Umbral Moonshine and K3 surfaces

Francesca Ferrari

Universiteit van Amsterdam

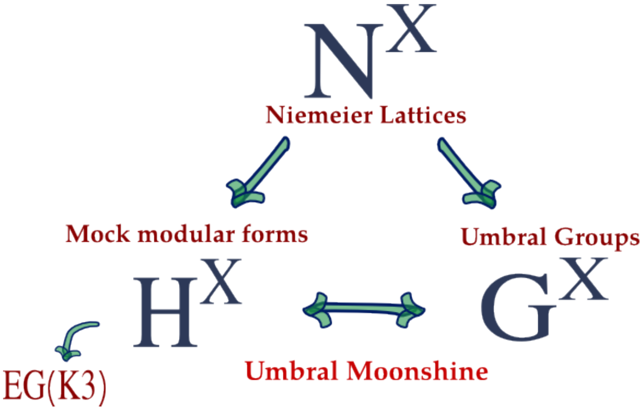
Joint Dutch-Brazil School

Monstrous Moonshine

Around 1978 a group of mathematicians, Mckay, Thompson, Conway and Norton, started to speculate on the **Monstrous Moonshine**. The subject was addressed as Moonshine because of the apparently illogical realtion between the **Monster group** and **modular functions**.

The sense of illicitness of the subject suggested to Conway, the image of American mountaineers producing illegally distilled spirits.

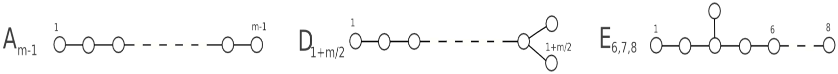
Umbral Moonshine



Niemeier Lattices & Finite groups

[Niemeier(1973)]

The **Niemeier lattices** are the unique 24-dimensional even self-dual lattices with total rank 24. They are uniquely determined by their root systems, X.



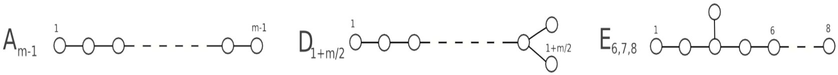
To each Niemeier lattice of type X it is possible to associate an **Umbral group** G^X, defined as the quotient of the automorphism group of the lattice by the Weyl group of X.

$$G^X = \frac{Aut(X)}{W(X)}$$

Niemeier Lattices & Finite groups

[Niemeier(1973)]

The **Niemeier lattices** are the unique 24-dimensional even self-dual lattices with total rank 24. They are uniquely determined by their root systems, X.



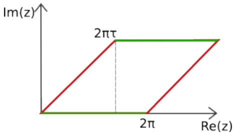
To each Niemeier lattice of type X it is possible to associate an **Umbral group** G^X, defined as the quotient of the automorphism group of the lattice by the Weyl group of X.

$$G^X = \frac{Aut(X)}{W(X)}$$

Partition function & Elliptic genus

[Witten (1987)]
The **partition function** for a bosonic string theory

$$Z(\tau) = Tr_{\mathcal{H}}(q^{L_0 - c/24}\bar{q}^{L_0 - c/24})$$



where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, depends on the Dedekind eta function,

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n).$$

$Z(\tau)$ turns out to be invariant under the **modular transformations**, $SL(2,\mathbb{Z})$.

The **elliptic genus** for an $\mathcal{N} = (2,2)$ superconformal theory is defined as

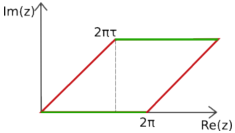
$$EG(\tau, z) = Tr_{\mathcal{H}_{\mathcal{RR}}}((-1)^F y^{J_0} q^{L_0 - c/24}\bar{q}^{-\bar{L}_0 - \bar{c}/24})$$

where $y = e^{2\pi iz}$, $z \in \mathbb{C}$ and $F = F_L - F_R$. It is a topological invariant and it transforms nicely under $SL(2,\mathbb{Z})$ transformations.

Partition function & Elliptic genus

[Witten (1987)]
The **partition function** for a bosonic string theory

$$Z(\tau) = Tr_{\mathcal{H}}(q^{L_0 - c/24}\bar{q}^{L_0 - c/24})$$



where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, depends on the Dedekind eta function,

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n).$$

$Z(\tau)$ turns out to be invariant under the **modular transformations**, $SL(2, \mathbb{Z})$.

The **elliptic genus** for an $\mathcal{N} = (2, 2)$ superconformal theory is defined as

$$EG(\tau, z) = Tr_{\mathcal{H}_{\mathcal{RR}}}((-1)^F y^{J_0} q^{L_0 - c/24}\bar{q}^{-\bar{L}_0 - \bar{c}/24})$$

where $y = e^{2\pi i z}$, $z \in \mathbb{C}$ and $F = F_L - F_R$. It is a topological invariant and it transforms nicely under $SL(2, \mathbb{Z})$ transformations.

Elliptic genus of a K3 surface

[Eguchi, Ooguri, Taormina, Yang (1988)]

Decomposing the **elliptic genus of a K3 surface** into irreducible characters of the $\mathcal{N} = 4$ superconformal algebra a q-series arises

$$EG(\tau, z; K3) = \frac{\theta_1(\tau, z)^2}{\eta^3(\tau)} \left\{ 24\mu(\tau, z) + 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + \dots) \right\},$$

where μ is the Appell-Lerch sum, while the second term coincides with the **mock modular form** defined by the root system $X = A_1^{24}$

$$H(\tau) = 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + \dots).$$

The striking feature of Moonshine is that the first coefficients in the q-series coincide with irreducible representations of the Umbral group M_{24}.

Elliptic genus of a K3 surface

[Eguchi, Ooguri, Taormina, Yang (1988)]

Decomposing the **elliptic genus of a K3 surface** into irreducible characters of the $\mathcal{N}=4$ superconformal algebra a q-series arises

$$EG(\tau,z;K3)=\frac{\theta_1(\tau,z)^2}{\eta^3(\tau)}\left\{24\mu(\tau,z)+2q^{-1/8}(-1+45q+231q^2+770q^3+\dots)\right\},$$

where μ is the Appell-Lerch sum, while the second term coincides with the **mock modular form** defined by the root system $X=A_1^{24}$

$$H(\tau)=2q^{-1/8}(-1+45q+231q^2+770q^3+\dots).$$

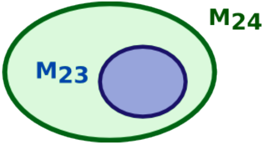
The striking feature of Moonshine is that the first coefficients in the q-series coincide with irreducible representations of the Umbral group M_{24}.

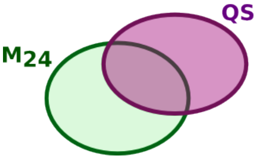
A question to be solved

What is the relation between Umbral moonshine and K3 surfaces?

[Mukai (1988), Kondo (1998)]

The geometric symmetries (symplectic automorphisms) of a K3 surface are a subgroup of $M_{23} \subset M_{24}$.



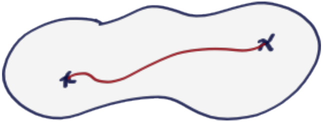


[Gaberdiel, Hohenegger, Volpato (2011)]

The supersymmetry-preserving automorphisms of any non-linear sigma-model on K3 are a subgroup of Co_1, the Conway group.

Conclusion

- [Taormina, Wendland (2013)]
  The geometric groups of symmetries of different K3 surfaces can be combined constructing an **"overarching" symmetry map** on the moduli space of complex structure of K3 surfaces.



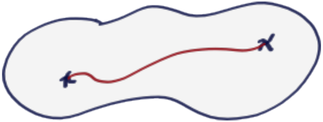
- [Cheng, Harrison (2013)]
  Through a uniform construction, the 23 cases of Umbral moonshine are built from the ADE-singularities classification of K3 surfaces.

With M. Cheng, S. Harrison, and N. Paquette, we are extending the analysis of K3 symmetries to other cases of umbral moonshine.

Thank you!

Conclusion

- [Taormina, Wendland (2013)]
  The geometric groups of symmetries of different K3 surfaces can be combined constructing an **"overarching" symmetry map** on the moduli space of complex structure of K3 surfaces.



- [Cheng, Harrison (2013)]
  Through a uniform construction, the 23 cases of Umbral moonshine are built from the ADE-singularities classification of K3 surfaces.

With M. Cheng, S. Harrison, and N. Paquette, we are extending the analysis of K3 symmetries to other cases of umbral moonshine.

Thank you!