Liouville Theory and the $S_1/Z_2$ Orbifold

Olga Papadoulaki

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Polyakov Path Integral

Using Polyakov formalism the String Theory partition function is:

\[
Z = \int \mathcal{D}g \mathcal{D}X \exp \left( -S[X; g] - \mu_0 \int d^2z \sqrt{g} \right) \tag{1}
\]

\[
S[X; g] = \frac{1}{4\pi} \int d^2z \, g^{ab} \partial_a X^I \partial_b X^I \tag{2}
\]

\(X^I\) are bosonic fields and \(I = 1, \ldots, d\)
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\(\mathcal{D}g, \mathcal{D}X\) are invariant under world-sheet diffeomorphism transformations but not under Weyl transformations such as:

\[
g_{ab} \rightarrow e^{\sigma} g_{ab}
\]

Under Weyl transformation the \(\mathcal{D}g X\) transforms as

\[
\mathcal{D}e^{\sigma} g X = e^{\frac{d}{48\pi} S_L(\sigma)} \mathcal{D}g X
\]
Liouville Action

$S_L$ is the Liouville Action.

$$S_L(\sigma) = \int d^2z \sqrt{g} \left( \frac{1}{2} g^{ab} \partial_a \sigma \partial_b \sigma + R \sigma + \mu e^\sigma \right)$$ (5)
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- The metric's integration measure $Dg$ is also not-invariant under Weyl transformations.
- To perform the integration with respect to the metric, we decompose the fluctuation of the metric $\delta g_{ab}$ into the diffeomorphism $u_a$, Weyl transformation $\sigma$ and the moduli $Y$.
- Dividing the path integral measure by the gauge (diffeomorphism) volume, we are left with the integration over the Weyl transformation and the moduli.
- The Jacobian for this change of variables can be calculated via the Fadeev-Popov method, introducing the ghost fields $b, \bar{b}, c, \bar{c}$. 
Critical String Case

\[
\int D\bar{b}bD\bar{c}c \exp \left( - \int d^2z \sqrt{g} \left( b\nabla c + \bar{b}\nabla \bar{c} \right) \right) \tag{6}
\]

and the transformation reads,

\[
D e^{\sigma} g (b\sigma) = e^{- \frac{26}{48\pi} S_L(\sigma)} D g (b\sigma) \tag{7}
\]

One can notice that in the case \( d = 26 \) the anomaly from \( D g X \) cancels the one from \( D g \).

The theory is Weyl invariant and we have the case of Bosonic Critical String Theory.
Critical String Case

\[ \int \mathcal{D}b \bar{b} \mathcal{D}c \bar{c} \exp \left( - \int d^2 z \sqrt{g} \left( b \nabla c + \bar{b} \nabla \bar{c} \right) \right) \]  \hspace{1cm} (6)

and the transformation reads,

\[ \mathcal{D}_{e^\sigma g} (bc) = e^{-\frac{26}{48\pi} S_L(\sigma)} \mathcal{D}_g (bc) \]  \hspace{1cm} (7)

- One can notice that in the case \( d = 26 \) the anomaly from \( \mathcal{D}_g X \) cancels the one from \( \mathcal{D}_g \).
- The theory is Weyl invariant and we have the case of Bosonic Critical String Theory.
- But we are interested in 2D String Theory (Non-Critical string theory).
- In this case, by choosing the conformal gauge \( g_{ab} \rightarrow e^\phi \mathring{g}_{ab} \), the string theory action takes the form:

\[ Z = \int dY \mathcal{D}_{e^\phi \mathring{g}} \phi \mathcal{D}_{e^\phi \mathring{g}} bc \mathcal{D}_{e^\phi \mathring{g}} X \exp \left( -S[X; \mathring{g}] - S[bc; \mathring{g}] \right) \]  \hspace{1cm} (8)
Renormalised Liouville Action

- The *Liouville* mode measure is diffeomorphism *invariant* by definition, thus it satisfies

\[ \|\delta\phi\|^2_g = \int d^2z \ (\delta\phi)^2 = \int d^2z \sqrt{g} e^\phi (\delta\phi)^2 \]  \hspace{1cm} (9)

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The Liouville mode measure is diffeomorphism invariant by definition, thus it satisfies

\[ \| \delta \phi \|^2_g = \int d^2 z \ (\delta \phi)^2 = \int d^2 z \sqrt{\hat{g} e^\phi} (\delta \phi)^2 \]  

(9)

One can notice that the measure is not Gaussian.

We want to bring it into Gaussian form, hard to do it explicitly.

We argue using locality, diffeomorphism invariance and conformal invariance, that the form of the Renormalised Liouville Action should be,

\[ S_{RL} = \frac{1}{4\pi} \int d^2 z \sqrt{g} \left( g^{ab} \partial_a \phi \partial_b \phi + Q R \phi + 4\pi \mu e^{2\phi} \right). \]  

(10)
What is $Q,b$

- The theory should be invariant under $g_{ab} \rightarrow e^{\sigma} g_{ab}$ and $\phi \rightarrow \phi - \frac{\sigma}{2b}$.
- Then the $c_{\text{tot}} = c_\phi + c_X + c_{gh} = 0 \Rightarrow c_\phi = 26 - d$.
- Using coulomb gas representation one can compute: $c_\phi = 1 + 6Q^2$
  where $Q = \sqrt{\frac{25-d}{6}}$. 

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Liouville Theory and the $S_1/Z_2$ orbifold
What is $Q,b$

- The theory should be **invariant** under $g_{ab} \rightarrow e^\sigma g_{ab}$ and $\phi \rightarrow \phi - \frac{\sigma}{2b}$.
- Then the $c_{tot} = c_\phi + c_X + c_{gh} = 0 \Rightarrow c_\phi = 26 - d$.
- Using coulomb gas representation one can compute: $c_\phi = 1 + 6Q^2$ where $Q = \sqrt{\frac{25-d}{6}}$.
- For the theory to be **conformal invariant** $e^{2b\phi}$ should be $(1,1)$ tensor then $\Delta = b(Q - b) = 1 \Rightarrow Q = b + b^{-1}$.
- For the **metric** to be real $c_X \leq 1$, this can be seen by finding the minimum $\frac{dQ}{db} = 0 \Rightarrow b^2 = 1 \Rightarrow Q_{min} = 2$, so from expression $Q = \sqrt{\frac{25-d}{6}}$ one can see that $d \leq 1$, for $b$ to be real, but $d = c_X$. 

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Liouville Theory and the $S_1/Z_2$ orbifold
One-loop calculation of the partition function

- We calculate the 1-loop partition function of the $c=1$ Liouville theory with a compactified target space with radius $R$.

- We can do the integration because we first integrate over the zero mode of $\phi$. Then the non-zero mode path integral becomes simply free. The only contribution from the zero-mode is given by the Liouville volume, $V_\phi$.

$$Z_{\text{circle}} = \int d[Y] \mathcal{D}X \mathcal{D}\phi \mathcal{D}b \mathcal{D}c \, e^{-S_0} \quad (11)$$

$$Z(R) = -V_\phi \frac{1}{2} \int d^2 \tau \left( \frac{\left| \eta(q) \right|^4}{2\tau_2} \right) \left( 2\pi \sqrt{2\tau_2} |\eta(q)|^2 \right)^{-1} Z_{\text{bos}}(R, \tau) \quad (12)$$

- $\eta(q)$ is the Dedekind eta function.

- $\tau_2$ is the imaginary part of the torus moduli.

- $V_\phi = -\frac{1}{2b} \log \mu$ is the volume of the Liouville direction.

- $q = e^{2\pi \tau}$

- $Z_{\text{bos}} = \frac{R}{\sqrt{\tau_2} |\eta(q)|^2} \sum_{m,n=-\infty}^{\infty} \left( -\frac{\pi R^2 |n-m\tau|^2}{\tau_2} \right)$
One-loop calculation of the partition function

- The $|\eta(q)|^4$ comes from the integration of the ghost oscillators.
- The $|\eta(q)|^{-2}$ comes from the integration of the Liouville mode.
- The $(2\pi \sqrt{2\tau_2})^{-1}$ is from the integration of the Liouville momentum.
One-loop calculation of the partition function

- The $|\eta(q)|^4$ comes from the *integration* of the ghost oscillators.
- The $|\eta(q)|^{-2}$ comes from the *integration* of the Liouville mode.
- The $(2\pi \sqrt{2\tau_2})^{-1}$ is from the *integration* of the Liouville momentum.
- After performing the *integration* of the torus moduli we find:

$$Z_{\text{circle}} = -\frac{1}{24} \left(R + \frac{1}{R}\right) \log \mu$$  \hspace{1cm} (13)

- This result is in *agreement* with the one coming from the matrix-model approach.
The $S_1/Z_2$ orbifold

- If $X$ is a smooth manifold with a discrete isometry group $G$. We can form the quotient space $X/G$.
- If the manifold has not fixed points under the action of $G$ then $X/G$ is a smooth manifold, otherwise it has conical singularities at those points.

Physical States on an orbifold:

- Untwisted are those that exist on $X$ and are invariant under the group $G$, $\Psi = g \Psi$, $g \in G$.
- Twisted states are new closed-string states that appear after orbifolding $X$.

$\sigma + 2\pi = -\sigma$
The $S_1/Z_2$ orbifold

- If $X$ is a *smooth manifold* with a *discrete isometry group* $G$. We can form the *quotient space* $X/G$.
- If the manifold has not fixed points under the action of $G$ then $X/G$ is a smooth manifold, otherwise it has conical singularities at those points.
- *Simplest example* of an orbifold is the $S_1/Z_2$.
- That is a *circle* $x \sim x + 2\pi$, on which we made the following identification $x \sim -x$.
- This has *transformed the circle* to an *interval*, from $[0, \pi]$, with 0, $\pi$ the two fixed points.
- *Physical States* on an orbifold:
  - Untwisted are those that exist on $X$ and are *invariant* under the group $G$, $\Psi = g\Psi$, $g \in G$.
  - Twisted states are new closed-string states that appear after orbifolding $X^\mu (\sigma + 2\pi) = -X^\mu (\sigma)$.
The modular partition function has the following form:

\[ Z_{orb}(R, \tau) = \frac{1}{2} Z_{cir}(R, \tau) + \left\{ \left| \frac{\eta(\tau)}{\theta_{00}(0, \tau)} \right| + \left| \frac{\eta(\tau)}{\theta_{01}(0, \tau)} \right| + \left| \frac{\eta(\tau)}{\theta_{10}(0, \tau)} \right| \right\} \]

(14)
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(14)

The full torus partition function comes from the coupling of the above with the \textit{Liouville} and the \textit{ghosts} and integrating over the \textit{torus moduli} \( \tau \).

\[ Z(R) = -\frac{1}{2} \int d^2 \tau \left( \frac{\left| \eta(\tau) \right|^4}{2\tau_2} \right) \left( 2\pi \sqrt{2\tau_2} |\eta(\tau)|^2 \right)^{-1} Z_{orb}(R, \tau) \]

(15)

Performing the Integration and using the fact that \( Z_{orb}(R = 1, \tau) = Z_{cir}(R = 2, \tau) \), one gets:

\[ Z_{orb} = -\frac{1}{48} \left( R + \frac{1}{R} \right) \log \mu - \frac{1}{16} \log \mu \]

(16)
Outlook

- We are going to do the previous steps for cases of $0B$ and $0A$ string theory.
- Perform the same computation using random matrices approach and see if they match.
- Analytic continuation of the Euclidean time to Lorentzian signature and interpretation of the results extracting information for toy model cosmology???

Y. Nakayama, "Liouville Field Theory, A decade after the revolution", hep-th/0402009
N. Seiberg, "Notes on Quantum Liouville Theory and Quantum Gravity"
A. Zamolodchikov, A. Zamolodchikov, "Lectures on Liouville Theory and Matrix Models"

Thank you!