Coherent Network Analysis

- Inverse problem for bursts
- Likelihood analysis
- Detection statistics
- Astrophysical and network constraints

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Recap

• Lecture 2 describes networks of detectors
  - detector & network response
  - fundamental network parameters and how they affect detection and reconstruction
  - polarization patterns
  - sky localization

• Lecture 3 describes analysis of a single detector
  - data conditioning & regression
  - time-frequency transformations
  - multi-resolution analysis
  - selection of excess power samples & clustering

• In this lecture we combine all this together in the framework of the coherent network analysis
Inverse Problem for GW transients

\[ X = F \times H + N \]

data = network x wave + noise

Data analysis questions:
1. Detection: Is GW signal present in X?
2. Reconstruction: What can we learn about H from X?

DA scenarios:
- Arrival time \( \tau \)
- Arrival direction \((\theta, \phi)\)
- GW waveforms

known  unknown
- ExtTrig  all-time
- ExtTrig  all-sky
- Template  unmodeled

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Likelihood Method

- **Likelihood ratio** (global fit to GW data):
  \[ \Lambda = \frac{p(X \mid h)}{p(X \mid 0)} \]

- **Noise model**: usually multivariate Gaussian noise
  \[ \Sigma \text{-noise covariance matrix} \]

- **signal model** (defined by detector response)
  \[
P(X \mid 0) \propto \exp[-X\Sigma^{-1}X^T]
  \]
  \[
  \tilde{\xi}[i] = h_+[i] F_+^\dagger + h_x[i] F_x^\dagger,
  \quad h_+(\Omega), h_x(\Omega), \quad \Omega - \text{signal model}
  \]

- **find GW polarizations** \((h_+, h_x)\) at maximum of \(\Lambda\)
- **find source sky location** by variation of \(\Lambda\) over \(\theta\) and \(\phi\)
- **Ambiguity due to a large number of free parameters**

Guersel&Tinto, 1998
lanagan & Hughes, 199
Finn, 2001
### Matched Filter

\[ L = 2 \ln \Lambda = 2 \sum_i \left( \tilde{X}[i] \cdot \tilde{\xi}[i,h] \right) - \sum_i \left( \tilde{\xi}[i,h] \cdot \tilde{\xi}[i,h] \right) \]

#### Modeled (Inspiral)
- \( \xi \) is calculated from theoretical waveforms \( h_+ h_x \) described by source parameters \( \Omega \)
- Parameter space \( \Omega \) is constrained by the model
- Sample \( \Omega \) with templates (explicit template banks)
- Find \( \tau, \theta, \phi, \Omega \) (thus \( \xi \)) from best matching template
- Increase \( \Omega \) by expanding models: spin, eccentricity, etc

#### Un-modeled (burst)
- Amplitudes \( h_+[i], h_x[i] \) are free source parameters
- Parameter space is constrained by signal duration and bandwidth
- Search through parameter space analytically.
- Find \( \tau, \theta, \phi, \xi \) at maximum of \( L \)
- Decrease parameter space by adding astrophysical constraints

Conceptually the same method, but approaches is radically different
“forward” approach

- Select source model
  - for example, non-spinning, non-eccentric BHs
- Select parameter space
  - range of total masses
  - range of mass ratios
  - ... other parameters for more complex models
- Construct template bank of detector responses covering the source parameter space, inclination angles and sky locations. Make sure there are no cracks in the coverage – overlap > 0.98 between nearby templates
- Find matching template (and thus source parameters) at max likelihood
  - Find nearby templates to estimate errors
- **Practical inspiral algorithms do not really work this way**
  - detection and reconstruction algorithms are quite different
  - optimal placement of templates is very non-trivial
  - To make sure that astrophysical sources (NSNS, NSBH, BHBH) are not missed, template bank should be expanded to cover the whole parameter space (17par)
Network response to a GW event

- Consider a GW event consisting of I TF samples

\[
\begin{bmatrix}
    \xi[1] \\
    \xi[2] \\
    \vdots \\
    \xi[I]
\end{bmatrix} =
\begin{bmatrix}
    f[1] & 0 & \ldots & 0 \\
    0 & f[2] & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & f[I]
\end{bmatrix}
\begin{bmatrix}
    h[1] \\
    h[2] \\
    \vdots \\
    h[I]
\end{bmatrix}
\]

\[\Xi = F \cdot H\]

- \(\Xi\) – network response to a GW event
- \(F\) – network matrix
- \(H\) – GW amplitudes

- Network event \(X\), where \(N\) is network noise.

\[X = F \cdot H + N\]

- Template search: events are matching waveforms in the bank
- Burst search: How do we define a network event?
Burst Network Event

- after conditioning detector data is transformed into WDM domain, whitened, excess power (above Gaussian noise) data samples are selected.

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**Standard likelihood solution for bursts**

**“inverse” approach**

- Select sky location \((\theta, \phi)\)
  - calculate network matrix \(F\) for TF “event” \(\{1, \ldots, I\}\)
  - Calculate data vector \(X\) by time-shifting data streams to synchronize detectors: \(X = \{\ddot{x}[1], \ldots, \ddot{x}[I]\}\)
- Parameterize GW signal: \(H = \{\dot{h}[1], \ldots, \dot{h}[I]\}, h[i] = (h_+[i], h_\times[i])\)
- Find likelihood and its derivatives

\[
L = 2 \ln \Lambda = X^T (FH) + (FH)^T X - (FH)^T (FH)
\]

\[
\frac{\partial L}{\partial h} = 0
\]

- Solution for \(H\) is coherent combination of \(X\)
- Repeat for all-sky locations maximizing \(L(H_s)\)
- Find waveforms \(H_m\) and \((\theta_m, \phi_m)\) at \(\max\{L\}\)
- Confront waveforms with source models

\[
H_s = \left( F^T F \right)^{-1} F^T X
\]

Moore-Penrose inverse

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Rank Deficiency of Network Matrix

\[ F = \begin{bmatrix} f[1] & 0 & \ldots & 0 \\ 0 & f[2] & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f[I] \end{bmatrix} \]

\[ \tilde{\xi}[i] = [\tilde{f}_+ [i], \tilde{f}_x [i]]^T \begin{bmatrix} h_+[i] \\ h_+ [i] \end{bmatrix} = f[i] \cdot h[i] \]

i – is a single sample of network response

- Multiply data \( \tilde{x} = \tilde{\xi} + \tilde{n} \) by the network pattern vectors (i is omitted)
  - DPF is assumed \( (\tilde{f}_+ \cdot \tilde{f}_x) = 0 \) - diagonalize network matrix

\[
\begin{align*}
(\tilde{\xi} + \tilde{n}) \cdot \tilde{f}_+ \\
(\tilde{\xi} + \tilde{n}) \cdot \tilde{f}_x
\end{align*}
\]

\[
\begin{bmatrix}
\tilde{x} \cdot \tilde{f}_+ \\
\tilde{x} \cdot \tilde{f}_x
\end{bmatrix} = \begin{bmatrix}
|\tilde{f}_+|^2 & 0 \\
0 & |\tilde{f}_x|^2
\end{bmatrix} \begin{bmatrix}
h_+ \\
h_x
\end{bmatrix} + \begin{bmatrix}
\tilde{n} \cdot \tilde{f}_+ \\
\tilde{n} \cdot \tilde{f}_x
\end{bmatrix}
\]

\[
\frac{\partial L}{\partial h_+} = 0, \quad \frac{\partial L}{\partial h_x} = 0
\]

- \( |f_x| << |f_+| \) (A<<1) - \( h_x \) can not be reconstructed from noisy data
- need regulators – un-modeled constraints

Rakhmanov (2006)

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Network projections

- To find detection statistic $L_{\text{max}}$ we do not need explicit $h_+ & h_x$
- $L_{\text{max}} = L_+ + L_x$

\[
L_+ = \frac{(\vec{x} \cdot \vec{f}_+)^2}{|\vec{f}_+|^2} = X^T P_+ X, \quad P_{+ij} = \frac{f_{+i} f_{+j}}{|\vec{f}_+|^2} = e_{+i} e_{+j}
\]

\[
L_+ = \frac{(\vec{x} \cdot \vec{f}_x)^2}{|\vec{f}_x|^2} = X^T P_x X, \quad P_{xij} = \frac{f_{x_i} f_{x_j}}{|\vec{f}_x|^2} = e_{x_i} e_{x_j}
\]

- Textbook detection: given $L_{\text{max}}$ calculate probability to mimic it by noise (significance), declare discovery of GWs if significance $>5\sigma$
Real-life Detection

- Data is non-stationary, non-gaussian and affected by artifacts
- Empirical background sample for estimation of FA probability
  - constructed by time-shifting data → may be biased wrt true background
  - need a massive background set (T observation x 10^6)

CQG 29 (2012) 155002
Coherent Statistics

- True GW signal should be in the $f_+, f_x$ plane

\[ L = E - N \]

- Detected (signal) energy
- Total energy
- Noise (null) energy

- Likelihood quadratic form

\[ L_{\text{max}} = X^T P X, \quad P_{nm} = e_n e_m + e_x e_m \]

\[ L = \sum_i \sum_{n,m} x_n[i] x_m[i] P_{nm}[i] = L_{i=j} + L_{i \neq j} \]

- L matrix
- Incoherent
- Coherent

- Detection statistics
  - Event ranking: characterize event strength, preferable if $\sim \text{SNR}$
  - Event consistency: significant null stream can be indication of a noise artifact
Two detector case

- no null space (any unconstrained event is admitted as GW!)
- $A \ll 1$ for significant fraction of the sky
- $L=\text{const}(\theta, \phi)$

\[ \xi_1 = x_1, \quad \xi_2 = x_2 \]

- Two detector paradox (Mohanty et al, CQG 21 S1831 (2004))
  - no $x$-correlation term in the likelihood matrix! $P_{12} = 0!$
  - contradict to the case of two co-aligned detectors where

\[ \xi_1 = \xi_2 = \frac{x_1 + x_2}{2}, \quad L_+ + L_x = \frac{1}{2} \left[ \langle x_1 x_1 \rangle + \langle x_2 x_2 \rangle + 2 \langle x_1 x_2 \rangle \right] \]

- What is meaning of coherent energy?
In-coherent/Coherent Energy

\[ L_+ = \sum_{i,j} x_i x_j P_{ij,+} = E_{+ (i=j)} + C_{+ (i \neq j)} \]
\[ L_\times = \sum_{i,j} x_i x_j P_{ij,\times} = E_{\times (i=j)} + C_{\times (i \neq j)} \]

- quadratic forms \( C_+ \) & \( C_\times \) depend on time delays between detectors and carry information about \( \theta, \phi \) – sensitive to source coordinates
- properties of the likelihood quadratic forms

\begin{align*}
& \text{arbitrary network} \\
\text{cov}(L_+ L_\times) &= 0 \\
\text{cov}(C_+ C_\times) &= -\sum e_{+i}^2 e_{\times i}^2 \\
\text{cov}(E_+ E_\times) &= \sum e_{+i}^2 e_{\times i}^2 \\
& \text{2 detector network} \\
& C_+ + C_\times = 0 \\
& E_+ + E_\times = x_1^2 + x_2^2 \\
& \text{E+}, \text{Ex}, C+, \text{Cx} \text{ are dependent} \\
& \text{How should we calculate “generalized” network x-correlation?}
\end{align*}
**THE Projection Operator**

- Construction of the projection operator
  \[ P_{nm} = e_{+n} e_{+m} + e_{\times n} e_{\times m} \]

  is ambiguous:
  \[ e_+ e_+ \rightarrow \text{rotation} \rightarrow e_+ e_+ \]

  \[ L_{\text{max}} = X^T P X = X^T P' X \]

- incoherent & coherent terms are not invariant

- Select the projection operator as
  \[ P_{nm} = u_n u_m \]

  (solves two-detector paradox)

- coherent/incoherent energies

  \[ C = X^T P_u (n \neq m) X \quad E_I = X^T P_u (n = m) X \]

\[ \vec{u} \cdot \vec{v} = 0 \]
Meaning of Coherent Energy

- Reconstructed network response
  \[ \tilde{\xi} = (\tilde{x} \cdot \tilde{u}) \tilde{u} \]

- Total signal energy
  \[ L_{\text{max}} = (x \mid \xi) = \sum_i (\tilde{x}[i] \cdot \tilde{u}[i])^2 \]

- Let's consider the case when \( x = \xi \)
  \[ L_{\text{max}} = \sum_i |\tilde{\xi}[i]|^2 \cdot (\tilde{u}[i] \cdot \tilde{u}[i])^2 = \sum_i |\tilde{\xi}[i]|^2 \sum_{nm} u_n^2[i] u_m^2[i] \]

- Coherent energy
  \[ C = \sum_i |\tilde{\xi}[i]|^2 \left[ 1 - \sum_n u_n^4[i] \right] \]
Response Index

\[ I_r = \sum_k u^4[k] \quad \bar{u} = \frac{\bar{\xi}}{|\bar{\xi}|} \]

- \(1/I_r\) - effective number of detectors contributing to total network SNR: distributed between 1 and K
- For GW signals response index correlates with network index
- For noise and glitches there is no correlation
- Describes how similar (coherent) are responses in individual detectors
- Great tool to distinguish signal from glitches

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Detection statistics: coherent – null energy

- **coherent energy**: sum of the off-diagonal elements of L matrix
  \[ E_{\text{coherent}} = \sum_{i \neq j} L_{ij} \]

- **null energy null**: energy of the reconstructed detector noise
Rejection of glitches

- **Coherent statistics**
  - Network correlation coefficient $cc$ - rejection of glitches
  - Network correlated amplitude $\eta$ – event ranking statistic

$$cc = \frac{E_{i \neq j}}{N + E_{i \neq j}}$$

$$\eta = \sqrt{\frac{cc \cdot E_{i \neq j}}{K - 1}}$$

Use also DQ and Veto: characterization of detector noise is one of the most challenging tasks in the GW experiment
Dual Stream Phase Transform

- Likelihood formalism is easily generalized for the dual data stream analysis
  - quadrature data stream contains the same information as $x$
  - network response can be presented as pairs of vectors $\tilde{\xi}, \tilde{\xi}$

- Phase transform
  - Apply phase transform to projections (don’t care about projections out of plane)

$$\xi = \xi' \cos(\lambda) + \tilde{\xi}' \sin(\lambda)$$
$$\tilde{\xi} = \tilde{\xi}' \cos(\lambda) - \xi' \sin(\lambda)$$

- With appropriate phase transformation the polarization pattern is revealed
Network Regulators & Constraints

- For existing LHV networks the standard projection is rarely an optimal solution

- Network regulators → construct P by guessing orientation of the projection vector $u$ (Klimenko et al, 2005)
  - hard regulator:
    $$\tilde{\xi} \rightarrow \tilde{\xi}_+, \quad \tilde{\nu} \rightarrow 0$$
    gives optimal solution for closely aligned networks
  - soft regulator:
    $$\tilde{\xi} \rightarrow \tilde{\xi}_+, \quad \tilde{\nu} \rightarrow \tilde{\nu}$$

after polarization phase transform
Polarization Constraints

- Linear
circular
elliptical

- Just fit data to a selected polarization pattern!
Other constrained likelihood solutions

- In addition to relatively simple network and polarization constraints, additional source models, even not accurate can be used to constrain the likelihood functional

\[
L' = X^T (Fh) + (Fh)^T X - (Fh)^T (Fh) + \lambda g(X, h), \quad g(X, h) = 0
\]

- \( g(X, h) = 0 \) is a constraint condition

- Conceptually simple, but could be very hard to solve – in most cases there is no analytical solution.

Lagrange multiplier
Probability Sky Map

**probability map:** obtained from the likelihood sky distribution

PSM shows how consistent are reconstructed waveforms and time delays as function of $\theta, \phi$. Source location is at PSM max.

- **detector plane**
- **constant delay rings for detector pairs**
Error Regions

- Source location is characterized by a spot in the sky (error region) rather than by a single \((\theta, \phi)\) direction
  - \(x\%\) error region - a sky area with the cumulative probability of \(x\%\)
- The coverage of error regions has to be validated with MonteCarlo

- Error regions can be reported for optical/radio followup \(\rightarrow\) multi-messenger observations
Objectives of Coherent Network Analysis

- Understand benefits and shortcomings of detector networks to detect sources and optimally capture science.
- Combine measurements from several detectors
  - confident detection; elimination of instrumental/environmental artifacts
  - reconstruction of GW polarizations
  - reconstruction of source coordinates
  - reconstruction of GW waveforms
- CAN is a unified approach to handle
  - arbitrary number of detectors at different locations and arm’s orientations
  - variability of detector responses as function of source coordinates
  - differences in the strain sensitivity of detectors
- Extraction of source parameters
  - confront measured waveforms with source models or include models
Reading Material

• LIGO/Virgo publications on burst searches:
  https://www.lsc-group.phys.uwm.edu/ppcomm/Papers.html
• Guersel, Tinto, PRD 40 v12, 1989
  ➢ reconstruction of GW signal for a network of three misaligned detectors
• Likelihood analysis: Flanagan, Hughes, PRD57 4577 (1998)
  ➢ likelihood analysis for a network of misaligned detectors
• Two detector paradox: Mohanty et al, CQG 21 S1831 (2004)
  ➢ state a problem within standard likelihood analysis
• Constraint likelihood: Klimenko et al, PRD 72, 122002 (2005)
  ➢ address problem of ill-conditioned network response matrix (rank deficiency)
  ➢ first introduction of likelihood constraints/regulators
• Rank deficiency of network matrix: Rakhmanov, CQG 23 S673 (2006)
• GW signal consistency: Chatterji et al, PRD 74 082005(2006)
• Coherent Burst search: S. Klimenko et al., Class. Quantum Grav. 25, 114029 (2008)
• Three figures of merit..., B. Schutz, CQG 28 125023(2011)