Universiteit Utrecht

Liouville Theory and the S_1/Z_2 Orbifold

Olga Papadoulaki

Supervised by Dr Umut Gursoy

Polyakov Path Integral

- Using Polyakov formalism the String Theory partition function is:

$$Z = \int \mathcal{D}g\mathcal{D}X \exp\left(-S[X;g] - \mu_0 \int d^2z\sqrt{g}\right) \tag{1}$$

$$S[X;g] = \frac{1}{4\pi}\int d^2z\, g^{ab}\partial_a X^I \partial_b X^I \tag{2}$$

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- $\mathcal{D}g$, $\mathcal{D}X$ are *invariant* under world-sheet diffeomorphism transformations but *not* under Weyl tranformations such as:

$$g_{ab} \to e^{\sigma} g_{ab} \tag{3}$$

Under Weyl transformation the $\mathcal{D}_g X$ transforms as

$$\mathcal{D}_{e^{\sigma}g} X = e^{\frac{d}{48\pi} S_L(\sigma)} \mathcal{D}_g X \tag{4}$$

Liouville Action

- S_L is the Liouville Action.

$$S_L(\sigma) = \int d^2z \sqrt{g} \left(\frac{1}{2} g^{ab} \partial_a \sigma \partial_b \sigma + R\sigma + \mu e^{\sigma} \right) \tag{5}$$

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- The *metric's integration measure* $\mathcal{D}g$ is also *not- invariant* under Weyl transformations.
- To perform the *integration with respect to the metric*, we decompose the *fluctuation of the metric* δg_{ab} into the diffeomorphism u_a,Weyl transformation σ and the moduli Y.
- *Dividing* the *path integral measure* by the gauge (diffeomorphism) volume, we are left with the *integration* over the Weyl transformation and the moduli.
- The *Jacobian* for this change of variables can be calculated via the Fadeev-Popov method, introducing the *ghost fields* $b, \bar{b}, c, \bar{c}$.

Critical String Case

- $$\int \mathcal{D}b\bar{b}\mathcal{D}c\bar{c}\, \exp\left(-\int d^2z\sqrt{g}\left(b\overline{\nabla}c + \bar{b}\nabla\bar{c}\right)\right) \quad (6)$$

  and the transformation reads,

  $$\mathcal{D}_{e^{\sigma}g}\left(bc\right) = e^{-\frac{26}{48\pi}S_L(\sigma)}\mathcal{D}_g\left(bc\right) \quad (7)$$

- One can notice that in the case $d = 26$ the anomaly from $\mathcal{D}_g X$ cancels the one from $\mathcal{D}g$.
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- The theory is Weyl invariant and we have the case of Bosonic Critical String Theory.
- But we are interested in 2D String Theory (Non-Critical string theory).
- In this case, by choosing the conformal gauge $g_{ab} \to e^{\phi}\hat{g}_{ab}$, the string theory action takes the form:

$$Z = \int dY \mathcal{D}_{e^{\phi}\hat{g}}\phi\mathcal{D}_{e^{\phi}\hat{g}}bc\mathcal{D}_{e^{\phi}\hat{g}}X\, \exp\left(-S[X;\hat{g}] - S[bc;\hat{g}]\right) \tag{8}$$

Renormalised Liouville Action

- The *Liouville* mode measure is diffeomorphism *invariant* by definition, thus it satisfies

$$\|\delta\phi\|_g^2 = \int d^2z\ (\delta\phi)^2 = \int d^2z\, \sqrt{\hat{g}} e^{\phi}\, (\delta\phi)^2 \qquad (9)$$

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- One can notice that the *measure* is not Gaussian.
- We want to *bring it* into Gaussian form, hard to do it explicitly.
- We argue using locality, diffeomorphism invariance and conformal invariance, that the form of the *Renormalised Liouville Action* should be,

$$S_{RL} = \frac{1}{4\pi}\int d^2z\,\sqrt{g}\left(g^{ab}\partial_a\phi\partial_b\phi + QR\phi + 4\pi\mu e^{2b\phi}\right). \tag{10}$$

What is Q,b

- The theory should be invariant under $\hat{g_{ab}} \rightarrow e^{\sigma}\hat{g_{ab}}$ and $\phi \rightarrow \phi - \frac{\sigma}{2b}$.
- Then the $c_{tot} = c_{\phi} + c_X + c_{gh} = 0 \Rightarrow c_{\phi} = 26 - d$.
- Using coulomb gas representation one can compute: $c_{\phi} = 1 + 6Q^2$ where $Q = \sqrt{\frac{25-d}{6}}$.

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- Using coulomb gas representation one can compute: $c_{\phi} = 1 + 6Q^2$ where $Q = \sqrt{\frac{25-d}{6}}$.
- For the theory to be conformal invariant $e^{2b\phi}$ should be $(1,1)$ tensor then $\Delta = b(Q-b) = 1 \Rightarrow Q = b + b^{-1}$.
- For the *metric* to be real $c_X \leq 1$, this can be seen by finding the minimum $\frac{dQ}{db} = 0 \Rightarrow b^2 = 1 \Rightarrow Q_{min} = 2$, so from expression $Q = \sqrt{\frac{25-d}{6}}$ one can see that $d \leq 1$, for b to be real, but $d = c_X$.

One- loop calculation of the partition function

- We calculate the *1-loop partition function* of the c=1 *Liouville theory* with a compactified target space with radius R.
- We can do the integration because we first integrate over the zero mode of ϕ. Then the non-zero mode path integral becomes simply free. The only contribution from the zero-mode is given by the Liouville volume, V_ϕ.

$$Z_{circle} = \int d[Y]\mathcal{D}X\mathcal{D}\phi\mathcal{D}b\mathcal{D}c\, e^{-S_0} \tag{11}$$

$$Z(R) = -V_\phi \frac{1}{2}\int d^2\tau \left(\frac{|\eta(q)|^4}{2\tau_2}\right)\left(2\pi\sqrt{2\tau_2}\,|\eta(q)|^2\right)^{-1} Z_{bos}(R,\tau) \tag{12}$$

- $\eta(q)$ is the Dedekind eta function.
- τ_2 is the imaginary part of the torus moduli.
- $V_\phi = -\frac{1}{2b}\log\mu$ is the volume of the Liouville direction.
- $q = e^{2\pi\tau}$
- $Z_{bos} = \frac{R}{\sqrt{\tau_2}|\eta(q)|^2}\sum_{m,n=-\infty}^{\infty}\left(-\frac{\pi R^2|n-m\tau|^2}{\tau_2}\right)$

One- loop calculation of the partition function

- The $|\eta(q)|^4$ comes form the *integration* of the ghost oscillators.
- The $|\eta(q)|^{-2}$ comes from the *integration* of the Liouville mode.
- The $\left(2\pi\sqrt{2\tau_2}\right)^{-1}$ is from the *integration* of the Liouville momentum.

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- After performing the *integration* of the torus moduli we find:

$$Z_{circle} = -\frac{1}{24}\left(R + \frac{1}{R}\right)\log\mu \tag{13}$$

- This result is in *agreement* with the one coming from the matrix-model approach.

The S_1/Z_2 orbifold

- If X is a *smooth manifold* with a *discrete isometry group* G. We can form the quotient space X/G.
- If the manifold has not fixed points under the action of G then X/G is a smooth manifold, otherwise it has conical singularities at those points.

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- If X is a *smooth manifold* with a *discrete isometry group* G. We can form the quotient space X/G.
- If the manifold has not fixed points under the action of G then X/G is a smooth manifold, otherwise it has conical singularities at those points.
- *Simplest example* of an orbifold is the S_1/Z_2.
- That is a *circle* $x \sim x + 2\pi$, on which we made the following identification $x \sim -x$.
- This has *transformed the circle* to an interval, from $[0, \pi]$, with $0, \pi$ the two fixed points.
- Physical States on an orbifold:
  - Untwisted are those that exist on X and are *invariant* under the group G, $\Psi = g\Psi, g \in G$.
  - Twisted states are new closed-string states that appear after orbifolding $X^{\mu}(\sigma + 2\pi) = -X^{\mu}(\sigma)$

Liouville Theory on S_1/Z_2 orbifold

- The modular partition function has the following form:

$$Z_{orb}(R,\tau) = \frac{1}{2} Z_{cir}(R,\tau) + \left\{ \left| \frac{\eta(\tau)}{\theta_{00}(0,\tau)} \right| + \left| \frac{\eta(\tau)}{\theta_{01}(0,\tau)} \right| + \left| \frac{\eta(\tau)}{\theta_{10}(0,\tau)} \right| \right\} \tag{14}$$

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- The full torus partition function comes from the coupling of the above with the *Liouville* and the *ghosts* and integrating over the *torus moduli* τ.

$$Z(R) = -\frac{1}{2} \int d^2\tau \left(\frac{|\eta(\tau)|^4}{2\tau_2} \right) \left(2\pi\sqrt{2\tau_2}\, |\eta(\tau)|^2 \right)^{-1} Z_{orb}(R,\tau) \tag{15}$$

- Performing the Integration and using the fact that $Z_{orb}(R=1,\tau) = Z_{cir}(R=2,\tau)$, one gets:

$$Z_{orb} = -\frac{1}{48}\left(R + \frac{1}{R}\right)\log\mu - \frac{1}{16}\log\mu \tag{16}$$

Outlook

- We are going to do the previous steps for cases of $0B$ and $0A$ string theory.
- Perform the same computation using random matrices approach and see if they match.
- Analytic continuation of the Euclidean time to Lorentzian signature and interpretation of the results extracting information for toy model cosmology???

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Thank you!