Higher-Order Calculations

Fixed-Order Corrections to Jet Processes

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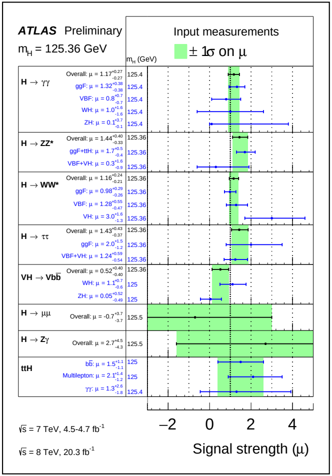
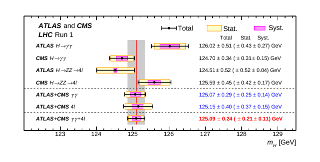
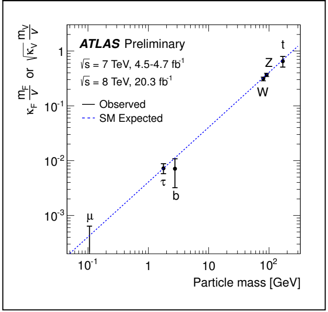
Higgs Properties

Is the boson found The Higgs Boson?



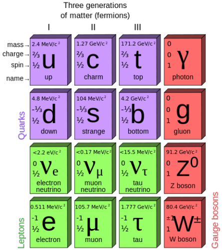




Measurements so far are consistent with the expectations in the SM

The Standard Model Lagrangian

A theory based on the
$SU(3) \times SU(2) \times U(1)$
gauge group and the
matter content:



$$\mathcal{L}_{SM} = \mathcal{L}_{YM} + \mathcal{L}_{D} + \mathcal{L}_{H} + \mathcal{L}_{\text{Yuk}}$$

Where: $\mathcal{L}_{YM} = \mathcal{L}_{QCD} + \mathcal{L}_{I_w} + \mathcal{L}_{Y}$

$$= -\frac{1}{4}\sum_{a=1}^{8} G^{a}_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}\sum_{i=1}^{3} F^{i}_{\mu\nu}F^{i\mu\nu}$$

$$-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

and for the fermions (no explicit mass term included):

$$\mathcal{L}_{D} = \Sigma_f \bar{\psi}_f \gamma^{\mu} \mathcal{D}_{\mu} \psi_f$$

At this level this simple model has only three parameters

But EW gauge bosons are massive and explicit mass terms are not gauge invariant!!

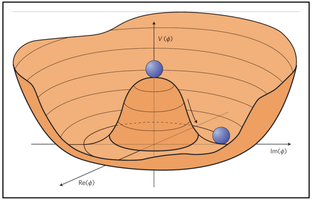
The Higgs Sector

EW SSB is achieved by adding complex scalar doublet:

$$\mathcal{L}_{SM} = \mathcal{L}_{YM} + \mathcal{L}_{D} + \mathcal{L}_{H} + \mathcal{L}_{\text{Yuk}}$$

with $\mathcal{L}_H =$

$$(\mathcal{D}_\mu H)^\dagger(\mathcal{D}^\mu H) - (\mu^2 H^\dagger H + \lambda(H^\dagger H)^2)$$



Yukawa Hff terms can economically generate masses for chiral fermions!

$$\mathcal{L}_{\text{Yuk}} \sim -\lambda_d \overline{\mathbf{Q}}_L H \mathbf{d}_R + h.c. \qquad \rightarrow \qquad m_d = \frac{\lambda_d v}{\sqrt{2}}$$

No masses for neutrinos have been considered, usually regarded as BSM!

The Standard Model Parameters

In all the SM have 19 free parameters!

Gauge couplings	3	Determine relative strength of forces
Higgs parameters	2	Higgs mass and quartic coupling
Fermion masses	9	After EWSSB generated from the Yukawa couplings
CKM parameters	3+1	Quark mixing angles and CP phase
QCD θ	1	Related to the QCD vacuum structure and the so called Strong CP problem

Observables and Parameter Fits

Choose a set $\{\mathcal{O}_i^{\rm expt}\}$ of observables measured and compute $\{\mathcal{O}_i^{\rm th}(\boldsymbol{\theta})\}$, where $\boldsymbol{\theta}$ is a subset of SM parameters. Then minimize the χ^2 function:

$$\chi^2(\boldsymbol{\theta}) = \sum_i \frac{(\mathcal{O}_i^{\rm expt} - \mathcal{O}_i^{\rm th}(\boldsymbol{\theta}))^2}{(\Delta\mathcal{O}_i^{\rm expt})^2}$$

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α^{expt}		$\alpha^{\text{th}} = \frac{e^2}{4\pi}$
G_F^{expt}		$G_F^{\text{th}} = \frac{1}{\sqrt{2}v^2}$
m_Z^{expt}	Tree level →	$m_Z^{\text{th}} = \frac{e^2 v^2}{4 s_W^2 c_W^2}$
m_W^{expt}		$m_W^{\text{th}} = \frac{e^2 v^2}{4 s_W^2}$
$\Gamma_{l^+l^-}^{\text{expt}}$		$\Gamma_{l^+l^-}^{\text{th}} = \frac{v}{96\pi} \frac{e^3}{s_W^3 c_W^2} [(-\frac{1}{2} + 2 s_W^2)^2 + \frac{1}{4}]$

The theory results are expressed in terms of e, s_W and v, with $g_I = e/s_W$, $g_Y = e/c_W$ and $v^2 = -\mu^2/\lambda$

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With very precise measurements from α, G_F and m_Z we get (tree level) predictions for m_W and $\Gamma_{l^+l^-}$!

The Global EW Fit

Modern fits use larger set of observables and most precise predictions!
Also, theoretical uncertainties are considered for the fit.

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Parameter	Input value	Free in fit
M_H [GeV]$^{(\circ)}$	125.14 ± 0.24	yes
M_W [GeV]	80.385 ± 0.015	–
Γ_W [GeV]	2.085 ± 0.042	–
$\overline{m}_c$ [GeV]	$1.27^{+0.07}_{-0.11}$	yes
$\overline{m}_b$ [GeV]	$4.20^{+0.17}_{-0.07}$	yes
m_t [GeV]	173.34 ± 0.76	yes
$\Delta\alpha_{\rm had}^{(5)}(M_Z^2)^{(\dagger\triangle)}$	2757 ± 10	yes
$\alpha_s(M_Z^2)$	–	yes

Gffiter Group, arXiv:1407.3792

M_Z [GeV]	91.1875 ± 0.0021	yes
Γ_Z [GeV]	2.4952 ± 0.0023	–
$\sigma^0_{\rm had}$ [nb]	41.540 ± 0.037	–
R^0_ℓ	20.767 ± 0.025	–
$A_{\rm FB}^{0,\ell}$	0.0171 ± 0.0010	–
A_ℓ $^{(\star)}$	0.1499 ± 0.0018	–
$\sin^2\theta^\ell_{\rm eff}(Q_{\rm FB})$	0.2324 ± 0.0012	–
A_c	0.670 ± 0.027	–
A_b	0.923 ± 0.020	–
$A_{\rm FB}^{0,c}$	0.0707 ± 0.0035	–
$A_{\rm FB}^{0,b}$	0.0992 ± 0.0016	–
R^0_c	0.1721 ± 0.0030	–
R^0_b	0.21629 ± 0.00066	–

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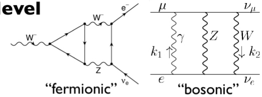
Calculations

Slide from R. Kogler

All observables calculated at 2-loop level

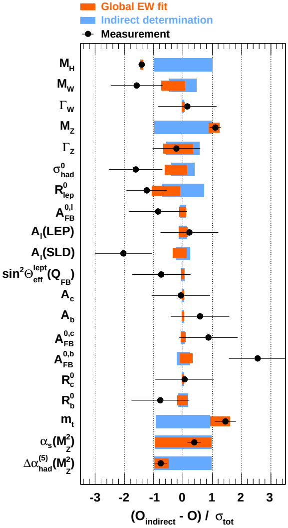


- **M_W** : full EW one- and two-loop calculation of fermionic and bosonic contributions
  [M Awramik et al., PRD 69, 053006 (2004), PRL 89, 241801 (2002)]
  + 4-loop QCD correction [Chetyrkin et al., PRL 97, 102003 (2006)]
- **$\sin^2\theta^l_{eff}$** : same order as M_W, calculations for leptons and all quark flavours
  [M Awramik et al, PRL 93, 201805 (2004), JHEP 11, 048 (2006), Nucl. Phys. B813, 174 (2009)]
- **partial widths Γ_f**: fermionic corrections in two-loop for all flavours (includes predictions for σ^0_{had}) [A. Freitas, JHEP04, 070 (2014)]
- **Radiator functions**: QCD corrections at N^3LO
  [Baikov et al., PRL 108, 222003 (2012)]
- **Γ_W** : only one-loop EW corrections available, negligible impact on fit
  [Cho et al, JHEP 1111, 068 (2011)]
- **all calculations**: one- and two-loop QCD corrections and leading terms of higher order corrections

Global Fits of SM Parameters

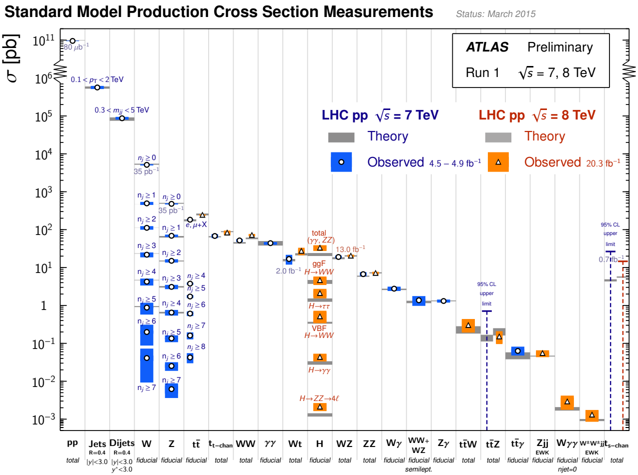
- Comparison of results in terms of corresponding uncertainty
- Indirect determination removes direct measurement constrain
- Good description of all observables
- Largest deviation observed in bottom forward-backward asymmetry

Gffiter arXiv:1407.3792



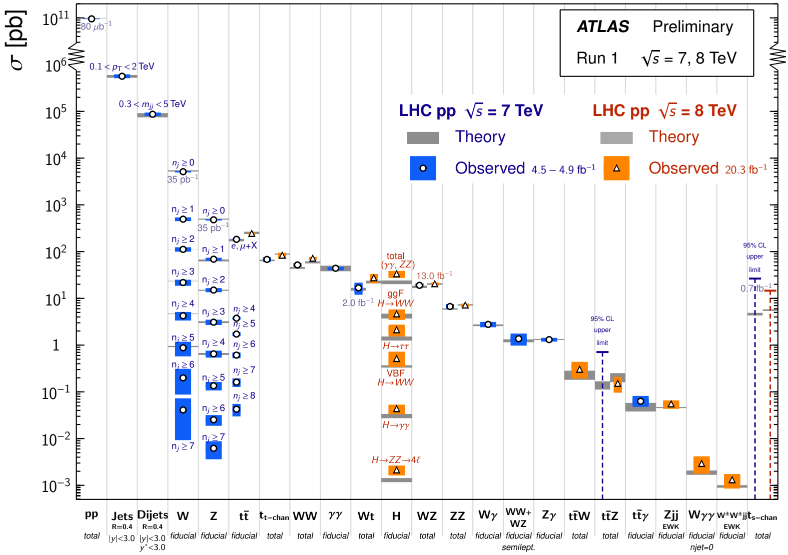
The $\chi^2_{\min}$ obtained in the fit is 18.2 for a total of 14 d.o.f.

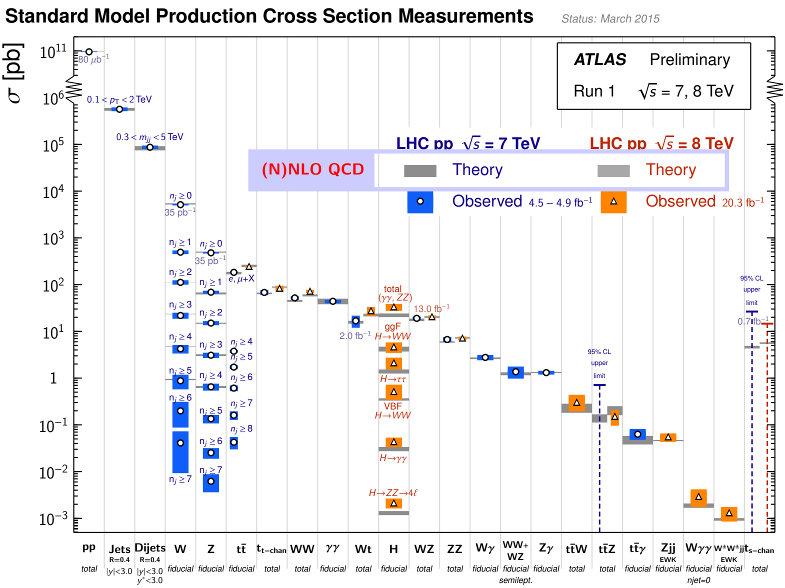
Full Power of SM at Hadron Colliders



- Summary plot of SM cross sections
- Impressive agreement between theory and experiment
- Smallest cross section from $W^{\pm}W^{\pm} + 2$ jets
- Similar results from CMS

Standard Model Production Cross Section Measurements





Standard Model Production Cross Section Measurements
Status: March 2015
ATLAS Preliminary
Run 1 √s = 7, 8 TeV
LHC pp √s = 7 TeV
LHC pp √s = 8 TeV
(N)NLO QCD
Theory
Theory
Observed 4.5 – 4.9 fb−1
Observed 20.3 fb−1
σ [pb]
pp total
Jets R=0.4 |y|<3.0
Dijets R=0.4 |y|<3.0 y*<3.0
W fiducial
Z fiducial
tt̄ fiducial
tt−chan total
WW total
γγ fiducial
Wt total
H fiducial
WZ total
ZZ total
Wγ fiducial
WW+WZ fiducial semilept.
Zγ fiducial
tt̄W total
tt̄Z total
tt̄γ fiducial
Zjj EWK fiducial
Wγγ fiducial njet=0
W±W±jj EWK fiducial
ts−chan total

Very Precise: NLO QCD Recent Progress

By now we have highly automated NLO QCD tools for up to $2 \to 4$ processes!
(Madgraph, OpenLoops, Gosam, ...)

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$W/Z + 4$-Jet Production	2010-2011	Berger, Bern, Dixon, FFC, Forde, Hoeche, Ita, Kosower, Maitre **(BlackHat)**
4-Jet Production - $\gamma\gamma + 3$ Jets	2013	Badger, Biedermann, Guffanti, Uwer, Yundin **(NJet)**
Off-shell $t\bar{t}H$	2015	Denner, Feger

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$2 \to 6$

$W + 5$-Jet Production	2013	Bern, Dixon, FFC, Hoeche, Ita, Kosower, Ozeren, Maitre **(BlackHat)**

Very Precise: (N)NNLO QCD Recent Progress

Higgs production (ggH)	N3LO QCD (2015)	Anastasiou, Duhr, Dulat, Furlan, Herzog, Gehrmann, Lazopoulos, Mistlberger
W^+W^-	NNLO QCD (2014)	Gehrmann, Grazzini, Kallweit, Maierhfer, von Manteuffel, Pozzorini, Rathlev, Tancreedi
$W+$ Jet	NNLO QCD (2015)	Boughezal, Focke, Liu, Petriello
$H+1$ Jet	NNLO QCD (2015)	Boughezal, Caola, Melnikov, Petriello, Schulze
$Z+1$ Jet	NNLO QCD (2015)	Gehrmann-De Ridder, Gehrmann, Glover, Huss, Morgan

And much more!

This Lecture

INTRODUCTION

The SM Global Fit, LHC Observables, Recent progress in pQCD

FACTORIZATION FOR HADRONIC XS's

PDF/Hard XS/Fragmentation, Perturbations, Interference Diags, $\delta\sigma^{\mathrm{(N)NLO}}$

QCD TREE LEVEL FACTORIZATION

Collinear and Soft Limits, Dim Reg, General Expressions

Following Lectures

2) SUBTRACTION METHODS

NLO algorithms, Tools, NNLO state of the art $\longrightarrow$ (de Florian)

3) DISSECTING HIGHER ORDERS

KLN Theorem, Scales, Numerical Stabilities $\longrightarrow$ (Kosower)

4) HANDS ON LAB SESSION

Computing observables at NLO in automated frameworks!

FACTORIZATION FOR HADRONIC XS's

PDF/Hard XS/Fragmentation, Perturbations, Interference Diags, $\delta\sigma^{\mathrm{(N)NLO}}$


$\mathcal{L}_{QCD}$

Partonic Cross Section in Perturbation Theory

$$\hat{\sigma}(\alpha_s, \mu_F, \mu_R) = [\alpha_s(\mu_R)]^{n_\alpha} \left[\hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi} \hat{\sigma}^{(1)}(\mu_F, \mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}(\mu_F, \mu_R) + \cdots \right]$$

LO NLO NNLO

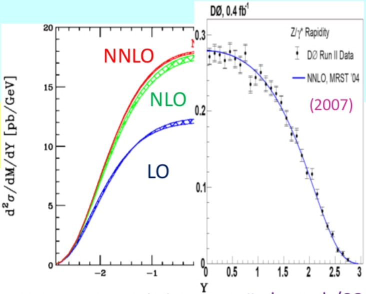
Problem: Leading-order, tree-level predictions only qualitative due to **poor convergence** of expansion in $\alpha_s(\mu)$ (setting $\mu_R = \mu_F = \mu$)

Example: *Z* production at Tevatron
Distribution in rapidity *Y*

$$Y = \frac{1}{2} \ln \left(\frac{E + p_z}{E - p_z} \right)$$

$\frac{d\sigma}{dY}$ has $n_\alpha = 0$

still ~50% corrections, LO → NLO



[Anastasiou, Dixon, Melnikov, Petriello hep-ph/0312266]

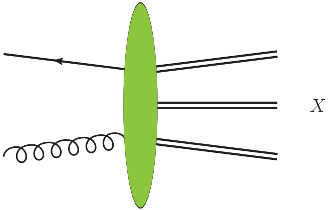

$\mathcal{L}_{QCD}$

QCD TREE LEVEL FACTORIZATION

Collinear and Soft Limits, Dim Reg, General Expressions

Producing X via a $\bar{q}g$ channel

Suppose you are studying some production channels of your preferred signal X

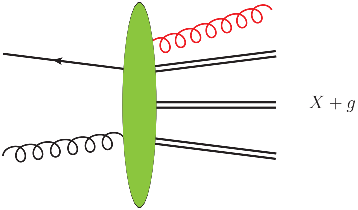


Start for computing the born level cross section, and then ask

how can I get extra radiation on on top of X?

Start with adding a gluon!

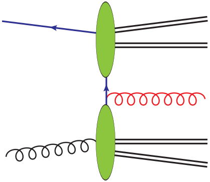
- $\mathcal{O}(\alpha_s)$ corrections to your signal
- Part of the *real* NLO corrections



Extra gluon emission $\bar{q}g \to X + g$

Pay attention to the diagrams in which the extra gluon couples to the external $\bar{q}$ line:

$\mathcal{A}_{\bar{q}g\to g+X} =$



$$+ \underbrace{\cdots}_{\text{Other diagrams with } g \text{ not coupling to } \bar{q} \text{ line}} = \sum_i D_i + \cdots$$

In the square of the amplitude we then find:

$$|\mathcal{A}_{\bar{q}g\to g+X}|^2 = \sum_i |D_i|^2 + \sum_{i\neq j} D_i^\dagger D_j + \cdots \qquad (1)$$

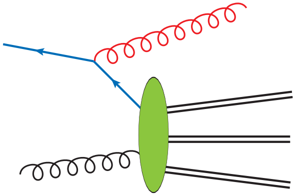
Notice that the propagator leading to the vertex that couples g and $\bar{q}$ in diagram D_j leads to a term like (we set $m_{\bar{q}} = 0$ for now!):

$$\frac{1}{(-p_{\bar{q}} + p_g + p_{X'_j})^2}$$

And so in Eq. 1 we find a potential divergent terms of the form $1/(2p_{\bar{q}} \cdot p_g)^2$!

Exploring Singularities of QCD Tree Amplitudes

These (most) singular terms come in $|\mathcal{A}_{\bar{q}g\to g+X}|^2$ from the square of the set of diagrams (let's call them D_1):



So let's explore in detail D_1 contributions!

First:

$$D_1 = g_s t^a \bar{v}(p_{\bar{q}})\gamma_\mu \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2}\ \tilde{\mathcal{A}}_{\bar{q}g\to X}\ \epsilon^{\mu *} \tag{2}$$

In the matrix element square, we need to deal with the sum over polarizations of the g. We introduce a light-like vector n^μ with $n \cdot p_g \neq 0$ and write:

$$\sum_{\text{polarizations}} \epsilon^{\mu *}\epsilon^{\nu} = -g^{\mu\nu} + \frac{p_g^\mu n^\nu + p_g^\nu n^\mu}{p_g \cdot n} \tag{3}$$

And then, in a sum over initial and final states degrees of freedom, we find:

$$\begin{aligned}\sum |D_1|^2 &= g_s^2 C_F \\ & \quad \mathrm{Tr}\left\{ \tilde{\mathcal{A}}^{\dagger}_{\bar{q}g\to X} \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2} \left[\gamma_\nu \not{p}_{\bar{q}} \gamma_\mu \right] \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2} \tilde{\mathcal{A}}_{\bar{q}g\to X} \right\} \\ & \qquad \left(-g^{\mu\nu} + \frac{p_g^\mu n^\nu + p_g^\nu n^\mu}{p_g \cdot n} \right) \\ &= g_s^2 C_F \\ & \quad \mathrm{Tr}\left\{ \tilde{\mathcal{A}}^{\dagger}_{\bar{q}g\to X} \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2} \left[-\gamma^\mu \not{p}_{\bar{q}} \gamma_\mu + \frac{\not{n}\not{p}_{\bar{q}}\not{p}_g + \not{p}_g \not{p}_{\bar{q}} \not{n}}{n \cdot p_g} \right] \right. \\ & \quad \left. \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2} \tilde{\mathcal{A}}_{\bar{q}g\to X} \right\} \qquad (4)\end{aligned}$$

Employing identities for Dirac's γ matrices (like $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$, etc) we obtain the compact expresion:

Exploring Singularities of QCD Tree Amplitudes

$$\sum |D_1|^2 = g_s^2 C_F \frac{2}{(2p_{\bar{q}} \cdot p_g)^2 (n \cdot p_g)} \mathrm{Tr}\Big\{ \tilde{\mathcal{A}}^{\dagger}_{\bar{q}g\to X} (\not{p}_g - \not{p}_{\bar{q}}) \Big[(n \cdot p_{\bar{q}}) \not{p}_g + (p_{\bar{q}} \cdot p_g) \not{n} \Big] (\not{p}_g - \not{p}_{\bar{q}}) \tilde{\mathcal{A}}_{\bar{q}g\to X} \Big\}$$

$$= g_s^2 C_F \frac{2}{(2p_{\bar{q}} \cdot p_g)(n \cdot p_g)} \mathrm{Tr}\Big\{ \tilde{\mathcal{A}}^{\dagger}_{\bar{q}g\to X} \Big[(n \cdot p_{\bar{q}}) \not{p}_{\bar{q}} + n \cdot (p_g - p_{\bar{q}})(\not{p}_g - \not{p}_{\bar{q}}) + (p_{\bar{q}} \cdot p_g) \not{n} \Big] \tilde{\mathcal{A}}_{\bar{q}g\to X} \Big\} \tag{5}$$

Here it comes the crucial step!

> If we explore the regions were our diagrams diverge (i.e. were $(2p_{\bar{q}} \cdot p_g) \to 0$), this occurs either because g is *soft* or because g turns *collinear* to $\bar{q}$!

Collinear Singularities in QCD

Characterize the collinear region with the help of the *Sudakov parameterization* ($k_\perp$ is a space-like vector $\perp$ to both p_g and $p_{\bar{q}}$):

$$p_g = (1-z)p_{\bar{q}} + \beta n^\mu - k_\perp^\mu \qquad (6)$$

where picking $\beta = -k_\perp^2/(2(1-z)(n \cdot p_{\bar{q}}))$ ensures $p_g^2 = 0$.

We are going to let $k_\perp$ go to zero, and with it have a measure of how collinear is our configuration! We get:

$$\begin{aligned}\sum |D_1|^2 \quad &= \quad g_s^2 C_F \frac{2}{(2p_{\bar{q}} \cdot p_g)(n \cdot p_g)} \, \mathrm{Tr}\Big\{ \tilde{\mathcal{A}}^\dagger_{\bar{q}g\to X} \\ &\left[\frac{(n \cdot p_g)}{(1-z)} \not{p}_{\bar{q}} \quad - \quad \frac{(p_g \cdot n) z}{(1-z)} (\not{p}_g - \not{p}_{\bar{q}}) - \frac{k_\perp^2}{2(p_g \cdot n)} \not{n}\right] \tilde{\mathcal{A}}_{\bar{q}g\to X} \Big\}\end{aligned} \qquad (7)$$

Collinear Singularities in QCD

Now, with the use of the simple identity:

$$\not{p}_{\bar{q}} = \frac{1}{z}\left(-(\not{p}_g - \not{p}_{\bar{q}}) - \not{k}_\perp - \frac{k_\perp^2}{2(1-z)(n\cdot p_{\bar{q}})}\not{n}\right)$$

we find:

$$\begin{aligned}\sum |D_1|^2 \quad = \quad & 2g_s^2 C_F \frac{-1}{k_\perp^2}\ \mathrm{Tr}\Big\{\tilde{\mathcal{A}}^\dagger_{\bar{q}g\to X} \\ & \Big[(-\frac{1}{z} - z)(\not{p}_g - \not{p}_{\bar{q}}) + \mathcal{O}(k_\perp^2)\Big]\tilde{\mathcal{A}}_{\bar{q}g\to X}\Big\} \qquad (8)\end{aligned}$$

And notice that in the collinear limit ($k_\perp^2$ going to zero), the singular piece approximates the full amplitude square:

$$\sum |\mathcal{A}_{\bar{q}g\to g+X}|^2 \overset{k_\perp^2\to 0}{\approx} \sum |D_1|^2 \qquad (9)$$

Collinear Singularities in QCD

And then we encounter an interesting relation!

$$\sum |\mathcal{A}_{\bar{q}g\to g+X}|^2 \overset{k_\perp^2\to 0}{\approx} 2g_s^2 C_F \frac{-1}{k_\perp^2}\frac{1+z^2}{z}\mathrm{Tr}\left\{\mathcal{A}^\dagger_{\bar{q}g\to X}(\not{p}_g - \not{p}_{\bar{q}})\mathcal{A}_{\bar{q}g\to X}\right\}$$
$$= 2g_s^2 C_F\left(-\frac{1}{k_\perp^2}\right)\frac{1+z^2}{z}\sum |\mathcal{A}_{\bar{q}g\to X}|^2 \quad (10)$$

Now suppose that you are interested in the behavior of the differential cross section around the collinear limit. Notice that you can factorize the Lorentz Invariant Phase-Space of the collinear gluon like:

$$\frac{d^3p_g}{(2\pi)^3}\frac{1}{2E_g} \overset{k_\perp^2\to 0}{\approx} \frac{1}{16\pi^2}\frac{dz}{(1-z)}d(-k_\perp^2)\frac{d\phi}{2\pi} = \frac{1}{16\pi^2}\frac{dz}{(1-z)}d(-k_\perp^2) \quad (11)$$

Where in the last step we integrated the azimuthal angle.

Collinear Factorization in QCD

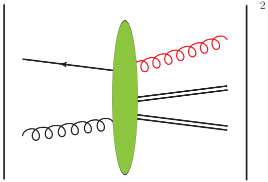
We arrive to this important *collinear* relation:

$$d\hat{\sigma}_{\bar{q}g\to g+X} \overset{k_\perp^2\to 0}{\approx} \frac{dk_\perp^2}{k_\perp^2}\frac{dz}{z}\frac{\alpha_s}{2\pi}\underbrace{\frac{1+z^2}{1-z}}_{\tilde{\mathrm{P}}_{\mathrm{qq}}(\mathrm{z})} d\hat{\sigma}_{\bar{q}g\to X} \tag{12}$$

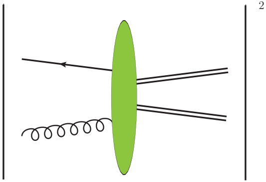
- The function $\tilde{P}_{qq}(z)$ is associated to the so called Altarelli-Parisi splitting function for a q to turn into a collinear q (and a g).
- Notice that as written, $\tilde{P}_{qq}(z)$ has a divergence for $z \to 1$, which is actually associated with a *soft* divergence.
- This is commonly regulated in order to avoid double counting when soft divergences are treated separately.

Collinear Factorization in QCD

We have found a picture of the factorization of our process $\bar{q}g \to g + X$ when the g goes collinear with the $\bar{q}$ like:



$$\overset{k_\perp^2 \to 0}{\propto} \tilde{P}_{qq}(z)$$



Comments

- If g goes collinear with the initial state gluon we find a similar result. Also for any other colored parton in the final state an associated relation is found.
- In such cases corresponding *Splitting functions appear*.
- Notice that integration over $dk_\perp^2/k_\perp^2$ is divergent, so there is need of a regularization procedure!

Mass regularization of Collinear Divergences

Consider a collinear splitting $g \to q'\bar{q}'$, and suppose the quarks q' have a mass $m > 0$. In such situation one finds that, up to powers of m^2, the singular transverse integral changes according to:

$$\frac{d|k_\perp^2|}{|k_\perp^2|} \xrightarrow{m>0} \frac{d|k_\perp^2|}{|k_\perp^2| + m^2} \tag{13}$$

Which then allows to integrate down to $k_\perp^2 = 0$, returning a $\log(Q^2/m^2)$ (Q^2 some large scale).

- The divergence is now explicit in the log of the (small) mass.
- Although a useful regularization procedure for collinear divergences with quark masses, we can't do the proper with gluon masses (as we would explicitly break gauge invariance).
- If the quark mass is of relevance for your studies (e.g. certain b quark studies) large logarithms might be present!
- Soft divergences are not regularized by m.

The $d = 4 - 2\epsilon$ Trick

A way to regularize divergences in gauge theories is the procedure called *Dimensional Regularization*. Preservation of gauge invariance, regularization of both soft and collinear divergences (and also UV!), extraction of divergences as poles in a Laurent series, are some of the properties that makes it a standard in perturbative calculation in gauge theories!

A simple idea...

$$\int d^3 r \frac{1}{|\vec{r}|^3} \to \int_{r_1}^{r_2} |\vec{r}|^2 d|\vec{r}| \frac{1}{|\vec{r}|^3} \to \log\left(\frac{r_2}{r_1}\right) \xrightarrow{r_1 \to 0} \infty$$

$$\Downarrow$$

$$\int d^{3-2\epsilon} r \frac{1}{|\vec{r}|^3} \to \int_{r_1=0}^{r_2} |\vec{r}|^{2-2\epsilon} d|\vec{r}| \frac{1}{|\vec{r}|^3} \xrightarrow{\epsilon<0} -\frac{1}{\epsilon} r_2^{|\epsilon|}$$

Volume Integrals in d Dimensions

But how to get a grasp of continuous dimensions?
(Most of the time) *Just don't!*

Recursive $(d-1)$ Solid Angle Calculation

- $d = 2 \Rightarrow \int d\Omega_1 = \int d\phi = 2\pi$, polar coordinates in $\mathbb{R}^2$
- $d = 3 \Rightarrow \int d\Omega_2 = \int d\phi \sin(\theta) d\theta = 4\pi$, spherical coord in $\mathbb{R}^3$
- $d = 4 \Rightarrow \int d\Omega_3 = \int d\phi \sin(\theta') d\theta' \sin^2(\theta) d\theta = 2\pi^2$
- $d \Rightarrow \int d\Omega_{d-1} = \int d\Omega_{d-2} \sin^{d-2}(\theta) d\theta = 2\pi^{d/2}/\Gamma(d/2)$

- The space dimension is then a parameter in your calculation and amplitudes become a Laurent series in ϵ
- By the KLN theorem, ϵ poles will cancel off phys. observables
- To keep integral dimensions correctly, one introduces a dimensionful parameter μ, the *regularization scale* (which gets identified with μ_r and μ_f), $d^4p \to \mu^{2\epsilon} d^{d=4-2\epsilon}p$

Spitting Functions in Dimensional Regularization

We can then go ahead and revisit our collinear factorization in d dimensions. We would find a similar picture, with the leading order, d dimensional, massless, unregulated, averaged over polarizations Splitting functions $\hat{P}_{ij}(z)$ for the spitting process $i \to jk$:

Altarelli-Parisi Splitting Functions

- $\hat{P}_{qq}(z) = C_F \left(\frac{1+z^2}{1-z} - (1-z)\epsilon \right)$
- $\hat{P}_{qg}(z) = C_F \left(\frac{1+(1-z)^2}{z} - (z)\epsilon \right)$
- $\hat{P}_{gq}(z) = T_R \left(1 - \frac{2z(1-z)}{1-\epsilon} \right)$
- $\hat{P}_{gg}(z) = C_A \left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right)$

QCD General Factorization in Soft and Collinear Limits

Some of the most important properties for tree level QCD amplitudes are indeed their factorizing behavior when soft and collinear limits are taken. We are ready to enunciate these relations (and you can prove them before the discussion session!)

- For a process like $a(p_a) + b(p_b) \to i_1(p_1) + \cdots + i_n(p_n)$ we write the QCD tree level amplitude like $\mathcal{A}(\{c_a, s_a, p_a\}, \{c_b, s_b, p_b\}; \{c_1, s_1, p_1\}, \cdots, \{c_n, s_n, p_n\}) \equiv \mathcal{A}_{2,n}$
- Construct a ket $|a, b; 1, \cdots, n\rangle_{2,n}$ in color and spin space such that the coefficient of a given element in color and spin space $|\{c_a, s_a\}, \{c_b, s_b\}; \{c_1, s_1\}, \cdots, \{c_n, s_n\}\rangle$ would be this amplitude
- With this notation you get the relation:

$$\sum_{\text{colors,spins}} |\mathcal{A}_{2,n}|^2 = {}_{2,n}\langle a, b; 1, \cdots, n | a, b; 1, \cdots, n\rangle_{2,n}$$

Collinear Limits

Consider the final state splitting $(ij) \to ij$. Employing the Sudakov parameterization:

$$p_i^\mu = zp^\mu + k_\perp^\mu - \frac{k_\perp^2}{2zp\cdot n} n^\mu, \quad p_j^\mu = (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{2(1-z)p\cdot n} n^\mu$$

We can then generalize our previous collinear relation to:

$$_{2,n+1}\langle a,b;1,\cdots,n+1|a,b;1,\cdots,n+1\rangle_{2,n+1} \xrightarrow{k_\perp^2\to 0}$$

$$\frac{4\pi\mu^{2\epsilon}\alpha_s}{p_i\cdot p_j}\,{}_{2,n}\Big\langle a,b;\underbrace{1,\cdots,n+1}_{i,\ j\ \text{replaced by}(ij)}\Big|\hat{P}_{(ij),i}(z,k_\perp,\epsilon)\Big|a,b;\underbrace{1,\cdots,n+1}_{i,\ j\ \text{replaced by}(ij)}\Big\rangle_{2,n}$$

Here $\hat{P}_{(ij),i}(z,k_\perp,\epsilon)$ can in general be polarization dependent (*spin correlations!*). If the splitting parton was in the initial state, we reproduce our previous result (with the extra $1/z$ factor).

Soft Limits in QCD

Soft divergences appear when a final state gluon momenta goes to zero. Let's introduce a dimensionless parameter λ to parameterize the soft limit:

$$p_j^\mu = \lambda q^\mu$$

Then, in the limit $\lambda \to 0$ it is found:

$${}_{2,n+1}\langle a, b; 1, \cdots, n+1 | a, b; 1, \cdots, n+1 \rangle_{2,n+1} \longrightarrow$$

$$-\frac{8\pi\mu^{2\epsilon}\alpha_s}{\lambda^2} \sum_i \frac{1}{p_i \cdot q} \sum_{k \neq i} \frac{p_k \cdot p_i}{(p_i + p_k) \cdot q}$$

$${}_{2,n}\left\langle a, b; \underbrace{1, \cdots, n+1}_{j \text{ removed}} \middle| \mathbf{T}_k \cdot \mathbf{T}_i \middle| a, b; \underbrace{1, \cdots, n+1}_{j \text{ removed}} \right\rangle_{2,n}$$

The last amplitude is a color correlated amplitude, in which the operator $\mathbf{T}_k \cdot \mathbf{T}_i$ represents an insertion of the color degrees of freedom of a gluon between the partons k on the left and i on the right.

IR Limits in QCD Processes

- After two partons go collinear, square of QCD amplitudes factorize into a lower point amplitudes times a divergent term and a Splitting function. Spin correlations remain.
- If a final state gluon goes soft, square of QCD amplitudes produce a divergent term times a color correlated amplitude.
- These divergences are commonly regulated using dimensional regularization.
- In the same spirit of what we studied, multi-particle divergences appear in QCD amplitudes. Later in this set of lectures we will employ them to further our understanding of gauge theory amplitudes!

Summary

- QCD Corrections necessary for hadron collider (precision) pheno
- Great progress over the last decade for QCD calculations
- Subtraction techniques used by automated tools
- Factorization properties basic to our understanding of QCD!