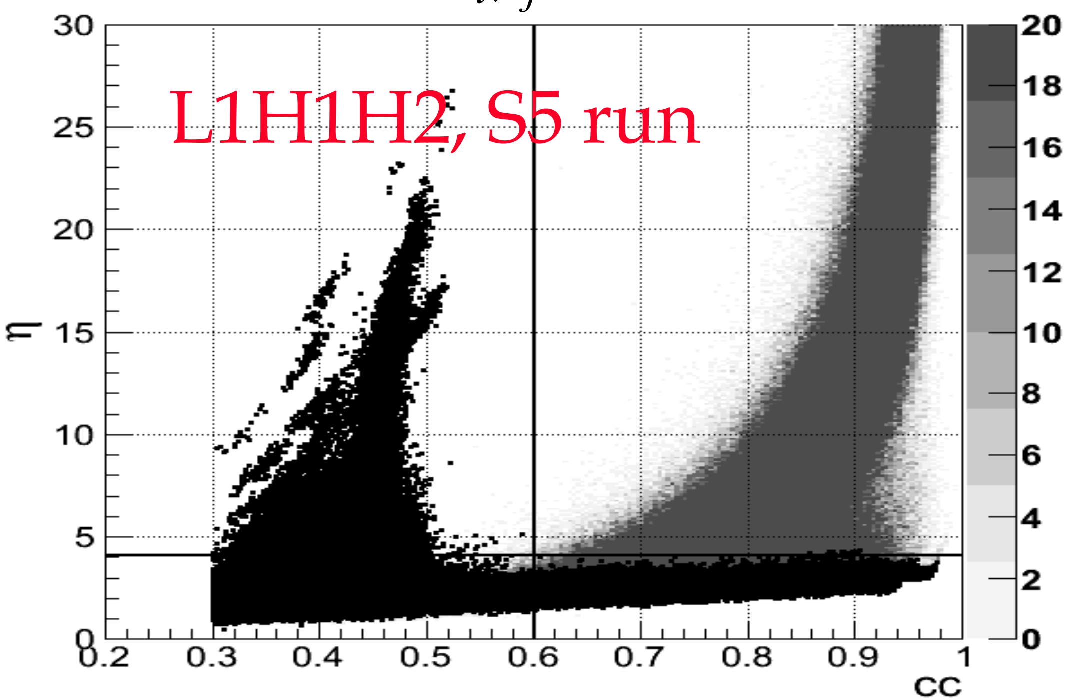
- **null energy *null*: energy of the reconstructed detector noise**



Rejection of glitches

- **Coherent statistics**
  - **Network correlation coefficient cc - rejection of glitches**
  - **network correlated amplitude η – event ranking statistic**

$$cc = \frac{E_{i \neq j}}{N + E_{i \neq j}} \qquad \eta = \sqrt{\frac{cc \cdot E_{i \neq j}}{K - 1}}$$



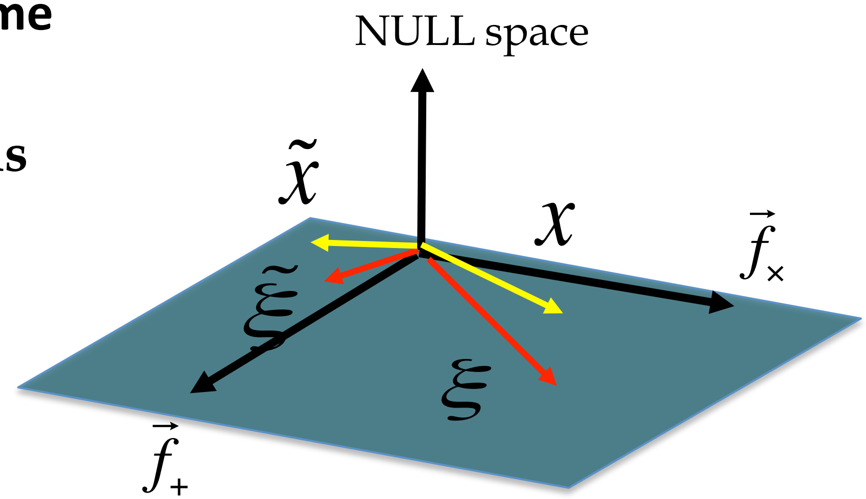
black - background
gray – GW injections

PRD 81 (2010) 102001

Use also DQ and Veto: characterization of detector noise is one of the most challenging tasks in the GW experiment

Dual Stream Phase Transform

- **Likelihood formalism is easily generalized for the dual data stream analysis**
  - **quadrature data stream contains the same information as x**
  - **network response can be presented as pairs of vectors** $\vec{\xi}, \vec{\tilde{\xi}}$



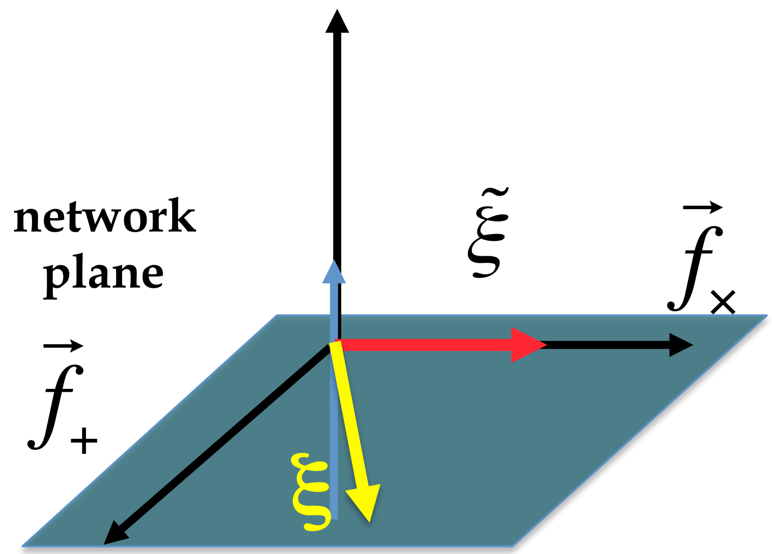
- **Phase transform**
  - **Apply phase transform to projections (don't care about projections out of plane)**

$$\xi = \xi' \cos(\lambda) + \tilde{\xi}' \sin(\lambda)$$
$$\tilde{\xi} = \tilde{\xi}' \cos(\lambda) - \xi' \sin(\lambda)$$

- **With appropriate phase transformation the polarization pattern is revealed**

Network Regulators & Constraints

- **For existing LHV networks the standard projection is rarely an optimal solution**



- **Network regulators → construct P by guessing orientation of the projection vector u (Klimenko et al, 2005)**
  - **hard regulator:**

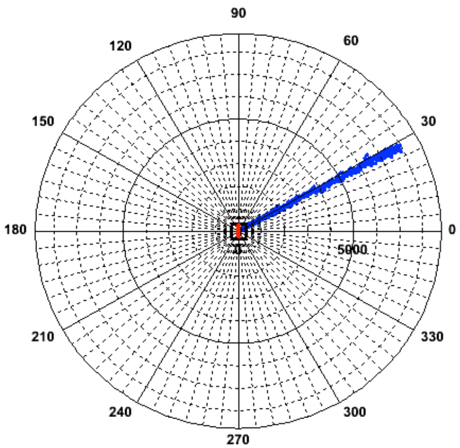
$$\vec{\xi} \to \vec{\xi}_+, \quad \vec{\tilde{\xi}} \to 0$$

gives optimal solution for closely aligned networks

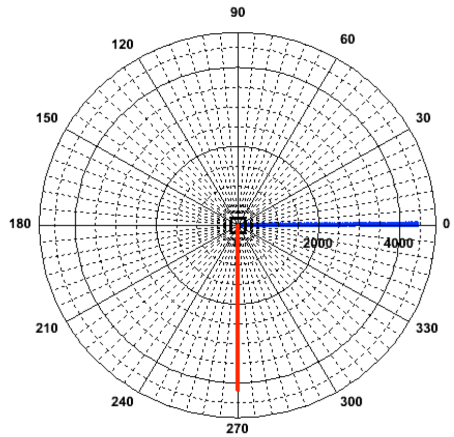
  - **soft regulator:** $\vec{\xi} \to \vec{\xi}_+, \quad \vec{\tilde{\xi}} \to \vec{\tilde{\xi}}$

after polarization phase transform

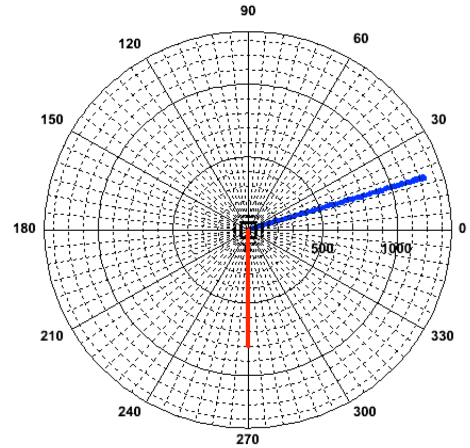
Polarization Constraints



linear

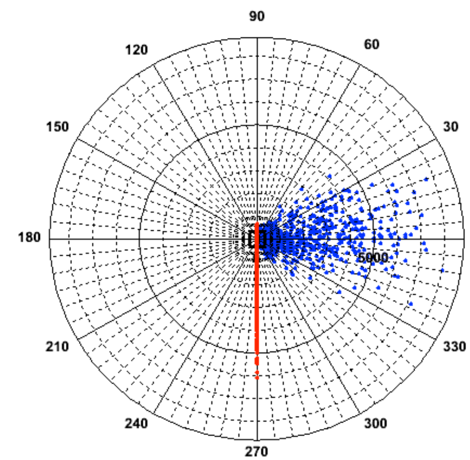


circular



elliptical

- **Just fit data to a selected polarization pattern!**



Other constrained likelihood solutions

- In addition to relatively simple network and polarization constraints, additional source models, even not accurate can be used to constrain the likelihood functional

$$L' = X^T (Fh) + (Fh)^T X - (Fh)^T (Fh) + \lambda g(X,h), \quad g(X,h) = 0$$

Lagrange multiplier

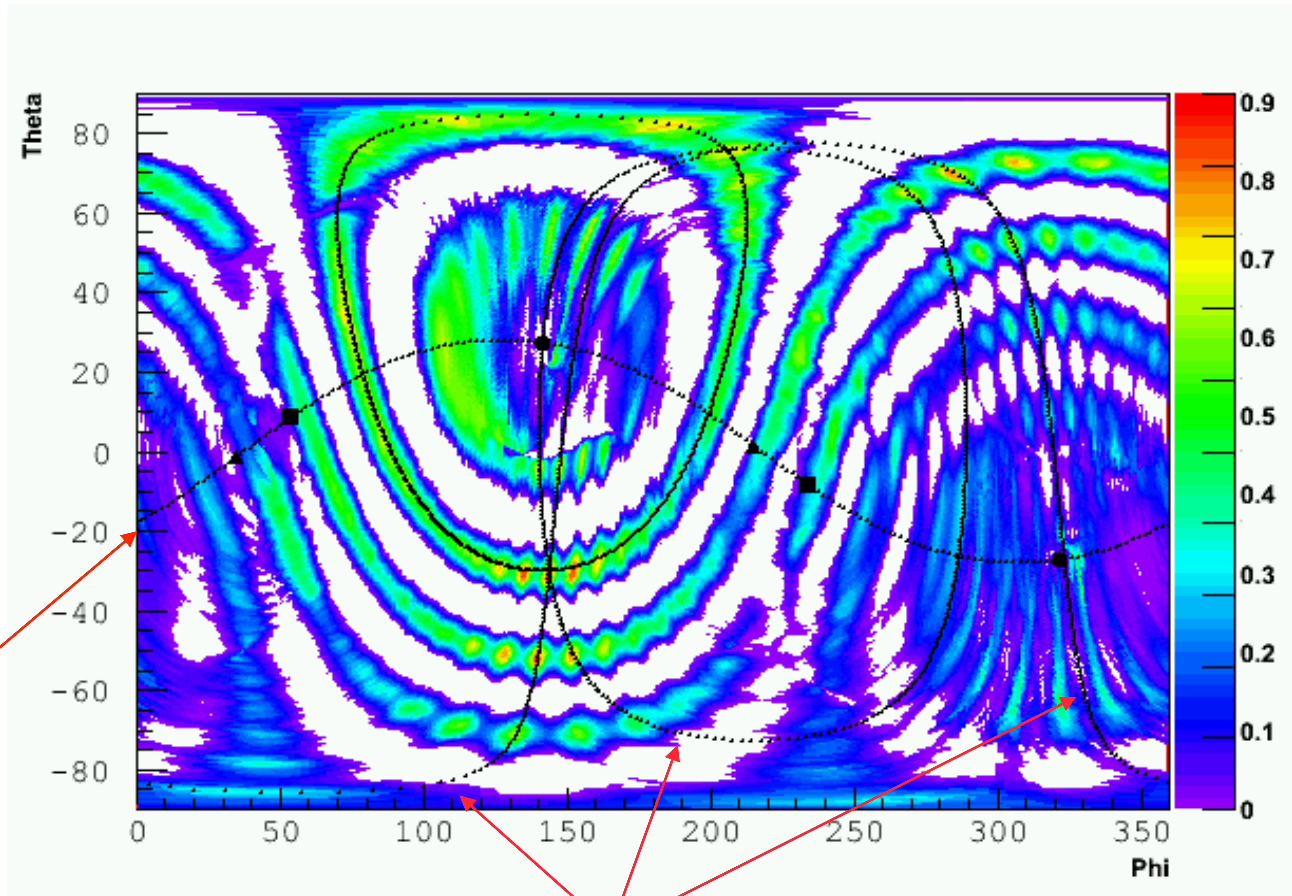
- g(X,h)=0 is a constraint condition

- Conceptually simple, but could be very hard to solve – in most cases there is no analytical solution.

Probability Sky Map

probability map: obtained from the likelihood sky distribution

PSM shows how consistent are reconstructed waveforms and time delays as function of θ,ϕ. Source location is at PSM max.

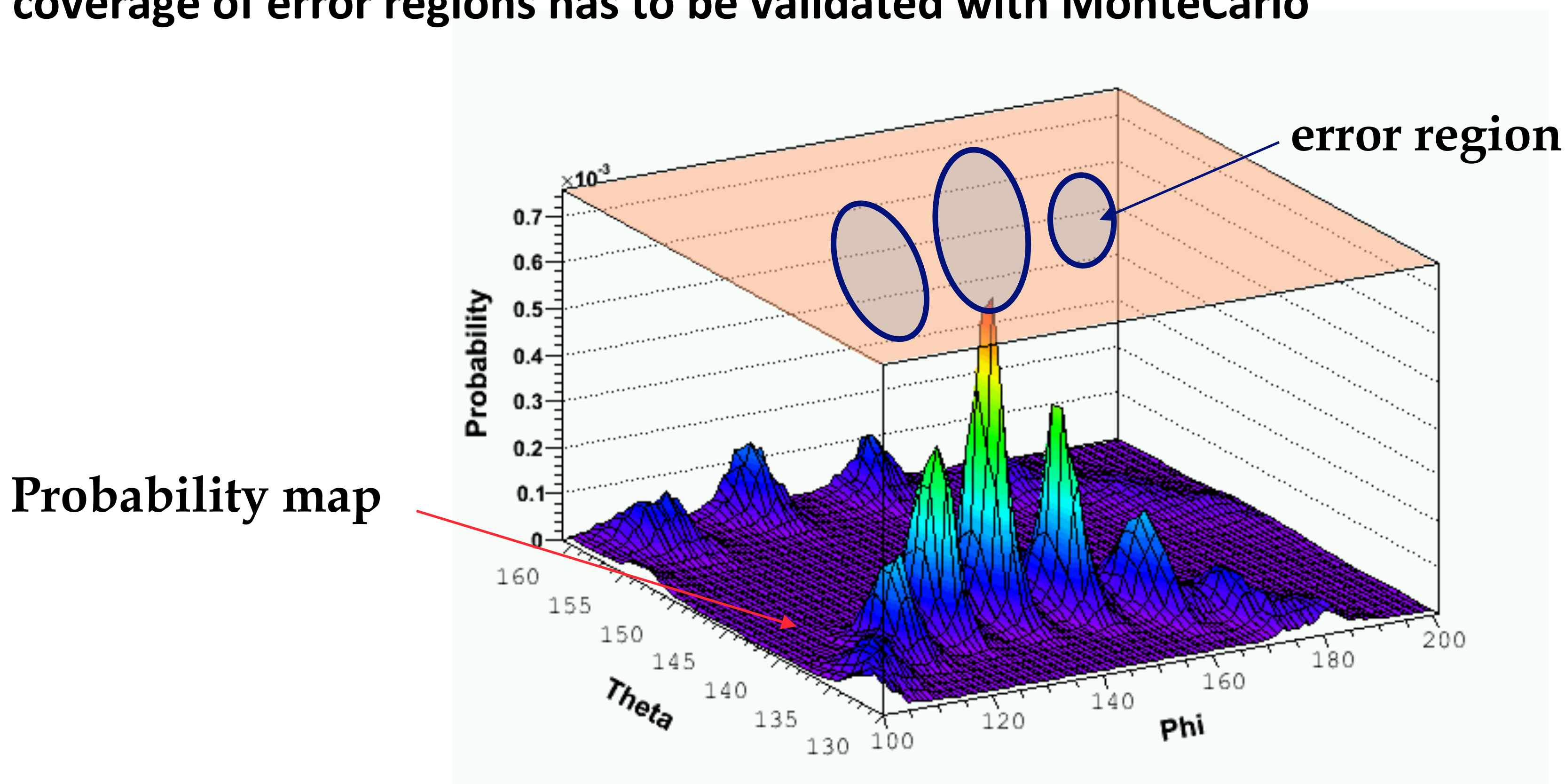


detector plane

constant delay rings for detector pairs

Error Regions

- **Source location is characterized by a spot in the sky (error region) rather than by a single (θ,ϕ) direction**
  - x% error region - a sky area with the cumulative probability of x%
- **The coverage of error regions has to be validated with MonteCarlo**



- **Error regions can be reported for optical/radio followup →multi-messenger observations**

Objectives of Coherent Network Analysis

- **Understand benefits and shortcomings of detector networks to detect sources and optimally capture science.**
- **Combine measurements from several detectors**
  - **confident detection; elimination of instrumental/environmental artifacts**
  - **reconstruction of GW polarizations**
  - **reconstruction of source coordinates**
  - **reconstruction of GW waveforms**
- **CAN is a unified approach to handle**
  - **arbitrary number of detectors at different locations and arm's orientations**
  - **variability of detector responses as function of source coordinates**
  - **differences in the strain sensitivity of detectors**
- **Extraction of source parameters**
  - **confront measured waveforms with source models or include models**

Reading Material

- LIGO/Virgo publications on burst searches: https://www.lsc-group.phys.uwm.edu/ppcomm/Papers.html
- Guersel,Tinto, PRD 40 v12,1989
  - reconstruction of GW signal for a network of three misaligned detectors
- Likelihood analysis: Flanagan, Hughes, PRD57 4577 (1998)
  - likelihood analysis for a network of misaligned detectors
- Two detector paradox: Mohanty et al, CQG 21 S1831 (2004)
  - state a problem within standard likelihood analysis
- Semi-coherent burst search. Klimenko S and Mitselmakher CQG 21 S1819 (2004)
- Constraint likelihood: Klimenko et al, PRD 72, 122002 (2005)
  - address problem of ill-conditioned network response matrix (rank deficiency)
  - first introduction of likelihood constraints/regulators
- Penalized likelihood: Mohanty et al, CQG 23 4799 (2006).
- Rank deficiency of network matrix: Rakhmanov, CQG 23 S673 (2006)
- GW signal consistency: Chatterji et al, PRD 74 082005(2006)
- Coherent Burst search: S. Klimenko et al., Class. Quantum Grav. 25, 114029 (2008)
- Sky localization with advanced network. S. Klimenko et al. PRD 83, 102001 (2011).
- Three figures of merit..., B.Schutz, *CQG* **28** 125023(2011)