Topological–anti-topological fusion

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We study some non-perturbative aspects of $N = 2$ supersymmetric quantum field theories (both superconformal and massive deformations thereof). We show that the metric for the supersymmetric ground states, which in the conformal limit is essentially the same as Zamolodchikov's metric, is pseudo-topological and can be viewed as a result of fusion of the topological version of $N = 2$ theory with its conjugate. For special marginal/relevant deformations (corresponding to theories with factorizable S-matrix), the ground state metric satisfies classical Toda/Affine Toda equations as a function of perturbation parameters. The unique consistent boundary conditions for these differential equations seem to *predict* the normalized OPE of chiral fields at the conformal point. Also the subset of $N = 2$ theories whose chiral ring is isomorphic to $SU(N)_k$ Verlinde ring turns out to lead to affine Toda equations of $SU(N)$ type satisfied by the ground state metric.

1. Introduction

$N = 2$ supersymmetric quantum field theories have recently undergone an intensive investigation from many different points of view: From the string point of view $N = 2$ superconformal models in 2 dimensions constitute the building blocks of $N = 1$ space-time supersymmetric string vacua [1]. From the point of view of classification of conformal theories, they are in a sense the simplest type to classify, and a nice subset of them, supersymmetric Landau–Ginzburg theories, is related to catastrophe theory [2–5]. From the point of view of topological characterization of the theory, they have a finite ring of operators (chiral primary fields) [4] which is believed to basically characterize them. There is a "twisted" version of these theories [6], the topological version, which has as its physical degree of freedom *only* these operators. These topological theories have been studied from the view point of 2d superconformal [7] and topological Landau–Ginzburg theories [8], and

from the viewpoint of their properties under coupling to topological gravity in [9–11].

From a slightly different point of view, $N = 2$ supergravity theories has also been studied in four dimensions, and it was found that for the construction of the theory a very special type of Kähler geometry is needed [12]. This in turn is related to the fact that in the type II superstring compactification, leading to 2-dimensional $N = 2$ superconformal theories, the metric on moduli space of a three-fold Calabi–Yau has special properties, and is basically characterized by a holomorphic, topological object (pre-potential) [13,14]. This geometry is called "special geometry". The metric on moduli space of Calabi–Yau is the same as the Zamolodchikov metric of the underlying $N = 2$ SCFT, thus relating geometry with SCFT correlation functions.

In the topological description of $N = 2$ theories, one of the two supersymmetry charges plays the role of a BRST operator and the physical operators of the theory get truncated to the chiral ring. In this way the computations can be performed in a more or less closed form. The topological correlation functions are basically combinatorial objects, holomorphic functions of moduli. In the case of special geometry these topological correlation functions serve as *coefficients* of differential equations characterizing Zamolodchikov's metric on moduli space, which thus makes the Zamolodchikov's metric pseudo-topological. The Zamolodchikov metric which appears for example in the low-energy *dynamics* of the effective field theory description of strings is thus characterized by purely *kinematical / combinatorial* topological data. In these cases one finds that the Zamolodchikov (Weil–Petersson) metric is Kähler and the Kähler potential is written as a finite sum of holomorphic and anti-holomorphic "blocks" (periods) in the moduli of target space.

In the context of supersymmetric quantum mechanics related to LG theories it was found in ref. [14] that the same system of differential equations that characterizes the ground state metric (basically the Zamolodchikov metric) at the conformal point and gave rise to special geometry are also valid off the conformal point. That naturally raises a question of whether there is a generalization of special geometry off the conformal point as well.

One of the aims of this paper is to uncover the special geometry for massive (i.e. non-conformal) theories as well, and explain the rationale for finding a pseudo-topological metric from the topological viewpoint for both massless and massive theories. Basically what we find is, that if one fuses a topological theory with itself, one ends up with topological objects such as the holomorphic pre-potential which arise in special geometry. If on the other hand we fuse a topological theory with its conjugate, which we call *anti-topological*, we end up with pseudo-topological objects such as Zamolodchikov's metric. The generalized notion of special geometry simply encodes this relation between topological–topological fusion versus topological–anti-topological fusion and their variation with respect to moduli. We

show from this viewpoint in precisely what sense they are topological and derive the equations that characterize them, thus generalizing the results derived in ref. [14] for Landau–Ginzburg theories to arbitrary $N = 2$ QFTs. In this way we find a *generalized special geometry* to be equally valid on- or off-criticality. Even though the equations are the same in the two cases, we find a sharp difference between the *solutions* to these equations on- and off-criticality. In both cases we find the metric to be a sum over a finite block of objects, but in the critical theory these objects (periods) are *holomorphic* while in the off-critical theory these objects (the generalized periods) are *not holomorphic* functions of moduli and are generically far too complicated to give in closed form. From the viewpoint of chiral rings the main reason for complication of solutions to special geometry in the massive case is that in this case the ring is *not* nilpotent.

These ideas are made more concrete using many explicit examples of massive deformations of $N = 2$ LG theories, which is the main reason for the unusual length of the present work. The special examples that we obtain, which are of the form of generalized affine Toda equations, bring a completely orthogonal direction of interest to the present work. Namely, many of our examples provide interesting non-singular solutions to some affine Toda equations in terms of correlations (the metrics) of $N = 2$ theories. In this way we can use the methods available to us from the $N = 2$ theories, to gain insight into the solutions of (self-similar) affine Toda equations, which one generally does not have a good handle on. Along the way we are able to reproduce some deep mathematical results for solutions to Painlevé III [15] and Bullough–Dodd [16], which had been obtained using isomonodromic deformation technique and generalize them to other affine Toda theories. We basically find that the OPEs of SCFT *solve* the boundary conditions needed for a non-singular solution to (self-similar) affine Toda equations.

As is the case with many works on integrable systems there are many mysteries which need explanation. We find a number of intriguing results which beg for a deeper understanding. In particular many of our $N = 2$ massive supersymmetric theories are themselves described by *quantum* affine Toda theories (some non-supersymmetric and some $N = 2$ supersymmetric affine Toda lagrangians). In these cases we find that the ground state metric, which could be viewed as some particular correlation functions in these theories, as a function of the overall coupling (temperature or scale parameter) satisfy ordinary *classical* affine Toda equations *of the same type* (or reductions thereof). This is somewhat reminiscent of the space-time–target duality obtained for critical $N = 2$ strings [17]. The magic is even more mysterious: some of the cases corresponding to $N = 2$ supersymmetric affine Toda lagrangians (the SU(N) case) turn out to be related to Verlinde's ring for SU(N)$_k$ RCFT [18].

The structure of this paper is as follows: In sect. 2 we review some topological aspects of $N = 2$ theories, and introduce the idea of topological–anti-topological fusion. In sect. 3 we derive some equations satisfied by the ground state metric by

considering a family of $N = 2$ theories. We also discuss some general properties of the metric. In sect. 4 we discuss the relation to renormalization group flows, the c-function and Zamolodchikov metric. In sect. 5 we discuss the reductions to SQM and in particular derive a rule which allows us to relate different models by non-invertible change of variables. Moreover we find a "period" decomposition of the metric which generalizes the known result at the conformal point to the massive theories. In sect. 6 we discuss some Lie-algebraic aspects of our equations, which are very helpful in a classification of their solutions. In sect. 7 we consider some examples related to minimal models and some special massive perturbations of them. In sect. 8 we consider a few of the examples discussed in sect. 7 in more detail, using properties of the solutions to Painlevé III and Bullough–Dodd [15,16]. In sects. 9 and 10 we study more tricky models related to Verlinde rings. In sect. 11 we present our conclusions. In appendices A and B the properties of the metric in the UV and IR are discussed respectively. Finally in appendix C the relationship with the "special coordinates" of special geometry is uncovered.

2. Topological aspects of $N = 2$ theories

In this section we review some of the background work which is needed for this paper. Our main interest for most of the paper is $N = 2$ Landau–Ginzburg theory, but many of our constructions are more general, and so in this section we will not commit ourselves to the Landau–Ginzburg theory, and consider the more general class of $N = 2$ quantum field theories. Moreover we do not make the assumption that the quantum field theory is conformal, and our treatment applies to both massive and massless (conformal) cases. We will be mostly interested in the 1- and 2-dimensional descriptions, but some of what we say generalizes in a simple way to higher dimensions (and in particular to Donaldson theory [19]).

In an $N = 2$ theory, there are two supersymmetry charges, which we label by Q^+ and Q^-. The main property of these supersymmetry charges is that

$$(Q^+)^2 = (Q^-)^2 = 0, \qquad \{Q^+, Q^-\} = H, \qquad (Q^+)^\dagger = Q^-, \qquad (2.1)$$

where H is the hamiltonian. Topological theory is obtained by declaring Q^+ to be a BRST operator [6] and by identifying the BRST cohomology of Q^+ with the physical Hilbert space (note that in the context of two-dimensional theories, this means that we put periodic boundary conditions on the circle in order to have a supersymmetry operator, i.e. we are in the Ramond sector)

$$Q^+ |\psi\rangle = 0, \qquad |\psi\rangle \sim |\psi\rangle + Q^+ |\rho\rangle.$$

We can fix the ambiguity of the topological theory in identifying the state, by using the Q^- operator and demanding that the physical states be also annihilated by

Q^-. This is the analog of picking a harmonic representative in the standard cohomology. As is clear from the standard arguments, this fixes the ambiguity of adding Q^+-exact states to the ground state. In fact using (2.1) we can identify the topological states with the ground states of the supersymmetric theory.

The topological operators ϕ_i are defined to be operators which commute with Q^+, i.e.

$$[Q^+, \phi_i] = 0. \tag{2.2}$$

These fields are called chiral fields. A field which itself is a Q^+-commutator acts trivially on the Hilbert space. It is obvious that chiral fields form a ring, because of OPE of two of them is Q^+-closed and so can be expanded in terms of chiral fields. But most of the elements that appear in the product are themselves Q^+-commutators, and thus are trivial operators in the topological theory. Since the translation operator is itself a Q^+-commutator (following from supersymmetry) the chiral fields and their translations differ by Q^+-trivial operators. Thus we see that in order to obtain the topological product of two chiral fields at different points it is sufficient to take their product *at the same point*. This will differ from the fields at different points by fields which are Q^+-commutators. So to specify this ring we do not have to specify the points at which we put the fields. If we choose a basis ϕ_i for the physical chiral fields, we get a ring

$$\phi_i \phi_j = C_{ij}^k \phi_k + Q^+\text{-commutator terms.}$$

This ring is in generic cases a finite ring. In the context of critical theory this ring, the chiral primary ring, was defined and studied in ref. [4].

The question arises as to whether there is a natural identification of the states with the operators in the topological theory. This would be obvious if we can identify a unique vacuum state in the topological theory which we denote by $|0\rangle$. Once we have such a state then we simply identify the states by the operation of ϕ_i on the vacuum

$$\phi_i |0\rangle = |i\rangle.$$

The property (2.2) guarantees that the resulting state is Q^+-closed and is thus itself a topological state. So the main question is how we identify the vacuum state. In general there are a number of ground states which all have zero energy (in the LG case the number is equal to Witten's index) and it might at first sound impossible to pick a "preferred" one. If we were dealing with the SCFT there is a canonical choice. Namely in that case we have two U(1) charges (the left and right charges) which labels the vacua and we look for the unique state with minimum (left and right) charge and identify that as $|0\rangle$. All the other states are obtained from it by applying the physical fields (chiral primary fields which all have positive U(1)

charge) on it. Here we have crucially used the properties of the conformal theory, and in particular the existence of an additional U(1) charge, which is the property of the critical theory. In the general massive case there is only one U(1) charge and that counts the fermion number (the difference between the left and right charges at the conformal point). In particular this is not enough to pick a unique state (for example in the LG theories all the ground states have equal left and right charges and thus are neutral under this charge).

One might be led to believe that a canonical choice for the ground state of the Ramond sector does not exist off criticality, but that turns out not to be so. To see this we can use the spectral flow to give an alternative definition of the vacuum state [4]. Consider the Hilbert space based on the NS sector, i.e. circle with antiperiodic boundary condition for fermions. The spectral flow is obtained by changing the boundary condition for fermions *continuously* from antiperiodic to periodic. This can be done because we do have a conserved fermion number in the theory even off criticality. In this way we can identify each state in the NS sector with a unique state in the Ramond sector. In particular the unique vacuum of the NS sector will flow to a unique ground state of the Ramond sector which we identify as $|0\rangle$. Note that this description of spectral flow is equally valid *whether or not the theory is conformal*. So in this way we see that there is a canonical isomorphism between the operators in the NS sector and the topological states (in the Ramond sector).

There is a nice way to implement spectral flow in the path-integral language which will be very useful for us: Consider a hemisphere with the standard metric and with some operators inserted on it. The boundary of the hemisphere is a circle on which we base our Hilbert space. The path integral will give us a state in the Hilbert space. Now if we were doing the standard $N = 2$ quantum field theory on the hemisphere, the fermion spin structure is trivial on it, but that induces an *antiperiodic* boundary condition for the fermions on the boundary. So the standard path-integral, if we do not put spin operators on the hemisphere, will give us a state in the NS sector as is familiar from the study of SCFTs. The trick is to consider the topological version of this path-integral. This is equivalent [7] to putting a background gauge field which couples to fermions number and is set to be half of the spin connection. In this background, over the sphere the fermion number is violated by one unit, and over half of the sphere the fermion number is violated by one half, which is precisely the flow from the NS to R sector. Put differently, the boundary condition for the fermions at the circle boundary of the hemisphere is still antiperiodic, but there is a U(1) Wilson line which couples to fermion number. We can get rid of the Wilson line by changing the boundary condition of fermions by the holonomy

$$\exp\left(i\int_{S^1} A\right) = \exp\left(i\int_{\text{hemisphere}} F\right) = \exp(i\pi) = -1,$$

i.e. it is equivalent to changing the boundary conditions from the NS to the R. This is the magic of topological theory: it automatically "knows" about spectral flow.

The topological description guarantees that as long as we put fields which commute with Q^+ (i.e. chiral fields) on the hemisphere we get a state at the boundary which is in the topological Hilbert space, i.e. it is Q^+-closed. In fact the topological nature of the theory guarantees that the topological state that we get will not depend on the precise metric we put on the hemisphere. Changing the metric has the effect of shifting the resulting state by the addition of a Q^+-closed state. If we wish to obtain the actual ground state representative we will have to choose the metric on the hemisphere which makes it look like the standard hemisphere with an infinite cylinder glued at the boundary to it. In this way the propagation by $\exp(-TH)$ for large T along the infinite cylinder will project the topological state onto the actual ground state of the hamiltonian.

In this way we see that for each chiral field ϕ_i we get a state $|i\rangle$ in the Ramond sector by doing the path integral with that chiral field on the hemisphere. In particular $|0\rangle$ is the state associated to the identity operator. The topological nature of the theory will guarantee independence of where we put that field precisely within the hemisphere. In particular we can move it to the boundary of the hemisphere, in which case by operator formulation we see that the state is the same as multiplication of the state by the field ϕ_i

$$|i\rangle = \phi_i|0\rangle,$$

thus agreeing with the previous definition. Note that in this equation by $|i\rangle$ we mean the topological class of the state, i.e. $|i\rangle$ may differ from an actual ground state of the theory by Q^+-closed states. Again if we wish to obtain an actual ground state we should propagate the state along a cylinder for a long time. From the above we also learn that

$$\phi_i|j\rangle = \phi_i\phi_j|0\rangle = C_{ij}^k\phi_k|0\rangle = C_{ij}^k|k\rangle, \qquad (2.3)$$

where again the equalities are modulo Q^+-trivial states. We can thus represent the action of the chiral fields in the subsector of vacuum states by the matrix C_i

$$(C_i)_j^k = C_{ij}^k$$

Everything we have said in the above can be repeated replacing everything by its adjoint. In particular this means replacing Q^+ by its adjoint Q^-, the chiral fields ϕ_i by their adjoint antichiral fields $\bar{\phi}_i$, and the chiral ring coefficients C_{ij}^k with their complex conjugate $C_{ij}^{\bar{k}} = (C_{ij}^k)^*$ for the antichiral ring. In the path-integral definition of the states, we introduce a background gauge field which is now *minus* half the spin connection. In this way we get another topological theory which is simply the conjugate one and we call it the *anti-topological* theory.

It turns out to be crucial for us to have a deeper understanding of the relation between these two topological theories. The crucial link between the two theories turns out to be the Ramond sector. Namely, the physical states in both theories are in one to one correspondence with the Ramond vacua, as we have discussed above. So now let $|i\rangle$ and $|j\rangle$ denote the *actual ground states* corresponding to the fields ϕ_i and ϕ_j respectively. In this way we have found two "preferred" bases for the Ramond ground states. Of course we can write one in terms of the other, so we must have

$$\langle \bar{i}| = \langle j| M_i^j. \tag{2.4}$$

The matrix M defined above is referred to as the *real structure*. It is crucial for us because it is precisely an intertwiner between the topological theory and its conjugate. In a sense it allows us to compare a topological theory with its conjugate. Since the Hilbert space is coming from a quantum field theory we have a *CPT* operator which is an anti-unitary operator of order 2. Acting on (2.4) with this operator and using its anti-unitarity one easily deduces that the real-structure matrix satisfies

$$MM^* = \mathbb{1}, \tag{2.5}$$

where M^* denotes the complex (not hermitian) conjugate matrix to M. In order to completely understand the structure of our Hilbert space, in addition to the operator content of the Hilbert space we also need to know its inner product. Since we have a natural $N = 2$ field theory underlying our constructions we automatically have an inner product. That is simply the inner product in the Ramond ground state. To write it down, we need to choose bases. In particular we can use the basis where the left and right states are taken to be the chiral basis

$$\langle j|i\rangle = \eta_{ij}, \tag{2.6}$$

or the chiral and antichiral basis

$$\langle \bar{j}|i\rangle = g_{ij}, \tag{2.7}$$

and the complex conjugate of the above inner products. Of course the two metrics η_{ij} and g_{ij} are related using the real-structure matrix M

$$g_{i\bar{k}} = \eta_{ij}M_k^j. \tag{2.8}$$

Note that we can deduce from eqs. (2.5) and (2.8) the very useful identity which relates g and η

$$\eta^{-1}g\left(\eta^{-1}g\right)^* = \mathbb{1}. \tag{2.9}$$

The inner product of immediate interest in $N = 2$ theories is the g metric, because when we take the inner product of states we take the *adjoint* of a state on the left and the adjoint of the state $|i\rangle$ is $\langle \bar{i}|$ and not $\langle i|$. In particular the metric η is not hermitian whereas g is obviously a hermitian metric. However, as we shall see η is much simpler to compute, and is in fact a purely topological object (it will be clear as we proceed that η is a symmetric matrix).

To understand the structure of these two metrics better we represent them by path-integrals. The path-integrals are represented by two hemispheres, one on the left and the other on the right, joined by an infinitely long cylinder. We need an infinitely long cylinder to project onto the ground states. In addition we have a background gauge field, which for the computation of η_{ij} is set equal to half the spin connection throughout the sphere, and we insert the operator ϕ_i on the right hemisphere and the operator ϕ_j on the left hemisphere. For the computation of g_{ij}, on the right hemisphere and the right half of the infinite cylinder we have a background gauge field which is half the spin connection while on the left hemisphere and the left half of the infinite cylinder we have a background gauge field which is minus half the spin connection. The fact the the region were the left and right meet is flat, means that the gauge fields glue smoothly from one to the other, and we have a well defined gauge field. Also we insert the field ϕ_i on the right hemisphere and the field ϕ_j on the left hemisphere.

From the above path-integral definition it follows that essentially both metrics are topological, where by topological we mean if we perturb the corresponding positions of inserted fields or the metric on the hemisphere, as long as there is an infinitely long intermediate cylinder, with a fixed perimeter β, the result of the path-integral does not change. This is due to the fact that local perturbations of this kind, as noted above, are equivalent to operations by Q^+ or Q^- on some state, and propagation along the cylinder of length T results in $\exp(-TH)Q^\pm$ which goes to zero as $T \to \infty$. However η is *more* topological in the sense that even if we change the length of the intermediate cylinder or even completely change its metric, or even move the positions of fields from one hemisphere to the other, the result will still not change. This follows from the usual definitions of the topological theory, as all such variations are Q^+-commutators and since the fields commute with Q^+ we immediately see that the variations do not change the result of path-integral. Note that since we can exchange the position of the operators between the two hemispheres η is symmetric. The fact that η is purely topological allows us to give a simple closed form for it in many cases. The result for the LG theory will be mentioned below. This general notion of topological invariance, i.e. without necessitating an infinitely long intermediate cylinder, would not work for g_{ij} because we get both Q^+ and Q^- variations on the right and left hemispheres respectively which does not allow us to complete the argument. In this sense g_{ij} which is obtained by "fusing" a topological theory with its complex conjugate is

only "partially" topological in the sense discussed above *. In particular it depends on the perimeter of the cylinder β. It turns out that changing β is equivalent to changing the scale of the theory, and the one-parameter family of metrics g_{ij} that we obtain can be viewed as the trajectory of the metric under RG flow. In the following we set $\beta = 1$, and implement the change of scale by flow in the coupling-constant space of the theory. Even though g looks only pseudo-topological, as we shall see later in this paper the purely topological correlations allow us to completely determine it.

It is also easy to see these constructions in the operator language. In particular in this way we can show that even though in the above definitions of η and g we have used the ground states themselves the metric η_{ij} is independent of which representative we choose. This follows by noting that changing for example the representative $\langle i |$ is equivalent to shifting it by $\langle \alpha | Q^+$, but this does not affect the inner product $\langle i | j \rangle$, because $Q^+ | j \rangle = 0$ (for any representative of $| j \rangle$). Note that the same argument to show independence of g_{ij} of the choice of the representative would fail and so this quantity *does* depend on the fact that we have to actually choose the precise ground states representing the cohomology classes.

So far we have been general. We will now illustrate these ideas in the context of $N = 2$ LG theories. These models are defined by taking a number of superfields X_i in two-dimensional space with two left and two right moving anti-commuting coordinates denoted by $\theta^{\pm}$ and $\bar{\theta}^{\pm}$. The superfields are taken to be chiral which means that

$$D^+ X_i = \left(\frac{\partial}{\partial \theta^+} + \theta^- \frac{\partial}{\partial z} \right) X_i = 0 = \bar{D}^+ X_i,$$

and similarly $\bar{X}_i$ is anti-chiral (and satisfies the above equations with θ^+ and θ^- exchanged). Then one writes down a lagrangian

$$\mathscr{L} = \tfrac{1}{2} \int \mathrm{d}^4 \theta \, K\left(X_i, \bar{X}_j \right) + \int \mathrm{d}^2 \theta \, W(X_i) + \text{h.c.,}$$

which has $N = 2$ supersymmetry. This consists of two terms, the term involving K, the D-term, and the F-term W, the superpotential, which is a holomorphic function of X_i. If we represent the operators corresponding to $D^{\pm}$ and $\bar{D}^{\pm}$ acting on the Hilbert space as

$$D^{\pm} \to Q_{\mathrm{L}}^{\pm}, \qquad \bar{D}^{\pm} \to Q_{\mathrm{R}}^{\pm},$$

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* The construction of topological–anti-topological fusion can be extended to arbitrary genus. We take a surface consisting of alternating regions, supporting the topological theory and its conjugate respectively, which are separated by infinitely long tubes. However, once we know η, g and C on the sphere for arbitrary β we can write down the corresponding answer at higher genus using simple ideas of sewing.

we can write the two supersymmetry operators $Q^{\pm}$ discussed above as

$$Q^{\pm} = Q_L^{\pm} + Q_R^{\pm}.$$

When W is quasi-homogeneous the IR fixed-point of the LG theory is believed to describe an $N = 2$ superconformal theory [2,3]. Here we will not make these restrictions, and our discussion is equally valid for the critical as well as the non-critical (massive) theory. It is easy to find the chiral ring $\mathscr{R}$ for LG models. In fact the chiral ring is generated by the X_i themselves. All we have to do then is find the relations in this ring, or put differently, which product of the X_i is Q^+-closed. These relations will come from the variations of the lagrangian, which are the equation of motion for this theory. Varying the action with respect to X_i and doing the θ^+, $\bar{\theta}^+$ integrals in the D-term gives us

$$\partial_i W(X_j) = -D^+ \bar{D}^+ \partial_i K(X_j, \bar{X}_j).$$

This means that the chiral fields containing $\partial_i W$ are Q^+-commutators and thus are trivial in the ring. Therefore we learn that the chiral ring of the theory is simply

$$\mathscr{R} = \mathbf{C}[X_i]/\partial_j W.$$

An important thing to note here is that W completely determines the ring (known as the singularity ring of W) and the D-term K does not affect the ring. In particular the D-term is trivial in the sense of both supercharges $Q^{\pm}$. This in particular implies that the variations of K is trivial in the sense of both the topological theory and its conjugate. Thus, it will not affect the metrics we defined above, and so the two metrics just depend on W. The metric η turns out to be particularly simple to compute and it is simply computable using the techniques of topological theories. A topological description of LG theories and the computation of its correlation functions is given in ref. [8]. Alternatively, one can apply dimensional reduction to supersymmetric quantum mechanics, and compute the metric η using properties of solutions to the supersymmetric Schrödinger equation [14]. The answer is

$$\eta_{ij} = \langle \phi_i \phi_j \rangle_{\text{top.}} = \langle i \mid j \rangle = \text{Res}_W[\phi_i \phi_j]$$

in terms of the Groothendieck residue symbol $\text{Res}_W[\cdot]$ defined by

$$\text{Res}_W[\phi] = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{\phi(X)\, dX^1 \wedge \ldots \wedge dX^n}{\partial_1 W \partial_2 W \ldots \partial_n W} = \sum_{dW=0} \phi(X) \mathfrak{H}^{-1}. \quad (2.10)$$

where $\mathfrak{H}$ denotes the hessian of W: $\mathfrak{H} = \det \partial_i \partial_j W$ and we are assuming that the critical points are non-degenerate in writing the last equality. Note that with the

above result, under field redefinition, the fields do not transform as scalars. This "anomalous" behaviour, is connected with the fermion zero-modes in the background gauge field which we discussed for the topological theory. This anomalous behaviour will also be explained geometrically below in the context of SQM.

The computation of g_{ij}, or equivalently the real-structure matrix M, turns out to be far more complicated and the study of its properties is the main focus of this paper. In order to study these we need to review some techniques developed for this purpose. This will be done in sect. 3.

3. General properties of the metric and its variation

The basic method to compute the metric g is to study its behaviour under perturbations which preserve the $N = 2$ supersymmetry. In this setup, using standard perturbation theory techniques, one can derive differential equations which are satisfied by g. The coefficients of these differential equations turn out to be completely fixed by the chiral ring $\mathscr{R}$ and thus, in the case of LG theories, they only depend on W as they should.

The idea that there should be a differential equation on the coupling-constant space is not surprising. In fact in the context of non-degenerate perturbation theory in quantum mechanics it is well known that there is a canonical curvature on the perturbation space, and the integral of this curvature leads to the Berry phase [20]. In the case of degenerate perturbation theory, this leads to non-abelian gauge fields on the coupling space [21]. Our case is generically of this type, with the added structure that we have a *holomorphic* parameter space and that gives us some additional structure.

We will discuss the idea in the general setting. Again, our considerations apply to conformal and non-conformal cases with equal validity. We consider changing the action by giving expectation value to chiral and anti-chiral operators. This means that we vary the action by

$$\Delta L = \int d^2\theta \, \Delta t_i \phi_i + \text{c.c.},$$

where t_i correspond to the (complex) couplings in the theory. As we change t_i the Ramond vacua change. In perturbation theory one usually defines the variation of the state to be orthogonal to itself (and to the other states with the same energy). It is however more convenient to first allow an arbitrary basis for the perturbation and introduce a connection in the space of vacua which projects out the components of the perturbed vacua which are not orthogonal to the vacuum states. Let us denote this covariant derivative by D_i. Its basic property is that

$$\langle \bar{b} \mid D_i \mid a \rangle = \langle \bar{b} \mid \partial_i - A_i \mid a \rangle = 0,$$

where a, b label the Ramond vacua in some unspecified basis and $\langle \bar{b} |$ denotes the state adjoint to $|b\rangle$. Put differently, we can define a gauge field A_i on the coupling constant space given by

$$A_{ia\bar{b}} = \langle \bar{b} | \partial_i | a \rangle. \qquad (3.1)$$

It is easy to see that under a coupling-constant-dependent change of basis for the vacua, the quantity A transforms as a gauge field. Similarly we can define $\bar{D}_i$ and A_i. Let $\mathscr{V}$ be the space (the vector bundle) of Ramond ground states over coupling-constant space on which g defines a hermitian metric. Then it is easy to see that g is covariantly constant with respect to the gauge field we just introduced. In fact, this is how we defined A. One obtains

$$D_i g_{a\bar{b}} = \partial_i g_{a\bar{b}} - A_{ia}^{\ c} g_{c\bar{b}} - A_{i\bar{b}}^{\ \bar{c}} g_{a\bar{c}} = 0 = \bar{D}_i g,$$

where

$$A_{ia}^{\ b} = A_{ia\bar{c}} g^{b\bar{c}}, \qquad A_{i\bar{b}}^{\ \bar{c}} = \left(A_{ib}^{\ c} \right)^*.$$

It is natural to compute the curvature of these connections. We find [*]

$$\left[D_i, D_j \right] = \left[\bar{D}_i, \bar{D}_j \right] = 0,$$

$$\left[D_i, \bar{D}_j \right] = - \left[C_i, \bar{C}_j \right]. \qquad (3.2)$$

Moreover one has

$$D_i C_j = D_j C_i, \qquad \bar{D}_i \bar{C}_j = \bar{D}_j \bar{C}_i, \qquad D_i \bar{C}_j = \bar{D}_j C_i = 0. \qquad (3.3)$$

where C_i and $\bar{C}_j$ are the matrices which represent the action of ϕ_i and ϕ_j on the vacuum states. Since in the topological phase we can moves fields around, it is clear in addition that

$$\left[C_i, C_j \right] = 0 = \left[\bar{C}_i, \bar{C}_j \right].$$

In the conformal limit this system of equations was derived and studied in the physics literature from many different view points [13,14], and gives rise to what is called "special geometry". In fact it has been shown in the context of Landau–Ginzburg models [14] that these very same equations remain valid even off the conformal point. The technique used there involves a careful study of the zero-en-

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[*] These equations are a natural generalization of equations studied by Hitchin corresponding to a reduction of self-dual Yang–Mills equations to two dimensions.

ergy solutions to the Schrödinger equation in the context of SQM. As we will see
these turn out to be quite general and apply to arbitrary $N = 2$ theories, regardless
of whether they come from LG theories and they can be easily derived from the
path-integral viewpoint of fusing the topological theory with the anti-topological
theory.

In the usual non-topological setup, one can derive incorrect "theorems" by a
naive treatment of supersymmetric Ward identities which would lead one to
believe that the metric is constant, independent of the coupling constants. The
"argument" goes as follows:

$$\frac{\partial}{\partial t^a} \langle \bar{k} \mid h \rangle \overset{?}{=} \int \mathrm{d}^2 z \, \mathrm{d}^2 \theta \, \langle \bar{k} \mid \phi_a \mid h \rangle = 0,$$

i.e. using the fact that the ground states are annihilated by both $Q^{\pm}$, the
Grassmann integral seems to kill the above term. However, this is incorrect. The
difficulty lies in ignoring contact terms. It fact it is shown in ref. [22] that in the
conformal case such terms are crucial in obtaining the correct Zamolodchikov
metric. In the critical $N = 2$ SCFT theories (corresponding to strings on Calabi–Yau
manifolds), the contact terms were found to be crucial in getting the correct
answer [23]. However, amazingly the topological theory allows us to be "naive"
about contact terms and ignore them and get the correct answer! This is precisely
because contact terms which are UV singularities have no invariant definitions in
the topological phase, as we can move fields around with no consequence for
correlation functions.

Before turning to the derivation of these equations let us describe their
interpretation. The first line in eq. (3.2) is telling us that the gauge connection is
unitary, and the second is telling us that its curvature is computable using the
commutators of the ring of the topological and anti-topological theory. Combined
with eqs. (3.3) one sees that we can introduce "improved" connections which are
actually flat, namely consider *

$$\nabla = \mathrm{d}t^i (D_i + C_i),$$

$$\bar{\nabla} = \mathrm{d}\bar{t}^i (\bar{D}_i + \bar{C}_i). \tag{3.4}$$

Then the new connection $\nabla + \bar{\nabla}$ is flat,

$$\nabla^2 = \bar{\nabla}^2 = \nabla\bar{\nabla} + \bar{\nabla}\nabla = 0. \tag{3.5}$$

$\nabla + \bar{\nabla}$ is the analog of the Gauss–Manin (GM) connection well known to mathe-
maticians [24] which in the physics language plays a role when we are dealing with

* One could as well consider the dual connection $\partial' = D - C$, $\bar{\partial}' = \bar{D} - \bar{C}$ which is also flat.

marginal instead of massive perturbations of conformal theory. Indeed, when the $N = 2$ theory is a LG theory which has a σ-model interpretation, it is the usual GM connection (see ref. [14] for details).

In order to prove eqs. (3.2) and (3.3) our strategy will be as follows: We will first show that it is possible to choose a holomorphic basis in which a, b run over chiral indices and with

$$A_{ij}^{\,k} = g^{k\bar{r}}A_{ij\bar{r}} = 0.$$

Once we show this (and similarly the conjugate version of it) the first line in eq. (3.2) will follow. Similarly, the fact that in this basis C_i is holomorphic implies that

$$\overline{D}_j C_i = \bar{\partial}_j C_i = 0,$$

which with its conjugate version will prove the second line of (3.3). For the other equations we will have to work harder.

Let us start by showing that in the chiral basis we can choose a holomorphic gauge, i.e. a gauge in which the antiholomorphic components of the gauge field are zero. As we shall see, the topological path-integral automatically picks this gauge. By definition of the gauge field we have to compute

$$A_{ij}^{\,k} = \eta^{kl}\langle l\,|\,\bar{\partial}_i\,|\,j\rangle. \tag{3.6}$$

The matrix element in the above equation can be conveniently represented by a path-integral: We represent the state $|j\rangle$ by a topological path-integral on a hemisphere with a long tube attached to it with the field ϕ_j inserted in it. This space (with the long tube attached) we call the right-hemisphere S_R. In order to find $\bar{\partial}_i\,|\,j\rangle$ all we have to do is to insert the operator

$$\int_{S_R} d^2z \; d^2\theta^+ \; \overline{\phi}_i = \int_{S_R} d^2z \; D^+\overline{D}^+\overline{\phi}_i$$

in the path integral. To compute the matrix element in (3.6) we can create the state $\langle l\,|$ by a topological path integral on a left-hemisphere S_L, with ϕ_l inserted, again with a long tube attached, and glue it to the path-integral on the right sphere S_R. This we will represent symbolically by

$$A_{ijl} = \left\langle \phi_l \left|\left(\int_{S_R} D^+\overline{D}^+\overline{\phi}_i \right) \phi_j \right.\right\rangle.$$

Since Q^+ is a symmetry of the topological theory and ϕ_j is closed under it, we can write this as

$$\left\langle \phi_l \left| Q^+ \left|\left(\int_{S_R} \overline{D}^+\overline{\phi}_i \right) \phi_j \right.\right.\right\rangle.$$

This vanishes because the topological theory on S_L produces a state which is annihilated by Q^+. Thus we have seen that the path integral in the chiral basis provides a holomorphic basis for the connection in which the anti-chiral components of the connection vanish. This concludes the first thing we wished to show.

Now we turn to harder parts of the derivation and show how the second line in eq. (3.2) can be derived. To do that we have to show that

$$\partial_j A_{ik}^l - \partial_i A_{jk}^l = \left[C_i, \overline{C}_j \right]_k^l, \tag{3.7}$$

where we have used that fact that in our basis the anti-holomorphic component of the connection vanishes, and thus there are no commutator terms on the left-hand side. In fact we know that even the second term on the left-hand side is identically zero, but we will keep this as it cancels some of the terms from the first term on the left-hand side and slightly simplifies our analysis.

Using the path-integral representation of the left-hand side of eq. (3.7) it is easy to see that, after some obvious cancellation between terms, we get a path integral on the sphere which symbolically can be represented by

$$\partial_j A_{ikl} - \partial_i A_{jkl} = \left\langle \phi_k \left(\int_{S_L} D^+ \overline{D}^+ \overline{\phi}_j \right) \middle| \left(\int_{S_R} D^- \overline{D}^- \phi_i \right) \phi_l \right\rangle$$

$$- \left\langle \phi_k \left(\int_{S_L} D^- \overline{D}^- \phi_i \right) \middle| \left(\int_{S_R} D^+ \overline{D}^+ \overline{\phi}_j \right) \phi_l \right\rangle \tag{3.8}$$

Now we will show that these two terms give $-\overline{C}_j C_i$ and $C_i \overline{C}_j$ respectively (up to terms which cancel between them). Let us concentrate on the first term

$$\left\langle \phi_k \left(\int_{S_L} D^+ \overline{D}^+ \overline{\phi}_j \right) \middle| \left(\int_{S_R} D^- \overline{D}^- \phi_i \right) \phi_l \right\rangle$$

Just as discussed before we can move D^+ to the right where it kills everything except for D^- acting on ϕ_i which converts that into ∂ (by using the fact that D^+ kills ϕ_i and using the (anti-)commutators of $N = 2$ algebra). Similarly we can move $\overline{D}^+$ to the right and again the net effect on the path-integral on S_R is to replace $\overline{D}^+$ with $\overline{\partial}$. So we are left with *

$$\left\langle \phi_k \left(\int_{S_L} \overline{\phi}_j \right) \middle| \left(\int_{S_R} \partial \overline{\partial} \phi_i \right) \phi_l \right\rangle.$$

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* In order to more D^+ and $\overline{D}^+$ we have used the fact that there are two topological charges: Q^+ and $\overline{Q}^+$.

Now we can do the integral of the field on the right hemisphere and get a contribution on the boundary circle C on the cylinder which separates the two regions $S_{L,R}$. We get

$$-\left\langle \phi_k\left(\int_{S_L}\bar{\phi}_j\right)\middle|\left(\oint_C \partial_n\phi_i\right)\phi_l\right\rangle,$$

where ∂_n denotes derivative in the normal direction to the circle C, i.e. in the infinitely-stretched direction of cylinder. We can replace

$$\partial_n\phi_i = [H, \phi_i].$$

Since $|\phi_l\rangle$ is the same as the vacuum state $|l\rangle$, it is killed by H and so we can write the above matrix element as

$$-\left\langle \phi_k\left(\int_{S_L}\bar{\phi}_j\right)\middle|\left(H\oint_C \phi_i\right)l\right\rangle$$

We will divide the integral on the left-hand side to two roughly equal parts each of which is infinitely stretched, the first part includes the field ϕ_k and contains the curved piece of S_L with roughly half the infinitely stretched cylinder, while the second part includes only the other half of the infinite cylinder of S_L. The integral on the part further on the left will not contribute to the above matrix element, because the state one gets propagates infinitely long on the second part of the space, and so the net effect is projection on ground state which is accomplished by the $\exp(-TH)$ for large T, and the final state we get on the circle C is thus killed by H in the above matrix element. We are thus left with the second part of the integral on the left which is on a very long cylinder. Let τ denote the long direction on this cylinder and let us take it to run from 0 to $T \gg 1$. Meanwhile the empty first part of the path integral will convert the path integral with the insertion of ϕ_k to an actual ground state given by $\langle k |$. So we are left with

$$-\langle k|\int d\tau \oint \bar{\phi}_j(\tau) H \oint \phi_i|l\rangle,$$

where we have written the integral on the cylinder as first running around the perimeter on the cylinder at a fixed time τ and then integrating over all τ. Since the H kills the ground state on the left, we can replace H with its commutator with $\oint\bar{\phi}_j(\tau)$ which gives us a $-\partial_\tau\oint\bar{\phi}_j$. Thus doing the integral over τ becomes easy and we get the contributions from the boundaries at $\tau = 0$, T. The contribution at $\tau = T$ is on the same circle as the one the operator $\oint\phi_i$ is inserted and is canceled by the same term from the second term of eq. (3.8). We are thus left with

$$-\langle k|\oint\bar{\phi}_j \exp(-TH)\oint\phi_i|l\rangle,$$

where we have to send $T \to \infty$. This has the effect of projecting the intermediate states to the ground states of the theory, and we recover the definition of the chiral ring matrices * and so we get

$$-\left(\overline{C}_j C_i\right)_{kl}$$

And similarly for the second term in eq. (3.8) we get the same as above with C_i and $\overline{C}_j$ exchanged places and with the opposite sign. We thus get the commutator on the right-hand side, thus completing the proof of the second line eq. (3.2). Using very similar techniques, which we hope the reader would be able to reproduce, one can verify the validity of the first line of eq. (3.3).

On closing this section let us note that in this holomorphic basis, we can write everything in terms of the metric g and the holomorphic chiral ring elements C_{ij}^k. Namely from the fact that g is covariantly constant and that the antiholomorphic component of the gauge field vanishes we have

$$A_{ik}^l = -g_{kj}\left(\partial_i g^{-1}\right)^{jl}.$$

Moreover, just from the definition of the basis we have

$$\left(\overline{C}_j\right)_k^l = \left(gC_j^\dagger g^{-1}\right)_k^l.$$

Putting everything together, the zero-curvature conditions (3.5) become differential equations for the metric g. We get

$$\bar{\partial}_i\left(g\partial_j g^{-1}\right) - \left[C_j, \, g(C_i)^\dagger g^{-1}\right] = 0, \qquad (3.9)$$

$$\partial_i C_j - \partial_j C_i + \left[g\left(\partial_i g^{-1}\right), C_j\right] - \left[g\left(\partial_j g^{-1}\right), C_i\right] = 0, \qquad (3.10)$$

all other conditions being either trivially satisfied, or consequences of these two together with known properties of the topological functions C_{jk}^h and η_{ij}.

As we shall see in more detail in subsequent sections these equations have "magical" properties, making them a natural generalization of the so-called *Special Geometry* which plays a key role in understanding the geometry of the moduli space of $N = 2$ conformal field theories (related to CY manifolds). One important miracle is already evident from this discussion: our non-linear differential equations are always in the form of a consistency requirement for a set of linear equations, i.e. they always admit a zero-curvature (Lax) representation. Instead of eq. (3.9) and (3.10), we can study the associated linear problem

$$\nabla \psi = \overline{\nabla} \psi = 0. \qquad (3.11)$$

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* We have taken the perimeter of the cylinder β to have unit length, otherwise the commutators will be accompanied with a factor of β^2.

In this abstract sense our equations are always solvable. As we shall see below, because of this Lax representation, for simple models the equations have a tendency to reproduce celebrated equations of mathematical physics. More surprisingly, generally speaking, to solvable models in the world-sheet sense, models which lead to infinitely many conserved currents and connected with factorizable S matrices, we find solvable (classical) systems for the dependence of ground state metric as a function of coupling-constant space (the "target" space). Moreover these equations tend to be of the sample type! (Quantum affine Toda theory as the world-sheet theory, and classical affine Toda theory of the same type (or its reductions) as the equations satisfied by the ground state metric!) This bizarre duality between world-sheet and target phenomena is reminiscent of what one finds in the case of critical $N = 2$ string theories [17].

Not all the solution to the above equations can be accepted as ground state metrics. There are other conditions to be satisfied. First of all, g should be a positive-definite hermitian matrix. Furthermore the metric should have all the symmetries of the problem and in particular in the LG case, it inherits all the (pseudo) symmetries of W. Moreover, as mentioned before we have the "reality constraint"

$$\eta^{-1} g \left(\eta^{-1} g \right)^{*} = \mathbb{1}.$$

There are some general properties of the metric which follows from the above equations. Take the trace of eq. (3.9) which gives us

$$\bar{\partial}_{\bar{i}} \partial_{j} \log \det g = 0,$$

i.e.

$$\det g = |f(t)|^{2} \quad \text{with} \quad f(t) \quad \text{holomorphic}.$$

In particular, we can find a holomorphic basis such that $\det g = 1$.

Another general property of g which should be consistent with our equations is that the metric should not depend on t^{0}, the coupling associated to the operator $\mathbb{1}$. Indeed, adding a constant to the lagrangian in chiral superspace does not change the model because the Grassman integration over superspace kills it. This is consistent with our equations. In fact, $C_{0} = \mathbb{1}$ and hence it commutes with everything. This simple remark has a very useful generalization. Sometimes the $N = 2$ theory has a (pseudo)-symmetry such that the space of vacua viewed as a representation of a subring $\mathscr{R}'$ of $\mathscr{R}$ generated by some ϕ_{i} decomposes into orthogonal representations. Then if in a given irreducible representation some non-trivial operator reduces to a multiple of unity, $\langle \bar{i} | j \rangle$ ($| i \rangle$ in the given representation) is (essentially) independent of the corresponding coupling.

At this point a natural question arises. Are these conditions sufficient to uniquely determine the metric or not? A priori, one would think that the above differential equation should be supplemented with boundary conditions in order to predict g. However, the analogy with the geometrical case (the variational Schottky problem) which is the geometrical interpretation of these in the context of marginal operators of conformal theories suggests that *generically* the above conditions already lead to a very overdetermined problem. Then just *one* boundary condition would give a solution satisfying all the requirements simultaneously. In this sense, the equations *predict* their own boundary conditions. Although we do not have a general proof of this statement *, below we shall show in many explicit models how the equations are strong enough to predict their highly non-trivial boundary conditions. In particular the OPE of conformal theories are predicted by consistency alone and they agree with the results previously obtained. As a by-product we shall also reproduce some deep mathematical results in the context of isomonodromic deformation theory (together with some generalizations).

4. RG flow, Zamolodchikov metric and c-function

In the context of perturbing quantum field theories one usually defines a one parameter family of quantum field theories related to each other by a change in scale. This defines a "flow" on the space of quantum field theories which is known as the renormalization group flow. Conformal theories are precisely the fixed points of this flow. For a given theory characterized by a point on the coupling constant space, one defines an UV (ultra-violet) and an IR (infra-red) fixed point defined as the short-distance, and long-distance limits of that given theory. Generically one starts with a theory which is obtained by relevant perturbations of conformal theory so that the UV fixed point is the theory we started with. The infrared fixed points are generically infinitely massive theories; however, if one chooses the perturbation of the quantum field theory judiciously, one can end up with another conformal field theory as an IR fixed point. The study of this kind of situation in 2-dimensional quantum field theories was given a big boost by the work of Zamolodchikov [25]. In that work a function was defined on the parameter space, the "c-function", which has the beautiful property of decreasing along the renormalization group flows, and whose critical points correspond to fixed points of RG flow, i.e. CFTs. Moreover Zamolodchikov defined a metric on the parameter space, using the two-point function of perturbing operators on the plane at a fixed distance.

As we have been studying perturbed $N = 2$ SCFTs in this paper, it is natural to ask how the RG flows look in this context. Some aspects of this has been studied

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* Even in the geometrical (conformal) case there is no general proof.

[26]. We will focus on the case of Landau–Ginzburg theories. The non-renormalization theorems of $N = 2$ theory come to our aid in the study of RG flows. These state that the superpotential W of $N = 2$ theories does not get corrected perturbatively. We will take this to be true non-perturbatively. In fact it appears that the non-perturbative non-renormalization theorem can be proven along the following line of argument. In flat space, where the spin-connection vanishes, the functional measure for the LG model is identical to that of the topological theory with a certain gauge-fixing term. The topological theory is not renormalized just because there are no local degrees of freedom [*]. Then its quantum effective action Γ should have the form

$$\Gamma = \Gamma_{\mathrm{cl}} + s\Delta\Gamma,$$

where s is the topological Slavnov operator. In the LG context this equation is interpreted as the $N = 2$ non-renormalization theorem. Indeed, the usual super-diagrammatic proof of this result [27] consists of a loop expansion of this equation. For other viewpoints, see sect. 4 of ref. [5]. Anyhow, some evidence for the validity of this kind of conjecture is the correctness of some of its consequences [2,3]. To be more precise, even though we take W not to change, the action will pick up the supervolume factor. If we take $z \to \lambda z$, $\theta \to \lambda^{-1/2}\theta$ we get

$$\int \mathrm{d}^2 z \; \mathrm{d}^2\theta \; W(X_i) \to \lambda \int \mathrm{d}^2 z \; \mathrm{d}^2\theta \; W(X_i).$$

This overall factor of λ can be gotten rid of in the leading terms of W (the highest degrees of fields) by a field redefinition with the effect of rescaling the rest of the couplings. In this way the rescaling of W by λ *generates* a flow. The IR limit is when $\lambda \to \infty$ and UV is obtained when $\lambda \to 0$. This we take as our working hypothesis as to what the RG flow is for us. Needless to say the D-terms are expected to get corrected in a much more severe way, but as we have seen in previous sections, luckily our computations for the ground state metrics are independent of that [**].

Now it is natural to see whether we can compute the form of the metric g in the UV and IR limits. These will also be a kind of "boundary condition" for the differential equations we have discussed, eqs. (3.9), (3.10). In the UV, as $\lambda \to 0$, we start from a conformal theory. In other words, in this limit we can take W to be quasi-homogeneous by rescaling of the fields. For $N = 2$ LG theories, this problem has been solved in ref. [14] which shows how the differential equations (3.9) and (3.10) and other basic properties of the metric discussed above lead to the answer.

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[*] The topological Green functions are computable. From their explicit form the non-renormalization is obvious.

[**] In the formulation of topological–anti-topological fusion of sect. 2, the perimeter β of the intermediate cylinder can be identified with λ.

It turns out that the answer can be written in a simple closed form that we will now discuss. Let $\phi_i(X_k)$ be a basis for the chiral primary fields of the LG theory. Then the metric can be given by finite-dimensional integrals over the variables X_i. For instance, if $\phi_i(X_k)$ is *relevant* (i.e. $q(\phi_i) < 1$) one has the very compact formula

$$g_{ij} = \langle \phi_i \bar{\phi}_j \rangle = \int \prod \mathrm{d}X_l \, \mathrm{d}\bar{X}_l \, \phi_i(X_k) \bar{\phi}_j(\bar{X}_k) \exp(W - \bar{W}). \qquad (4.1)$$

We have to be a little careful with this integral. For one thing for large values of fields it is typically a highly oscillatory integral. Of course our intuition says that these highly oscillatory parts should not contribute appreciably to the integral. This intuition can be made more precise by defining the above integral using surfaces of constant W. Alternatively, we can define the above integrals by demanding Riemann bilinear identities to hold: Let $B^{\pm} \subset \mathbb{C}^n$ denote the asymptotic regions in $\mathbb{C}^n$ where Re $W \to \pm \infty$. Here n denotes the number of variables. Let $\gamma_i^{\pm}$ label a basis of equivalence class of the n-chains in $\mathbb{C}^n$, whose boundary $\partial\gamma_i^{\pm} \subset B^{\pm}$, in other words they define a basis of the relative homology classes

$$\gamma_i^{\pm} \in \mathrm{H}_n(\mathbb{C}^n, B^{\pm}).$$

Moreover, let C_{ij} denote the intersection matrix between these cycles

$$C_{ij} = \gamma_i^{+} \cap \gamma_j^{-}.$$

Then applying the idea of the Riemann bilinear identity to the above integral we come up with the following result [*]:

$$g_{ij}\big|_{\text{relev.}} = \int_{\gamma_l^-} \phi_i(X_k) \, \exp(W) C^{lm} \int_{\gamma_m^+} \bar{\phi}_j(\bar{X}_k) \, \exp(-\bar{W}). \qquad (4.2)$$

The residue can also be described in this way. One has

$$\eta_{ij} = \int_{\gamma_l^-} \phi_i(X_k) \, \exp(W) C^{lm} \int_{\gamma_m^+} \phi_j(X_k) \, \exp(-W)$$

$$\text{for} \quad q(\phi_i) + q(\phi_j) \leqslant c/3$$

$$= 0 \quad \text{otherwise}. \qquad (4.3)$$

Note that the above integrals are well defined by the choice of the cycles on which we integrate them. In appendix A we derive these formulas, by showing why they provide solutions to eqs. (3.9) and (3.10). It is important to notice that eq. (4.2) is

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[*] Technically speaking, the symbols $\gamma_k^{\pm}$ represent locally-constant families of homology cycles rather than given cycles. This remark applies throughout the paper.

valid *only* at the conformal point where W is quasi-homogeneous. For more general W the story is far more complicated and cannot be described by such a simple integral. However, using SQM, even in those cases one can write similar expressions but one has to replace the fields in the above by the exact solutions to Schrodinger equation. This will be discussed in sect. 5. Instead eq. (4.3) is valid for *arbitrary* W's. More precisely, the general formula is

$$\eta_{ij} = \int_{\gamma_l^-} \phi_i(X_k) \, \exp(W) C^{lm} \int_{\gamma_m^+} \phi_j(X_k) \, \exp(-W). \qquad (4.4)$$

However here there is a subtlety. Whereas both sides of these equations transform the same way under a change of basis in $\mathscr{R}$, they transform differently under a change of the representative of BRST-classes

$$\phi_i(X_k) \to \phi_i(X_k) + \sum_l h_i^l(X_k) \, \partial_l W.$$

Then eq. (4.4) holds only for *special* representatives. The special operators ϕ_i are those associated to the special coordinates of TFT [10,28]. These coordinates are discussed in appendix C. There eq. (4.4) is proven. With generic representatives, the r.h.s. of eq. (4.4) would differ from η because of spurious mixings of the operators of charge q with those of charge $q - k$ (k a positive integer). Modifying the definition as in eq. (4.3) we disentangle this mixing. Then eq. (4.3) holds for all choices of the operators ϕ_i. See also ref. [29].

In the case that $W = 0$ defines a Calabi–Yau manifold in weighted projective space these results are all consistent with what is known as special geometry. In fact the integral representation of the metric (4.1) is very reminiscent of the period integrals of special geometry, but now in the context of general LG theory. We will see more connections below.

We can vary W by marginal operators, and remain in the class of conformal theories. Then it is natural to ask what is the relation between the g we have computed, and Zamolodchikov's definition, which gives a natural metric on moduli space of conformal theory. As we have discussed spectral flow relates chiral operators to the ground states, and so the metrics that we have computed must be related to the metric that Zamolodchikov defines. This relation is quite precise in the case that the perturbations are marginal and preserve the conformal properties of the theory. In particular using conformal Ward identities it is easy to show that what we have computed in this case is

$$g_{ij} = \left\langle \phi_i(0) \overline{\phi}_j(1) \right\rangle$$

evaluated on the sphere. This is not precisely the metric that Zamolodchikov defines for two reason: The important reason is that ϕ_i and $\overline{\phi}_j$ are *not* themselves

the perturbing operators, but rather $\int d^2\theta \phi_i$ and the complex conjugate of it are the perturbing operators. That is easy to implement, as again the superconformal Ward identities relate these to the above computation by multiplication by a factor of q_i^2 where q_i is the U(1) charge of the field ϕ_i (which we assume to have equal left and right charge – otherwise we would get $q_{i,L}q_{i,R}$). Note in particular that the identity operator gets projected out once integration over Grassmann coordinates is performed. For marginal operators the charges are all 1 and so this does not affect the above metric at all. The other point to bear in mind is that Zamolodchikov's definition is the expectation value of two operators, and we need to choose a correct normalization for the vacuum by dividing out by $\langle 0 | 0 \rangle$. So for conformal deformations we see that the Zamolodchikov metric G is related to our g simply by (the index 0 labels the identity operator)

$$G_{i\bar{j}} = g_{i\bar{j}}/g_{0\bar{0}}, \qquad (4.5)$$

where $i, \bar{j}$ run over the marginal directions.

It turns out that quite generally, one can show that the metric G for the metric on moduli space of $N = 2$ SCFTs is Kähler. This is in fact true for arbitrary $N = 2$ SCFTs and not just LG theories. In the conformal limit we have an extra U(1) symmetry, with respect to which all chiral primary states, except for the identity operator which is neutral, have positive charge. Then by charge conservation we have

$$g_{0\bar{k}} = g_{k\bar{0}} = 0 \quad \text{for} \quad k \neq 0,$$

$$\left(gC_i^\dagger g^{-1} \right)_k{}^0 = 0 \quad \text{for} \quad k \neq 0.$$

Let the indices i, j correspond to *marginal* perturbations, i.e. chiral primary fields of charge $q = 1$. Then from (3.9) we find

$$-\partial_{\bar{j}}\partial_i \log\langle 0 | 0 \rangle = \left[\partial_{\bar{j}} \left(g\partial_i g^{-1} \right) \right]_0{}^0 = (C_i)_0{}^k g_{k\bar{l}} C_{\bar{j}\bar{0}}^{*\bar{l}} g^{\bar{0}0} = g_{i\bar{j}}/g_{0\bar{0}} = G_{i\bar{j}},$$

where we used that $C_{i0}^k = C_{0i}^k = \delta_i^k$. Let $|\rho\rangle$ be the Ramond state of maximal charge dual to $|0\rangle$ with respect to the pairing $\eta_{\bar{j}k}$. Using eq. (2.9) we see that

$$\partial_{\bar{j}}\partial_i \log\langle \rho | \rho \rangle = -\partial_{\bar{j}}\partial_i \log\langle 0 | 0 \rangle.$$

So we get

$$G_{i\bar{j}} = \partial_i\partial_{\bar{j}} \log\langle \rho | \rho \rangle.$$

Thus we find that in the $N = 2$ case the Zamolodchikov metric (along the marginal directions) is Kähler with potential $K = \log\langle \rho | \rho \rangle$. This is a result due to Periwal and Strominger [30].

In the case of LG theories the integral representation of the metric (4.1) implies that we can write the Kähler potential as an integral

$$g_{0\bar{0}} = e^{-K} = \int \prod dX_i \, d\bar{X}_i \, \exp(W - \bar{W}).$$ (4.6)

In the case that W is of a form to be directly related to Calabi–Yau manifolds [31], i.e. with integer $\hat{c}$ and the number of variables $n = \hat{c} + 2$, then doing the integral above with respect to one of the variables (with a suitable change of variables) results in $\delta(W)$ in the integrand. It was observed by Greene that if one continues this formal integration one more step one ends up with $\int \omega \wedge \bar{\omega}$, where ω is the representative of the $(\hat{c}, 0)$-form on the manifold $W = 0$ defined in weighted projective space. So in this case we have

$$e^{-K} = \int \omega \wedge \bar{\omega},$$

which is a well-known result due to Tian [32]. One should emphasize that (4.6) is valid *regardless* of a Calabi-Yau interpretation of the LG theory. From the other equations in (3.9) we get additional constraints on this Kähler potential. It is easy to show that they reproduce the conditions valid for a variation of Hodge structure on the algebraic hypersurface $W = 0$ in weighted projective space, which may or may not be a CY manifolds. This was discussed at length in ref. [14].

All we have said so far is only valid at the conformal point, i.e. the limit where $\lambda \to 0$. Now we wish to discuss what is the form of the metric in the IR, i.e. when $\lambda \to \infty$. In such a case the critical points of W, i.e. $dW = 0$ which are the minima of energy, become infinitely separated from each other, and to leading order do not see each other. In other words to leading order the metric becomes diagonal in basis of chiral fields corresponding to excitations near the vacua. So we can base our physical vacuum by shifting fields to correspond to each one of the vacua we wish to study. If the critical point is not a simple zero of dW, then the field configurations near that critical point will still describe a (massless) conformal theory and what we said above about the computation of g remains valid for this part of the metric. However at the critical points of W for which dW has a simple zero, we end up with a massive theory. In the limit that $\lambda \to \infty$ the mass goes to infinity proportional to λ. Again in this case the metric is trivial to compute using free massive field theory.

These vacua will not completely decouple from each other, in the sense that there are instanton corrections which tunnel from one vacuum to another and provide off-diagonal elements for the metric which are exponentially small as $\lambda \to \infty$. In order to describe this situation, let us take the case where all the critical points of W are simple, i.e. that they all give rise to massive theories. It is

convenient to use the "point" basis for $\mathcal{R}$. Two holomorphic functions $f_1(X)$ and $f_2(X)$ represent the same element in $\mathcal{R}$ iff

$$f_1(X_k) = f_2(X_k) \quad \forall X_k,$$

where X_k are the critical points; this follows from the residue formula (2.10). So we can label each equivalence class by its values at the critical points. We denote by ϕ_k the class such that we get 1 at X_k and 0 at X_h $(h \neq k)$. In this basis, as $\lambda \to \infty$ we get

$$g_{k\bar{h}} \simeq \frac{\delta_{kh}}{|\mathfrak{H}(X_k)|},$$

where $\mathfrak{H}$ denotes the hessian of λW evaluated at the critical point. In the case of one field, one can also give a general form for the first correction to this classical limit. One finds that if there is a primitive soliton connecting the two vacua, the condition for the existence of which has been studied in ref. [33], one obtains a correction of the form

$$\frac{g_{k\bar{h}}}{\left(g_{k\bar{k}}g_{h\bar{h}}\right)^{1/2}} \simeq \alpha_{k\bar{h}}(4\pi z_{kh})^{-1/2} \exp[-2z_{kh}] \quad k \neq h, \qquad (4.7)$$

where

$$z_{kh} = \lambda |W(X_k) - W(X_h)|,$$

and $\alpha_{k\bar{h}}$ is a phase factor. Here $2z_{kh}$ is equal to the mass of the soliton connecting the two vacua. This result is discussed in appendix B.

Having discussed the two limiting cases of UV and IR, it is natural to ask what can be said in general about the properties of the flow in between. In particular, does there exist a natural "c-function" for us? What is the relation of Zamolodchikov's metric to our ground state metric g away from the conformal point? We will now address these questions in turn.

The central charge of the SCFT is proportional to the maximum charge in the ring of chiral primary fields [2,4]. Indeed

$$c/3 = \hat{c} = q_{\max}.$$

In the Ramond sector the charges are shifted by $-\hat{c}/2$, and they are symmetrically distributed between $-\hat{c}/2$ to $\hat{c}/2$. It is natural to try to define this charge, even off criticality, and view it as a "c-function". We should in fact be able to do more: The charges q_k of the chiral primary fields are all on the same footing from an abstract point of view. So we must be able as well to define q-functions corresponding to the charges of all these operators. In fact there is a theorem in *Singularity Theory*

[34] stating that all these functions would satisfy a "c-theorem". More precisely, suppose we perturb a singularity (which corresponds to a given $N = 2$ critical theory) in order to get a simpler singularity (which is interpreted as the corresponding IR fixed point). Let Δ denote the number of chiral primary fields. Order the charges of the chiral primary operators in a non-decreasing sequence

$$0 = q_1 \leqslant q_2 \leqslant \ldots \leqslant q_\Delta = c/3,$$

then one has

$$q_k \leqslant q_k' + \tfrac{1}{6}(c - c') \leqslant q_{k+\delta}, \tag{4.8}$$

where the primed quantities refer to the IR fixed point and $\delta = \Delta - \Delta'$ is the difference of Witten indices between the UV and the IR theories.

Motivated by these observations one naturally looks for a definition of a "charge" matrix. Note that by a change of phase of the Grassmann variables, we see that the phase of λ is not a physical degree of freedom and all quantities depend on $|\lambda|$. Let

$$\lambda = e^\tau.$$

In other words, the metric and all the other physical quantities depend on τ through its real part $\tau + \bar{\tau}$. Now we are to define a notion of a charge matrix, using the only quantity available to us, namely the ground state metric g. Near a conformal point g becomes diagonal in a basis of ground state vacua with definite charge. One can easily see using the Ward identities that, in the basis defined by our path integral, as $\lambda \to 0$ g behaves as [*]

$$g_{ii} \sim (\lambda \bar{\lambda})^{-q_i - n/2},$$

where here q_i denotes the charge of the ith Ramond vacuum. We thus see that near the critical point the matrix

$$g\partial_\tau g^{-1} - n/2$$

is a diagonal matrix with eigenvalue equal to the charges of the Ramond vacua. In particular the maximum eigenvalue of this matrix reproduces $c/6 = \hat{c}/2$ near criticality. So let us define the Ramond charge matrix q as

$$q = g\partial_\tau g^{-1} - n/2, \tag{4.9}$$

i.e. the "gauge connection" in the direction of flow minus the "anomalous" part.

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[*] The shift of q_i by $n/2$ is related to the behaviour of η under a rescaling of W (which is a kind of "anomaly" arising from the Fermi zero modes). Indeed, from eqs. (2.9) and (2.10) we have $\det[g] = |\det[\eta]| \sim |\lambda|^{-n\Delta}$.

This q has a simple field-theoretical interpretation. Since nothing depends on the D-terms, we fix them to be the "standard" ones

$$K = \sum_i \overline{X}_i X_i.$$

If W is quasi-homogeneous we have a conserved U(1) current J_μ, and we must have

$$q_{h\overline{k}} = \langle \overline{k} \,|\oint J_0(\sigma) \, \mathrm{d}\sigma \,|\, h \rangle.$$

Noether's theorem gives the following expression [*] for J_μ:

$$J_\mu = J_\mu^5 + U \,|\, A_\mu,$$

where

$$J_\mu^5 = -\tfrac{1}{2} \sum_i \overline{\psi}_i \gamma_\mu \gamma_5 \psi_i,$$

$$U = \sum_i q_i \overline{X}_i X_i.$$

Since $U \,|\, A_\mu$ is a Q-commutator, we have

$$q_{h\overline{k}} = \langle \overline{k} \,|\oint J_0 \,|\, h \rangle = \langle \overline{k} \,|\oint J_0^5 \,|\, h \rangle.$$

Consider now a generic superpotential W. The current J_μ^5 is still partially conserved. Indeed, it is only softly broken by the superpotential W

$$-\partial_\mu J_\mu^5 = i \int \mathrm{d}^2\theta^+ W - i \int \mathrm{d}^2\theta^- \overline{W}. \qquad (4.10)$$

Hence it makes sense to consider its matrix elements. Then the natural definition of the off-critical charge is

$$q_{h\overline{k}} = \langle \overline{k} \,|\oint J_0^5 \,|\, h \rangle.$$

This definition agrees with the previous one, eq. (4.9). To see this we compare (4.10) with the path-integral definition of the connection. In our context, eq. (4.10) should be modified. Indeed, in order to produce the correct vacuum state we have

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[*] $U \,|\, A_\mu$ means the vector component of the superfield U.

introduced a background gauge field in the right hemisphere. Then the axial current develops an anomaly

$$-\partial_\mu J_\mu^5 = i\int d^2\theta^+ W - i\int d^2\theta^- \overline{W} + (n/2\pi)F.$$

Consider the connection along the flow

$$A_{\tau kh} \equiv \langle h | \partial_\tau | k \rangle = \langle h | (\partial_\tau - \partial_{\bar\tau}) | k \rangle.$$

It has the following functional representation:

$$i\langle \phi_h | \left(\int_{S_R} D^+ \overline{D}^+ W - \int_{S_R} D^- \overline{D}^- \overline{W} \right) \phi_k \rangle$$

$$= -\langle \phi_h | \left(\int_{S_R} \left[\partial_\mu J_\mu^5 + (n/2\pi)F \right] \right) \phi_k \rangle$$

$$= -\langle \phi_h | \oint J_0^5 | \phi_k \rangle - \tfrac{1}{2}n\langle \phi_h | \phi_k \rangle,$$

which shows that the two definitions agree. This also guarantees the "gauge independence" of the eigenvalues of q, which is not manifest from eq. (4.9). Under a "gauge transformation" the variation of the anomalous term compensates the change in the connection. From the QFT viewpoint it is manifest that the spectrum of q is real and symmetric about zero. This follows most clearly in a basis where $\eta = \eta^* = \eta^{-1}$. Then from eq. (2.9) we see that

$$q\eta = -\eta q.$$

Now we can show that the criticality of q as a function of couplings occurs only at the conformal points. This is an easy consequence of eq. (3.9), namely we have *

$$\bar\partial_i q = \left[C_\tau, gC_i^\dagger g^{-1} \right],$$

and at the conformal point the matrix C_τ is represented by multiplication by W, and since at the conformal point W is quasi-homogeneous, it follows that W itself is in the ideal generated by dW and thus is trivial in $\mathscr{R}$. Therefore $C_\tau = 0$ precisely at the conformal point and thus from the above equation we see that q is critical precisely at these points. This is also true the other way around, namely, $C_\tau = 0$ implies W is quasi-homogeneous [35]. This is the algebraic characterization of a fixed point, in the sense that when this happens the chiral ring has the properties

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* Because of reality of the eigenvalues it is enough to check stationarity with respect to the couplings $\bar t_i$.

prescribed for a critical point. Whether it is actually a fixed point is a more tricky question depending, of course, on the D-term too.

At criticality eq. (3.10) reduces to

$$[C_i, q] = C_i,$$

which merely states that only perturbations by operators of charge 1 are compatible with conformal invariance.

From this definition of the q-function it is not obvious that this quantity satisfies a "c-theorem". This should be globally true, in the sense that the inequalities (4.8) between the eigenvalues at the UV and IR points hold true. What is not manifest, is that pointwise along the "RG-trajectory" the derivative of these quantities has a definite sign. However, experience with concrete models suggests this is also true. Moreover, using the connection with Special Geometry it is easy to show that $\dot{c}$ is non-positive near a critical point. So, at least our version of the "c-theorem" holds in perturbation theory.

There is another way of getting the q-function which is more convenient since it holds in an arbitrary basis (provided the operators ϕ_k do not depend explicitly on the t's) without need of a compensating "anomalous" term. Consider the matrix

$$Q_k{}^h = G_{k\bar{l}}\partial_\tau(G^{-1})^{\bar{l}h},$$

where G is the above normalized metric. It is easy to see that near the critical point this definition of charge Q gives the list of the charge of chiral primary fields and in particular the range of the eigenvalues goes from 0 to $c/3 = \hat{c}$. Three times the maximal eigenvalue is then a candidate c-function. Obviously, the two definitions agree. We will refer to this function as *algebraic c-function*. It would be interesting to see what is the precise relationship of this c-function with that of Zamolodchikov.

Now we turn to the question of the relation between the Zamolodchikov metric off criticality with the ground state metric g. If we wished to write the Zamolodchikov metric for both marginal and relevant perturbations, *at the conformal point* all we have to do is to multiple G by factors of charge mentioned above. It is now clear that we cannot expect a simple relation between our metric g and Zamolodchikov's metric G off-criticality, because we already see that even near criticality we have to know the charges of fields in order to relate the two, and the notion of $U(1)$ charges of fields is well defined only at criticality. It is natural to suspect that given the off-critical definition of charge discussed above there might be a way to define a natural metric which is related to Zamolodchikov's definition. Even though there are some obvious guesses, we leave a carefully study of this for the future.

5. Reduction to SQM

There are other useful points of view about the ground-state metric. In ref. [5] it was shown that g can be computed by dimensional reduction to one dimension (i.e. in Supersymmetric Quantum Mechanics). Roughly, this follows from the fact that one can find a susy (but not Lorentz) invariant D-term which suppresses all the non-zero modes in the Fourier expansion of the fields. Thus, independence from the dimensions is a special instance of independence from the Kähler potential. Although the computations can be done directly in 2 dimensions, the reduction to SQM is useful for two reasons: first of all, here one has an explicit construction for the isomorphism of primary fields and states in the Ramond sector in terms of the wave functions of the SQM vacua. This also naturally encodes in a geometric way the "anomalous" transformation under field redefinitions, which as we mentioned is related to the violation of fermion numbers in the topological description of the theory. The second reason is that we can give a general solution to the linear problem (3.11) in terms of the vacuum wave functions. This also turns out to be very closely related to the generalization of special geometry in the context of massive theories. As customarily, we identify SQM wave functions with differential forms via

$$\Phi_{i_1 i_2 \ldots i_r \bar{k}_1 \bar{k}_2 \ldots \bar{k}_s}(X_j)\psi^{i_1}\ldots\psi^{i_r}\overline{\psi}^{\bar{k}_1}\ldots\overline{\psi}^{\bar{k}_s}|0\rangle$$

$$\to \Phi_{i_1 i_2 \ldots i_r \bar{k}_1 \bar{k}_2 \ldots \bar{k}_s}(X_j)\, dX^{i_1}\wedge\ldots\wedge dX^{i_r}\wedge d\overline{X}^{\bar{k}_1}\wedge\ldots d\overline{X}^{\bar{k}_s}.$$

Then in the Schrödinger representation, Q_R^+ is represented as

$$Q_R^+ = \bar{\partial} + dW \wedge$$

and Q_L^- is represented as

$$Q_L^- = \partial + d\overline{W} \wedge .$$

The isomorphism between the realizations of $\overline{W}$-cohomology on fields and states becomes

$$\phi_k \to \frac{1}{(-2\pi)^{n/2}}\phi_k\, dX^1\wedge\ldots\wedge dX^n + Q_R^+\xi_k.$$

Note that this isomorphism takes into account the topological violation of fermions number mentioned before. In fact from the path-integral description of sect. 2 it should be clear that once we see why the identity operator can be represented cohomologically by $dX^1\wedge\ldots\wedge dX^n$ the above follows, and that representation of the identity operator can be shown by taking a very tiny hemisphere, represented

by a little disc and perform the topological path integral. In the language of SQM it is manifest that under a field redefinition the Ramond state representing ϕ_k should transform as a $(n,))$-form rather than as scalar. This is the origin of the "anomalous" jacobian. Clearly,

$$Q = Q_L^+ + Q_R^- \equiv \exp\left[-W(X) - \overline{W}(\overline{X})\right] \, d \, \exp\left[W(X) + \overline{W}(\overline{X})\right],$$

where d is the exterior differential. Since the vacuum wave-forms ω_k are annihilated by Q and its adjoint $Q^\dagger$, the modified forms

$$\hat{\omega}_k \equiv \exp\left[W(X) + \overline{W}(\overline{X})\right]\omega_k,$$

$$\tilde{\omega}_k \equiv \exp\left[-W(X) - \overline{W}(\overline{X})\right] * \omega_k$$

are d-closed. They represent some kind of cohomology of the d-operator. Obviously this cannot be the usual deRham one, since for $\mathbb{C}^n$ it is trivial. In fact, these forms are representative of *relative* deRham classes. For $\tilde{\omega}_k$ the relevant cohomology is $H^n(\mathbb{C}^n, B)$, where $B \subset \mathbb{C}^n$ is the region where Re W is greater than a certain (large) value. The $\hat{\omega}_k$ correspond to the dual cohomology space. This dual space * can be identified with (equivalence classes of) n-chains γ_i^+ such that on $\partial\gamma_i^+$ we have Re $W = +\infty$. We put

$$\Pi_k^j = \int_{\gamma_j^+} \tilde{\omega}_k. \tag{5.1}$$

One checks that Π_k^j is finite and $\det[\Pi] \neq 0$. From ref. [14] one sees that there exists $\lambda_{i,k}$ such that

$$D_i\omega_k = (\partial + d\overline{W} \wedge)\lambda_{i,k},$$

where

$$(\bar{\partial} + dW \wedge)\lambda_{i,k} = \partial_i W\omega_k - (C_i)_k^h\omega_h.$$

Then one gets

$$D_i\Pi_k^j = -(C_i)_k^h\Pi_h^j, \qquad \overline{D}_a\Pi_k^j = -(\overline{C}_i)_k^h\Pi_h^j,$$

that is

$$\nabla\Pi = \overline{\nabla}\Pi = 0. \tag{5.2}$$

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* This dual space can be viewed as providing an *integral* basis for the vacua.

The matrix Π gives the general solution to the linear problem (3.11). These remarks give a simple description of the geometry of the bundle over the parameter space discussed in sect. 3. Indeed, we see that the vacuum wave-forms, after projection into the relevant relative homology, represent sections of the bundle discussed there.

The real-structure matrix M_k^j has a simple meaning in SQM. The Schrödinger equation is real, and hence the complex conjugate of a vacuum wave-form ω_k is again a vacuum wave-form and should be a linear combination of the ω_h. If the ω_k corresponds to the basis ϕ_k, we have

$$(\omega_k)^* = M_{\bar k}^h \omega_h,$$

from which the reality constraint is obvious. In particular, we have

$$\Pi^* = M\Pi \quad \Rightarrow \quad M = \Pi^* \Pi^{-1}, \tag{5.3}$$

which gives an alternative way of computing the metric from the solution of the linear problem.

In SQM, eq. (2.8) follows from the definition of the ground-state metric

$$\langle \bar k \mid h \rangle = \int * \omega_k^* \wedge \omega_h, \tag{5.4}$$

and the cohomological identity

$$\int * \omega_k \wedge \omega_h = \mathrm{Res}_W[\phi_k \phi_h] \equiv \eta_{kh}, \tag{5.5}$$

which is a consequence of the Bochner–Martinelli theorem (see the appendix of ref. [14] for details). In analogy with (5.1) we write

$$\hat\Pi_k^j = \int_{\gamma_j^-} \hat\omega_k,$$

where γ_j^- are cycles with $\mathrm{Re}\, W = -\infty$ at the boundary. Using the fact that

$$D_i \hat\Pi = C_i \hat\Pi, \qquad \overline{D}_i \hat\Pi = \overline{C}_i \hat\Pi,$$

one can easily show, using the uniqueness of solutions to linear differential equations, that

$$\hat\Pi_k^j = \eta_{kh} \rho^{ji} (\Pi^{-1})_i^h,$$

where ρ^{ji} is some pairing * of the above cycles which is independent of the couplings t^i. Then (5.3) gives

$$\eta = \hat{\Pi}\rho\Pi^{\mathrm{T}}, \qquad g = \hat{\Pi}\rho\Pi^{\dagger},$$

which are a kind of Riemann bilinear identities for the integrals (5.4) and (5.5).

This SQM viewpoint is quite suggestive of the geometry of a variation of Hodge structure (*special geometry* in the physics languages). Indeed, the matrix Π is just the period map for the relative classes $\tilde{\omega}_k$. Note though, the similarities are somewhat misleading in that the period matrix which is holomorphic in the case of special geometry (or variation of Hodge structure) has the distinctive property of *not* being holomorphic in terms of couplings t_i. And even though we have an integral representation for the metrics in terms of solutions of Schrödinger equation, it is *not* possible to give a closed form answer for them as integrals of simple objects, as it was the case in the quasi-homogeneous (conformal) case discussed in the previous section. In this sense the problem is much more difficult to solve in the massive case. We have already mentioned that ∂ can be identified with the Gauss–Manin connection. In fact eq. (5.2) can be seen as the defining property of the GM connection in terms of periods. So, the structure arising out of $N = 2$ susy is a generalization of *special geometry*.

The SQM viewpoint is very useful from another view point, and that arises when one considers changes of variables. Indeed it turns out that one can do non-invertible field redefinitions and still be able to relate the metrics between the two models. That this is possible is essentially why the formal arguments in ref. [31] relating LG theories to geometry of CY can be justified – at least as far as the metrics on the moduli space is concerned [14]. Moreover this will also justify, to the extent of getting the same moduli metric, the more recent work on relating different LG theories with each other by non-invertible changes of variables [36]. It turns out that for many of the applications that we will consider this is a very important technique.

The simplest way to understand how it works is in the language of SQM. We will use a mathematical language as it is most convenient to describe it in that setting, where we sometimes refer to the nice properties of non-invertible changes of variables as "functoriality with respect to branched coverings". Let ω_k ($k = 1, \ldots, \Delta$) be the vacuum wave-forms for some superpotential $W(X)$. In this superpotential we make a substitution

$$X_i = f_i(Y_j),$$

---

* Just as in the conformal case, ρ^{ij} is simply the inverse of the intersection matrix $\gamma_i^+ \cap \gamma_j^-$. It is possible to show this by multiplying the integrand in eq. (5.4) by one represented by $\exp(W + \overline{W}) \exp(-W - \overline{W})$, and using the Riemann bilinear identity.

where the map f is holomorphic but not globally invertible (otherwise we would get just an irrelevant field redefinition). Then consider the new superpotential

$$W_f(Y_j) = W(f_i(Y_j)) \equiv f^*W.$$

For the supercharges one has

$$Q_{R,f}^+ = \bar{\partial} + dW_f \wedge \equiv f^*Q_R^+,$$

$$Q_{L,f}^- = \partial + d\overline{W}_f \wedge \equiv f^*Q_L^-,$$

so that the forms $f^*\omega_k$ satisfy

$$Q_{R,f}^+ f^*\omega_k = Q_{L,f}^- f^*\omega_k = 0.$$

IN the case of just one field, these equations imply that $f^*\omega_k$ ($k = 1, \ldots, \Delta$) are vacuum wave-forms for the superpotential W_f. (Recall that if $n = 1$ the wave forms are independent of K as form, not just as cohomology classes). In the general case, the new wave functions are

$$\Omega_k = f^*\omega_k + Q_{R,f}^+ Q_{L,f}^- \mathfrak{H}^{-1} \Lambda f^*\omega_k.$$

where the dependence on the Kähler metric is hidden in Λ and $\mathfrak{H}$. The Ω_k's are manifestly cohomologous to the pullbacks of the forms ω_k. Indeed, if W is not degenerate, $\mathfrak{H}^{-1}$ is a continuous operator in the $(n-2)$-form sector. Of course, these functions are just a subset of all vacuum wave-forms for W_f since $\Delta_f > \Delta$ for a branched cover. Now for $n = 1$, one has simply

$$\langle \bar{k} | h \rangle |_{W_f} = \int * \overline{f^*\omega_k} \wedge f^*\omega_h$$

$$= (\deg f) \int * \overline{\omega_k} \wedge \omega_k = (\deg f) \langle \bar{k} | h \rangle |_W. \qquad (5.6)$$

(for $n = 1$, the Hodge dual $*$ on 1-forms depends on the complex structure only). The equality is true for the general case as well, the only difference being that in order to prove it one has to use the full machinery of the cohomological computation for overlap integrals, see ref. [14]. Alternatively, functoriality follows from the (conjectural) uniqueness of the solution to our equations. Indeed, the topological functions η_{ij} and C_{ij}^k are trivially functorial, and hence the equations themselves behave as expected under non-invertible change of variables. Therefore, if we know the ground-state metric for W_f we can get the metric for W just by

restricting ourselves to the cohomology classes $\phi_k \, dY^1 \wedge \ldots \wedge dY^n$ (with $\phi_k \in \mathscr{R}_f$) which can be written as

$$\phi_k \, dY^1 \wedge \ldots \wedge dY^n = f^*\left(\psi_k \, dX^1 \wedge \ldots \wedge dX^n\right), \quad \psi_k \in \mathscr{R}. \qquad (5.7)$$

Note that in this way we automatically reproduce the "anomalous" jacobian.

The presence of a jacobian in the transformation has another implication. Suppose that both W and f are quasi-homogeneous. Then so is W_f. Both models are critical and we can speak of their central charge. Then using the fact that hessian is the maximum charged element in the ring with charge $c/3$, eq. (5.7) implies

$$c = c_f - 6q_f(J), \qquad (5.8)$$

where $q_f(J)$ is the U(1) charge of the jacobian

$$\det\left[\partial f_i / \partial Y_j\right] \in \mathscr{R}_f.$$

The insertion of the jacobian just soaks up the excess of vacuum charge of the branched model with respect to the original one. Note that we can use this technique to relate different conformal theories even with *different* central charges, as far as the metric on chiral primary fields are concerned. It would be interesting to investigate the precise relation between the full conformal theories in such cases.

6. Lie-algebraic aspects

Our equations have an interesting group-theoretical meaning. This is well known in the conformal case where W is quasi-homogeneous, where it is related to the Lie-algebraic aspects of the period map of the corresponding hypersurface (or the Lie-algebraic structure of the Variation of Hodge structure). It turns out that the Lie-algebraic point of view is very useful even for *massive* perturbations of our theories as well and they help us understand the geometrical content of the equations as well as to actually solve them. Our discussion here is modelled on the classical one for the topology of algebraic hypersurfaces (which arises in the conformal limit). This case we will refer to as the "geometrical case" below.

We begin by discussing the reality condition on the metric (2.9). One can find a "special" holomorphic basis such that the residue pairing is independent of the couplings t^i and

$$\eta^* = \eta^{-1} = \eta.$$

Such special bases have been considered before in the context of topological field theories [10]. Their existence is a deep property of TFT and they are also technically convenient. See appendix C for details. In such a basis, the reality constraint on the metric becomes

$$g \eta g^{\mathrm{T}} = \eta,$$

i.e. g is orthogonal with respect to the real metric η. Then, $g \, \partial g^{-1}$ belongs to the corresponding Lie algebra, namely $g(\partial g^{-1})\eta$ is antisymmetric. Thus the first term in eq. (3.9) is skew-symmetric with respect to η. This is consistent with our equations. Indeed, the topological 3-point functions C_{ij}^{k} are η-symmetric (that is, $C_{ijk} \equiv C_{ij}^{l}\eta_{lk}$ is symmetric. So is $g C_{j}^{\dagger}g^{-1}$ (since $\eta M = (M^{-1})^{\mathrm{T}}\eta^{*}$). Then $[C_{i}, g C_{j}^{\dagger}g^{-1}]$ is also η-antisymmetric. Note that, without loss of generality, we can choose $\eta = \mathbb{1}$, so g is orthogonal in the standard sense. Of course, g belongs to the complexified orthogonal group, not to the usual compact form.

To go on with the discussion, it is better to rewrite the linear problem (3.11), in a more convenient way. Let $g = \exp[\mathscr{G}]$ and put $e = \exp[\mathscr{G}/2]$. We perform the gauge transformation

$$\psi \rightarrow eT.$$

Then the linear problem becomes

$$\left[\partial - (\partial e)e^{-1} + e^{-1}Ce\right]T = 0,$$

$$\left[\bar{\partial} + e^{-1}(\bar{\partial}e) + eC^{\dagger}e^{-1}\right]T = 0. \tag{6.1}$$

From now on, by T we mean the fundamental solution, i.e. T is the matrix solution such that $T(0) = \mathbb{1}$. By adding an irrelevant constant to W, we can assume that tr $C = 0$. Then from (6.1) it is manifest that T belongs to SL($\varDelta$, $\mathbb{R}$). This is similar to what one finds in the geometrical case, where however there are additional algebraic restrictions coming from the topology. They reflect the so-called Riemann bilinear relations. Under certain circumstances, similar restrictions apply to the massive case as well. They are quite important, since restricting the Lie group in which T takes values is a crucial step in solving the equations for particular models. Let G be this group and H be the subgroup gauges by the connection for D, $\bar{D}$. One had H $\subset$ K, where K is the maximal compact subgroup of G (this follows from the fact that the connection is metric – or put differently, from the eq. (6.1) and recalling that e is hermitian). Of course, g (and e) belong to H$_{\mathbb{C}}$ (i.e. they are complex gauge transformations). The importance of identifying G and H is best understood by realizing what the equations become for special G and H.

Suppose we have a family of superpotentials depending on just a single coupling t. This will be the case of most interest for us in the rest of this paper. The simplest

case is when H is the maximal torus of G. In this case eqs. (3.5) are just the usual Lax-representation of a Toda system (indeed, consistency alone implies that the matrix C_i is the sum of an *admissible* set of roots for G). Then eqs. (3.5) reduce to the standard equations of Toda field theory.

The Gauss–Manin equations for the variation of Hodge structure for an algebraic manifold X (of dimension m) are also of the form (6.1) with

$$G = SO(b_m^+, b_m^-), \qquad H = SO(h^{k,k}) \bigotimes_{p=0}^{k-1} U(h^{m-p,p}) \quad m = 2k,$$

$$G = Sp(b_m, \mathbb{R}), \qquad H = \bigotimes_{p=0}^{k} U(h^{m-p,p}) \quad m = 2k+1,$$

where $h^{p,q}$ (resp. b_m) are the Hodge (resp. Betti) primitive numbers and

$$b_m^+ + b_m^- = b_m, \qquad b_m^+ - b_m^- = \tau \quad \text{(Hirzebruch signature)}.$$

In this case C_i is the class in $H^1(\Theta)$ of the complex deformation corresponding to an infinitesimal variation of the parameter t^i, seen as the matrix of the endomorphism in $H^m(X)$ induced by wedge product (where Θ represents the tangent bundle).

In particular, if we have a Hodge (sub-)structure such that for some integer a

$$h^{m-p,p} = 1 \quad \text{for} \quad |m - 2p| \leqslant a$$

$$= 0 \quad \text{otherwise},$$

and $(\partial_t W)^a \neq 0$ in $\mathcal{R}$, then the GM equations reduce to those of the G-Toda molecule (i.e. the non-affine version). The simplest example of this state of affairs is the torus. The σ-model on a torus is equivalent to an orbifold of the LG theory [31] with superpotential

$$W = X_1^3 + X_2^3 + X_3^3 + t X_1 X_2 X_3.$$

in this case $h^{1,0} = h^{0,1} = 1$ and hence G = Sp(2, $\mathbb{R}$) and H = U(1). In other words, in this case the monomial $X_1 X_2 X_3$ generates a *nilpotent* subring of order 2, and that is how we end up with Sp(2, $\mathbb{R}$). Solving the linear problem one gets (for details, see ref. [37])

$$\frac{\langle q = 1/2 \,|\, q = 1/2 \rangle}{\langle q = -1/2 \,|\, q = -1/2 \rangle} = \frac{1}{4 \,|\mathrm{Im}\, \tau(t)|^2} \left| \frac{d\tau}{dt} \right|^2,$$

for some holomorphic function $\tau(t)$. This is precisely the general (real) solution to the SL(2)-Toda equation, i.e. the Liouville equation

$$\frac{\langle q = 1/2 \mid q = 1/2 \rangle}{\langle q = -1/2 \mid q = -1/2 \rangle} = \exp[\phi_{\text{Liouville}}].$$

However, in the LG language the function $\tau(t)$ is further restricted by the boundary conditions. It turns out that this function is just equal to the period for the torus $W = 0$ as it should from the general correspondence between LG theories and geometry [2,31]. Indeed, one can use the degeneration structure of the algebraic surface to find out what $\tau(t)$ exactly is.

This example can be generalized. Take the CY manifolds $\mathscr{X}_n$ associated to the superpotentials

$$W = X_1^n + X_2^n + \ldots + X_n^n + tX_1X_2\ldots X_n$$

and consider the Hodge substructure (i.e. the subset of $\mathscr{R}$) corresponding to the subspace of $H^{n-2}(\mathscr{X}_n)$ invariant under the automorphisms

$$X_j \to \exp[2\pi ia_j/n] X_j \quad \sum_j a_j = 0 \bmod n.$$

(It is precisely modding out the LG theory by this symmetry that has been shown to be a beautiful example of mirror symmetry [38].) The ring invariant under the above transformation is generated by $X_1\ldots X_n$. In this case the equations one gets for the metric g is the same as Toda molecule with $G = Sp(n - 1, \mathbb{R})$ (resp. $SO(n/2, n/2 - 1)$) for n odd (resp. even). These follow very easily from eq. (3.9). In particular these Toda theories emerge as a $\mathbb{Z}_2$ reduction of $sl(n - 1)$ Toda, with

$$\left\langle \left(\bar{X}_1\ldots\bar{X}_n\right)^r \middle| \left(X_1\ldots X_n\right)^r \right\rangle = \exp(q_r)$$

with $0 \leqslant r \leqslant n - 2$, and one identifies the vector v in $q_r - q_{r-1} = q_iv^i$ with a simple root of $sl(n - 1)$. The $\mathbb{Z}_2$ reduction follows from (2.9) implying $q_r + q_{n-2-r} = 0$. It is the nilpotent structure of the ring generated by the symmetric monomial $X_1\ldots X_n$ which directly reflects the $sl(n - 1)$ Toda molecule structure in these equations.

The general case of arbitrary deformations of algebraic hypersurfaces is a very natural generalization of the Toda molecule. In ref. [14] the ground state metric for quasihomogeneous superpotentials was written in terms of holomorphic countour integrals. This explicit representation is just the extension to the more general case of the standard Leznov–Saveliev algorithm to solve Toda equations [39]. (This is common knowledge in Algebraic Geometry). Indeed, this algorithm reduces the

solution of the Toda molecular to finding a triangular holomorphic matrix Π satisfying

$$\partial_t \pi_i{}^k = C_{ti}^h(t) \Pi_h{}^k.$$

The period integrals of ref. [14] (after filtration á la Griffiths) give the special Π matrix satisfying the correct boundary condition. Of course, this method works for all variations of Hodge structure, even if H is not abelian and we have a multiparameter family.

Now we come back to the more general case of massive perturbations, and wish to determine G and H. There is a simple method to determine H. Decompose the vacuum subspace $\mathscr{V}$ of the Hilbert space into orthogonal subspaces corresponding to different irreducible representations of the (pseudo)-symmetries of W. A priori from the above discussion it is clear that H is a subgroup of product of $U(N_R)$ where N_R denotes the dimension of the representations in question. However η, which is of order 2, acts on the representations, and because of the eq. (2.9) relates the $U(N_R)$ for each pair and so cuts the number of $U(N_R)$ by half. Also, if η maps a representation to itself, eq. (2.9) implies that the corresponding H is in $SO(N_R)$. Put differently, an irreducible representation we call *real* if it is real with respect to the real-structure M. Then, a real subspace of dimension N_r contributes a factor $SO(N_r)$ to H, and a conjugate pair of complex subspaces of dimension N_c contribute a factor $U(N_c)$. I.e.

$$H \subset \bigotimes_{\text{pairs}} U(N_c) \bigotimes_{\text{real}} SO(N_r).$$

In particular, H is abelian if all complex subspaces have dimension 1 and the real ones at most dimension 2. In the geometrical case H is given by this recipe with $\mathscr{V} \simeq H^m(X)$, the relevant subspaces being $H^{p,q}(X)$ and under complex conjugation $p \leftrightarrow q$.

The problem of determining G is more deep. A typical case when we have special restrictions on this group is in the presence of a special $\mathbb{Z}_2$ symmetry P; this occurs in a theory which has the property that for all values of the coupling t,

$$PW = -WP. \tag{6.2}$$

Such a symmetry operator P appears in the geometrical case as well and is called the "Weil operator" [40]. This operator is order 2 as far as the NS is concerned, but since the vacua are in the Ramond sector and two Ramond states produce an NS state P^2 acting on Ramond states can end up being ± 1. Since the spectral flow from NS to NS is equivalent to product of two Ramond vacuum states, and this is accomplished by the hessian of W, we learn that the phase of P^2 is simply the same as the phase of $\mathfrak{H}$ under P. Let us write

$$P^2 = (-1)^m.$$

Working in the holomorphic basis we represent P by

$$P \mid k \rangle = P_k^h \mid h \rangle.$$

Note that we have

$$\eta P = (-1)^m P^\mathrm{T} \eta.$$

This follows from the fact that a state and its dual with respect to η transform under P the same way up to the phase $(-1)^m$ which is the way the spectral flow (given by the hessian) relates them. We thus see that

$$\Omega^{ij} = \eta^{ik} P_k^{\ j}$$

is symmetric for m even and antisymmetric (a symplectic form) for m odd. Now if we consider

$$\Phi = \psi^\mathrm{T} \Omega \psi,$$

where ψ is the solution to the linear problem in the holomorphic basis (3.11), and note that eq. (6.2) implies that $PC_i = -C_i P$ we see that [*]

$$\partial \Phi = \bar{\partial} \Phi = 0 \quad \Rightarrow \quad \Phi = \Omega.$$

Then for m even (resp. odd) ψ is orthogonal (resp. symplectic) with respect to the constant pairing Ω. If the signature of Ω is (r, s), $G \subset SO(r, s)$. The geometrical case is just of this type, with $\Delta = b_m$, $r = b_m^+$ and $s = b_m^-$ (of course we can rewrite all these in the other gauge for T).

7. Minimal models perturbed by the most relevant operator and related models

In the remaining sections of the paper we shall discuss particular classes of Landau–Ginzburg models for which the computation of the ground state metric can be done explicitly. We do this both for the intrinsic interest of the "solvable" models in various physical applications and also in order to illustrate the general phenomena of the previous sections (in particular, the overdeterminate nature of the problem).

Among the perturbations of conformal theories by relevant operators Zamolodchikov [41] found a technique to find which directions give rise to integrable models. The integrability is in the sense of having factorizable S-matrices for the massive excitations of the resulting theory. The idea is to look for an infinite

[*] We are mimicking the geometrical case. In that case the bilinear form Ω is the intersection in H^m $(X, \mathbb{R})$.

number of conserved currents which survive the perturbations away from the conformal point. These ideas were applied to $N = 2$ minimal superconformal theories in ref. [33] where it was found that these models perturbed by the (last component of the) chiral primary field of lowest (non-trivial) dimension, i.e. most relevant operator, leads to an integrable theory. Moreover it was found that there is a beautiful interplay between the structure of the superpotential W and the solitons and their masses. Then essentially self-consistency alone fixes the S-matrix in these models. It was shown in ref. [42] how these models (and their generalizations) can be realized in terms of quantum affine Toda field theories with very specific couplings. Also, the geometry of solitons and their conservation laws for specific perturbations of certain Kazama–Suzuki models (and in particular the grassmannians) has been uncovered in an interesting recent paper [43].

As we will see it turns out that precisely these perturbations (and some natural generalizations to be mentioned below) which can be described by $N = 0$ *quantum* (affine) Toda field theories [42] lead to equations for the ground state metric which as a function of the perturbing parameter t (which can be identified with RG flow parameter) satisfy *classical* (affine) Toda equations of the *same type* (and their natural reductions). This is an intriguing connection between the quantum theory and the correlation functions of that quantum theory, which begs for a deeper understanding. That we should get Toda equations is already clear from the discussion of sect. 6. In fact that discussion will help us organize what we should expect for our equations. The general arguments of sect. 6 can be explicitly verified in the concrete examples we study in this section. The models of the present section are basically the ones for which the equations can be recast in a Toda form by elementary tricks. In sect. 9 and 10 we shall consider other model which are related to Verlinde rings whose equations are reduced to Today by more sophisticated techniques.

Here we limit ourselves to a discussion of the relevant equations. However, the real magic of the subject stems from the unique properties of the *solutions* corresponding to the actual metric rather than from the fact that the *equations* themselves are among the nicer ones in mathematical physics. Part of the magic will be discussed in some detail in sect. 8.

7.1. THE A_n SERIES

In the LG approach, the A_n minimal model corresponds to the superpotential $W = X^{n+1}/(n+1)$. The (non-trivial) chiral field of lowest dimension is X. Then we consider the superpotential

$$W(X, t) = \frac{X^{n+1}}{n+1} - tX, \tag{7.1}$$

and look for the dependence of the ground state metric g on t. As a basis on

$$\mathscr{R} \equiv \mathbf{C}[X]/(X^n - t)$$

we choose

$$1, X, X^2, \ldots, X^{n-1}.$$

The vacuum state associated to X^k will be denoted by $|k\rangle$.

The model described by (7.1) has the discrete symmetry

$$X \to \exp[2\pi i/n]X, \tag{7.2}$$

under which the state $|k\rangle$ picks up a phase $\exp[\pi i(2k + 1 - n)/n]$. Then $\langle k|h\rangle = 0$ for $k \neq h$, i.e. g is diagonal in this basis (from here till the end of the paper we have changed our notation and take $\langle k|$ to be the adjoint of $|k\rangle$). Therefore the group H defined in sect. 6 is abelian. From the discussion there it follows that our equations are of the Toda type. This system is rather peculiar in that the metric belongs to an abelian group just on symmetry grounds, i.e. before using the reality constraint to further reduce the number of independent elements of g. Imposing the reality constraint will lead to a consistent truncation of the Toda system to one with less degrees of freedom. Such consistent truncations are well known in the Toda theory [44] and are understood algebraically as foldings of the corresponding Dynkin diagrams.

To start with, ψ takes values in SL(n) and hence the equation for the t-dependence is that of some A_{n-1} Toda system. Which one depends on the admissible root system to which C_t corresponds. Multiplication by operator X is denoted by the matrix C_t given in the above basis of vacuum as

$$C_t = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ t & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

i.e. (up to conjugacy) C_t is the sum of primitive roots of sl(n) minus the longest root. Then we get the affine A_{n-1} equation.

To see what the truncated "real". Today system is, it is better to distinguish between even and odd n. If n is even $(= 2m)$ we have a "Weil operator" P. This is just the generator of the symmetry $X \to -X$. This is an element of the group in (7.2). From the phase a state picks up under such a transformation, we see that $P^2 = -\mathbb{1}$. Then, according to the discussion in sect. 6 we have

$$G = \mathrm{Sp}(2m, \mathbb{R}),$$

i.e. we get the $\hat{C}_m$ Toda equations (for $m = 1$ this is $\hat{A}_1$ and for $m = 2$ this is $\hat{B}_2$). This can be checked explicitly using eq. (3.9) as we will show below.

The situation for $n = 2m + 1$ odd is less simple. The truncated Toda equations are associated to a root system (denoted by $\widehat{BC}_m$) which do not correspond to any Lie algebra. The corresponding equations are called the generalized Bullough–Dodd equations, since the first equation in the series is precisely the usual BD equation.

Let us see how they arise. In our basis, the residue pairing is independent of t. The only non-vanishing entries are

$$\eta_{k,n-1-k} = 1.$$

Then the reality constraint reads

$$\langle k \,|\, k \rangle \langle n - 1 - k \,|\, n - 1 - k \rangle = 1.$$

In particular, if n is odd ($n = 2m = 1$) one has

$$\langle m \,|\, m \rangle = 1 \quad \text{for all } t.$$

In this way we reduce to $[n/2]$ unknown functions, namely $\langle k \,|\, k \rangle$ for $k = 0, 1, \ldots, [n/2] - 1$. In particular, for $n = 2$ or 3 we have a *single* unknown function. Writing

$$\varphi_i = \log\langle i \,|\, i \rangle, \quad i = 0, \ldots, n - 1,$$

and using the explicit form of C_t, eq. (3.9) becomes

$$\partial_t \partial_{\bar{t}} \varphi_0 + e^{(\varphi_1 - \varphi_0)} - |t|^2 \, e^{(\varphi_0 - \varphi_{n-1})} = 0,$$

$$\partial_t \partial_{\bar{t}} \varphi_i + e^{(\varphi_{i+1} - \varphi_i)} - e^{(\varphi_i - \varphi_{i-1})} = 0, \quad i = 1, \ldots, n - 2,$$

$$\partial_t \partial_{\bar{t}} \varphi_{n-1} + |t|^2 \, e^{(\varphi_0 - \varphi_{n-1})} - e^{(\varphi_{n-1} - \varphi_{n-2})} = 0. \tag{7.3}$$

To put these equations in standard form, we put ($i = 0, \ldots, n - 1$)

$$\varphi_i = q_i + \frac{2i - n + 1}{2n} \log |t|^2,$$

$$z = \frac{n}{n + 1} t^{(n+1)/n}.$$

We extend the definition of q_i to all i's by setting

$$q_{i+n} \equiv q_i.$$

Then eqs. (7.3) take the standard form for $\hat{A}_{n-1}$ Toda equations

$$\partial_z \partial_{\bar{z}} q_i + e^{(q_{i+1} - q_i)} - e^{(q_i - q_{i-1})} = 0. \tag{7.4}$$

However, we have still to use the reality constraint which in the new variables reads

$$q_i + q_{n-i} = 0.$$

If n is even ($n = 2m$), using this constraint we reduce to the $\hat{C}_m$ Toda theory. To write it in the canonical form, just write (notations as in ref. [45])

$$q_i = -2\phi_{i+1} + \frac{2i + m}{2(m - 1)} \log 2,$$

$$z \to 2^{1/2(m-1)}z.$$

Then eqs. (7.4) become

$$2\partial\bar{\partial}\phi_1 = e^{2(\phi_1 - \phi_2)} - 2\,e^{-4\phi_1},$$

$$2\partial\bar{\partial}\phi_j = e^{2(\phi_j - \phi_{j+1})} - e^{2(\phi_{j-1} - \phi_j)} \quad j = 2, \ldots, m - 1.$$

$$2\partial\bar{\partial}\phi_m = 2\,e^{4\phi_m} - e^{2(\phi_{m-1} - \phi_m)}.$$

For n odd ($n = 2m + 1$), the redefinition

$$q_i = -2\phi_{i+1} - \frac{1}{2}\left(\frac{i + 1}{2m + 1} - 1\right) \log 2$$

$$z \to 2^{-1/2(2m+1)}z$$

puts the reduced equations into the canonical $\widehat{BC}_m$ form

$$2\partial\bar{\partial}\phi_1 = e^{2(\phi_1 - \phi_2)} - 2\,e^{-4\phi_1},$$

$$2\partial\bar{\partial}\phi_j = e^{2(\phi_j - \phi_{j+1})} - e^{2(\phi_{j-1} - \phi_j)}, \quad j = 2, \ldots, m - 1,$$

$$2\partial\bar{\partial}\phi_m = e^{2\phi_m} - e^{2(\phi_{m-1} - \phi_m)}.$$

Of course, not all solutions to the above equations are acceptable as ground state metrics. At least two additional conditions are needed: first of all, $\langle k \,|\, k \rangle$ should be real, positive, and regular for all values of the couplings, and second the solution should not depend on the phase of the coupling t since this phase can be re-absorbed by the field redefinition

$$t \to e^{i\varphi}t, \quad X \to e^{i\varphi/n}X, \quad \theta \to e^{-i(n+1)\varphi/2}\theta.$$

Then only solutions invariant under rotations of z are acceptable. This property applies to all models we consider in the present section.

There is strong evidence that these two conditions uniquely fix the solutions. This will be discussed in sect. 8.

7.2 THE D_n SERIES

In the D_n case the most relevant perturbation of superpotential reads

$$W = \frac{X^{n-1}}{n-1} + XY^2 - tX.$$

As basis for $\mathscr{R}$ we choose

$$1, Y, Y^2, X, X^2, \ldots, X^{n-3}.$$

This model has two symmetries, namely

$$X \to \exp[2\pi i/(n-2)]X \quad Y \to Y,$$

$$X \to X \qquad\qquad\qquad Y \to -Y.$$

It follows, that in this basis the only non-vanishing off-diagonal element of g is $\langle Y^2 | 1 \rangle$. One has

$$\mathrm{Res}[X^a] = \tfrac{1}{2}\delta_{a,n-2}, \qquad \mathrm{Res}[Y^bX^c] = 0 \quad \text{for} \quad b, c \neq 0,$$

$$\mathrm{Res}[Y^{2k+1}] = 0, \qquad\qquad \mathrm{Res}[Y^2] = -\tfrac{1}{2}, \qquad \mathrm{Res}[Y^4] = -\tfrac{1}{2}t. \qquad (7.5)$$

Then, decomposing $\mathscr{R}$ according the representations of these symmetries, for n even (resp. odd) we have $n/2 - 1$ (resp. $(n-1)/2 - 1$) one-dimensional complex orthogonal subspaces, 1 (resp. 2) one-dimensional real subspace, and 1 two-dimensional real subspace spanned by $(1, Y^2)$. Then (cf. sect. 6)

$$H = SO(2) \otimes U(1)^{[(n-2)/2]},$$

is abelian and we get again a Toda system.

If $n = 2m + 2$ is even, the general arguments of sect. 6 uniquely fix the Toda system our equations correspond to. Indeed, we have a "Weil symmetry" P,

$$P : X \to -X.$$

This time $P^2 = \mathbb{1}$. Indeed, the hessian of W is even with respect to P, not odd as in the A-case. On $\mathscr{R}$ (neglecting the "decoupled" state $|Y\rangle$) the $+1$ eigenvalue of P has multiplicity $m + 1$. Then,

$$G = SO(m + 1, m),$$

and we have the $\hat{B}_m$ Toda system. Instead, the Toda for n odd does not correspond to a root system and cannot be deduced by symmetry arguments alone.

Explicitly the reality constraint reads

$$\langle X^a \mid X^a \rangle \langle X^{n-2-a} \mid^{n-2-a} \rangle = \tfrac{1}{4}, \quad a = 1, \ldots, n-3,$$

$$\langle Y \mid Y \rangle = \tfrac{1}{2}, \qquad \langle 1 \mid Y^2 \rangle = \tfrac{1}{2} t \langle 1 \mid 1 \rangle, \qquad \langle Y^2 \mid 1 \rangle = \tfrac{1}{2} \bar{t} \langle 1 \mid 1 \rangle,$$

$$2 \langle Y^2 \mid Y^2 \rangle = \frac{1}{2 \langle 1 \mid 1 \rangle} + \frac{|t|^2}{2} \langle 1 \mid 1 \rangle.$$

The coefficients C_t are

$$X \mid X^a \rangle = \mid X^{a+1} \rangle, \quad a = 0, \ldots, n-4,$$

$$X \mid Y \rangle = X \mid Y^2 \rangle = 0,$$

$$X \mid X^{n-3} \rangle = t \mid 1 \rangle + \mid Y^2 \rangle.$$

Let $n = 2m + 2 - s$ with $s = 0, 1$. The independent entries of g are $\langle X^a \mid X^a \rangle$ for $a = 0, 1, \ldots, m-1$. In terms of these variables, our equations become

$$-\partial_t \partial_{\bar{t}} \log \langle 1 \mid 1 \rangle = \frac{\langle X \mid X \rangle}{\langle 1 \mid 1 \rangle} - |t|^2 \langle 1 \mid 1 \rangle \langle X \mid X \rangle,$$

$$-\partial_t \partial_{\bar{t}} \log \langle X \mid X \rangle = \frac{\langle X^2 \mid X^2 \rangle}{\langle X \mid X \rangle} - \frac{\langle X \mid X \rangle}{\langle 1 \mid 1 \rangle} - |t|^2 \langle X \mid X \rangle \langle 1 \mid 1 \rangle,$$

$$-\partial_t \partial_{\bar{t}} \log \langle X^a \mid X^a \rangle = \frac{\langle X^{a+1} \mid X^{a+1} \rangle}{\langle X^a \mid X^a \rangle} - \frac{\langle X^a \mid X^a \rangle}{\langle X^{a-1} \mid X^{a-1} \rangle},$$

$$(a = 2, \ldots, m-2),$$

$$-\partial_t \partial_{\bar{t}} \log \langle X^{m-1} \mid X^{m-1} \rangle = \left(\frac{1}{2 \langle X^{m-1} \mid X^{m-1} \rangle} \right)^{1+s} - \frac{\langle X^{m-1} \mid X^{m-1} \rangle}{\langle X^{m-2} \mid X^{m-2} \rangle}.$$

$$(7.6)$$

To put these equations in canonical form, we define

$$\phi_{\pm} = \log \langle X \mid X \rangle \pm \log \langle 1 \mid 1 \rangle \mp F(|t|) + (1 \pm 1) \log |t|,$$

$$\phi_j = \log \langle X^{j+1} \mid X^{j+1} \rangle - \log \langle X^j \mid X^j \rangle + F(|t|), \quad (j = 2, \ldots, m-2),$$

$$\phi_{m-1} = -(1+s) \log \langle X^{m-1} \mid X^{m-1} \rangle + (1+s)(m-1) F(|t|) - (1+s) \exp |t|,$$

$$z = \left(\frac{1+s}{2^{2+s}} \right)^B \frac{t^{1+(1+s)B}}{1 + (1+s)B},$$

where

$$B = \frac{1}{2[1 + (m - 1)(1 + s)]},$$

and

$$F(|t|) = 2B[(1 + s) \log(|t|/2) + \log(1 + s) - \log 2].$$

Then eqs. (7.6) become

$$\partial\bar{\partial}\phi_{\pm} = 2 e^{\phi_{\pm}} - e^{\phi_1},$$

$$\partial\bar{\partial}\phi_1 = 2 e^{\phi_1} - e^{\phi_+} - e^{\phi_-} - e^{\phi_2},$$

$$\partial\bar{\partial}\phi_a = 2 e^{\phi_a} - e^{\phi_{a-1}} - e^{\phi_{a+1}} \quad (a = 2,\ldots,m - 3),$$

$$\partial\bar{\partial}\phi_{m-2} = 2 e^{\phi_{m-2}} - e^{\phi_{m-3}} - (2/(1 + s)) e^{\phi_{m-1}},$$

$$\partial\bar{\partial}\phi_{m-1} = 2 e^{\phi_{m-1}} - (1 + s) e^{\phi_{m-2}}.$$

In general, the Toda equations can be written in the form [44]

$$\partial\bar{\partial}\phi_a = C_{ab} e^{\phi_b},$$

where C_{ab} is the Cartan matrix of some root system. From the above explicit formula, we see that for $s = 0$ (n even) we get the Cartan matrix of $\hat{B}_m$, as expected from the general argument. Instead for $s = 1$ (n odd) we get the transpose Cartan matrix. This is the Toda system denoted by $D^T(SO(2m + 1))$ in ref. [44].

7.3. THE E-SERIES

The only new model is E_7, since E_6 and E_8 can be obtained as tensor products of A minimal models. In the E_7 case the most relevant perturbation of the superpotential reads

$$W = \tfrac{1}{3}X^3 + \tfrac{1}{3}XY^3 - tY.$$

As basis in $\mathscr{R}$ we take ($i = 1,\ldots,7$)

$$\phi_i = \{1, Y, X, Y^2, XY, X^2, X^2Y\}. \tag{7.7}$$

This model has a $\mathbb{Z}_7$ symmetry

$$X \to eX, \quad Y \to e^3Y, \quad e^7 = 1.$$

Under this symmetry no two fields in (7.7) transform the same way, and hence the metric g is diagonal. So H is abelian and we have again a Toda theory.

In the above basis the residue pairing reads

$$\eta_{ij} = (1 - 4\delta_{i,4})\delta_{i+j,8},$$

and the reality constraint reads

$$\langle i|i\rangle\langle 8-i|8-i\rangle = 1, \quad i \neq 4,$$

$$\langle 4|4\rangle = 3.$$

Then H = U(1)^3. The non-vanishing elements of C_t are

$$C_1^{\ 2} = C_2^{\ 4} = C_3^{\ 5} = C_6^{\ 7} = 1, \qquad C_5^{\ 1} = C_7^{\ 3} = t, \qquad C_4^{\ 6} = -3.$$

Putting

$$2\varphi_1 = \log\langle 3|3\rangle + \tfrac{1}{7}\log|t|^2 + \tfrac{1}{7}\log 24,$$

$$2\varphi_2 = -\log\langle 1|1\rangle - \tfrac{4}{7}\log|t|^2 - \log 2 + \tfrac{3}{7}\log 24,$$

$$2\varphi_3 = -\log\langle 2|2\rangle - \tfrac{2}{7}\log|t|^2 + \log 6 - \tfrac{2}{7}\log 24,$$

$$z \equiv \frac{7}{8\sqrt{2}}(24)^{1/7}t^{8/7}.$$

one gets the equations in the form

$$2\partial\bar{\partial}\varphi_1 = e^{2(\varphi_1-\varphi_2)} - 2\,e^{-4}\varphi_1,$$

$$2\partial\bar{\partial}\varphi_2 = e^{(2\varphi_2-\varphi_3)} - e^{2(\varphi_1-\varphi_2)},$$

$$2\partial\bar{\partial}\varphi_3 = e^{2\phi_3} - e^{2(\varphi_2-\varphi_3)},$$

which is the $\widehat{BC_3}$ Toda in the notations of ref. [45] (i.e. GD(H$_3$) in the language of [44]).

7.4. THE A$_n$ MODELS PERTURBED BY NEXT RELEVANT OPERATOR

Next we consider the models

$$W = \frac{X^{n+1}}{n+1} - t\frac{X^2}{2}.$$

For $\mathcal{R}$ we use the basis $1, X, \ldots, X^{n-1}$. These models have the discrete symmetry

$$X \to \exp[2\pi i/(n-1)]X, \qquad \theta \to \exp[-\pi i(n+1)/(n-1)]\theta.$$

This implies

$$\langle k \mid h \rangle = 0 \quad \text{for} \quad k \neq h \quad \text{except for} \quad \langle n-1 \mid 0 \rangle \quad \text{and} \quad \langle 0 \mid n-1 \rangle.$$

Since the two-dimensional subspace spanned by 1 and X^{n-1} is *real*, H is still abelian and therefore we get again a Toda system. In fact one has

$$H = SO(2) \otimes U(1)^{[(n-2)/2]}.$$

In the present case the residue pairing is

$$\eta_{kh} = \delta_{k+h,n-1} + t\delta_{k,n-1}\delta_{h,n-1},$$

so the reality constraint becomes

$$\langle k \mid h \rangle \langle n-1-k \mid n-1-k \rangle = 1 \quad \text{for} \quad k \neq 0, n-1,$$

$$\langle 0 \mid n-1 \rangle = \tfrac{1}{2}t\langle 0 \mid 0 \rangle, \qquad \langle n-1 \mid 0 \rangle = \tfrac{1}{2}t\langle 0 \mid 0 \rangle,$$

$$\langle n-1 \mid n-1 \rangle = \frac{1}{\langle 0 \mid 0 \rangle} + \frac{|t|^2}{4}\langle 0 \mid 0 \rangle.$$

If $n+1$ is even ($=2m$) the model can be reduced to already solved ones. Indeed,

$$W(X) = W_0(X^2)$$

with

$$W_0(Y) = \frac{Y^m}{n+1} - \frac{t}{2}Y,$$

so the "odd'" states

$$|2k+1\rangle \quad (k = 0, 1, \ldots, m-2),$$

are just the pullbacks of the vacua for the A_{m-1} minimal model perturbed by the most relevant operator. For our purposes, these states decouple from the others and, by functoriality, the corresponding ground state metric

$$\langle 2k+1 \mid 2h+1 \rangle$$

is the solution to a $Sp(m-1)$ or a $\widehat{BC}_{m-1}$ Toda system according to whether m is odd or even.

Instead, the metric for the "even" states $|2k\rangle$ is equal to that of the D_{m+1} model. This follows from the fact that the D models are the orbifolds of the A_{2k} ones with respect to the symmetry

$$X \to -X.$$

Then for the even states we get $\hat{B}_{(m-1)/2}$ or $D^T(SO(m+1))$ Toda according whether m is odd or even.

On the contrary, when n is even ($= 2m$) we have no "Weil operator" and hence we expect a Toda theory associated to a generalized Cartan matrix. Indeed, let

$$q_i = -\log\langle 2(i-1)|2(i-1)\rangle \quad \text{for} \quad i = 1, 2, \ldots, \left[\frac{m+1}{2}\right]$$

$$= \log\langle 2(m-1)+1|2(m-i)+1\rangle \quad \text{for} \quad i = \left[\frac{m+1}{2}\right] + 1, \ldots, m.$$

Then the equations become

$$\partial\bar{\partial}q_1 = \tfrac{1}{4} e^{(q_1-q_2)} - \tfrac{1}{16}|t|^2 e^{-(q_1+q_2)},$$

$$\partial\bar{\partial}q_2 = \tfrac{1}{4} e^{(q_2-q_3)} - \tfrac{1}{4} e^{(q_1-q_2)} - \tfrac{1}{16}|t|^2 e^{-(q_1+q_2)},$$

$$\partial\bar{\partial}q_i = \tfrac{1}{4}\left[e^{(q_i-q_{i+1})} - e^{(q_{i-1}-q_i)}\right] \quad (i = 3, \ldots, m-1),$$

$$\partial\bar{\partial}q_m = \tfrac{1}{4}|t|^2 e^{2q_m} - \tfrac{1}{4} e^{(q_{m-1}-q_m)},$$

which, after an obvious re-interpretation of the symbols, is the same as eqs. (7.6). Then by a redefinition of the variables it can be recast in the standard $D^T(SO(2m+1))$ Toda form.

7.5. PERTURBED GRASSMANNIAN COSET MODELS

The Landau–Ginzburg description of some of the superconformal models proposed by Kazama and Suzuki [46] has been found in ref. [4] [*]. As another application of our techniques, we will focus on an interesting subclass of such models given by the level-1 superconformal grassmannian coset models

$$\mathcal{G}/\mathcal{H} = SU(n+m)/SU(m) \otimes U(n), \qquad c = \frac{3nm}{n+m+1},$$

perturbed by the most relevant operator. Again, these models are solvable as quantum field theories and related to $N = 0$ quantum Toda systems [42].

[*] Actually this has been conjectured for many cases but not proven in full generality yet.

Let us summarize it in a way convenient for our purposes. We assume, with no loss of generality, that $m \geqslant n$. We start with n fields Y_k $(k = 1, \cdots, n)$ with charge $q = 1/(n + m + 1)$ and consider the elementary symmetric functions

$$X_i = \sigma_i(Y_k) \equiv \sum_{1 \leqslant l_1 < l_2 < \ldots l_i \leqslant n} Y_{l_1} Y_{l_2} \ldots Y_{l_i} \quad (i + 1, \ldots, n). \tag{7.8}$$

Then take the function

$$W_f(Y_k) = \frac{1}{n + m + 1} \sum_k Y_k^{n + m + 1}.$$

By the fundamental theorem on symmetric functions, it can be rewritten (in a unique way) as a quasi-homogeneous polynomial in the $\sigma_i(Y)$, i.e. in terms of t he X_i one finds

$$W_f(Y_k) = f^*W(X),$$

where the map f is given by eq. (7.8). The function $W(x)$ so obtained is the superpotential for the grassmannian model. Thus the canonical branched covering of the grassmannian model is just n copies of the A_{n+m} minimal model. To check this picture of coset models, let us compute their central charge, using the formula for the change of c under covering maps, eq. (5.8). One has

$$J = \det\left(\frac{\partial X_i}{\partial Y_j}\right) = \Delta(Y_1, \ldots, Y_n),$$

where $\Delta(Y_i)$ is the Vandermonde determinant. Then

$$q_f(J) = \frac{n(n-1)}{2(n + m + 1)} \quad \Rightarrow \quad c = \frac{3nm}{n + m + 1},$$

as it should.

As perturbed superpotential we take

$$W(X_i, t) = W(X_i) - tX_1. \tag{7.9}$$

By going to the canonical covering, we get

$$W_f(Y_k, t) = f^*W(X, t) = \sum_{k=1}^{n} \left(\frac{Y_k^{n+m+1}}{n + m + 1} - tY_k\right).$$

Thus the perturbed model goes over to n copies of the already solved perturbed A_{n+m} minimal model. The ground state metric for W_f is just the product of the known one for each factor.

Now the metric for the grassmannian models can be obtained using change of variables. Let $P_r(X_i)$ $(r = 1, \ldots, (n + M)!/n!m!; i = 1, \ldots, m)$ be a set of polynomials making up a basis for the chiral ring $\mathscr{R}$ of the models in (7.9). Then eq. (5.6) gives,

$$\langle P_s \mid P_r \rangle = (1/n!)\langle \Delta(Y)P_r(\sigma_i(Y)) \mid \Delta(Y)P_s(\sigma_j(Y)) \rangle_f$$

(here $\langle \cdot \mid \cdot \rangle_f$ denotes the known metric for W_f).

By the same token, we can also solve the grassmannian models perturbed by the operator $(X_1^2 - 2X_2)$. Indeed,

$$f^*\left[W(X_i) - t\left(X_1^2 - 2X_2\right)\right] = \sum_{k=1}^{N} \left[\frac{Y_k^{n+m+1}}{n+m+1} - tY_k^2\right]$$

and we are reduced to n copies of the model we solved in subsect. 7.4.

7.6. PARTIALLY ABELIAN MODELS

In addition to the models that can be reduced to Today systems there are those for which the ground state metric decomposes in two "non-interacting" sectors one of which can be recast in a Toda form. Many of these models can be related to theories leading to Toda equations by a simple change of variables. Then the sector arising as the pull-back of the simpler theory "decouples" and has the Toda form.

There are however, other more interesting examples. We make no attempt to completeness, but we merely mention an example to show how it works.

Consider the model

$$W = X^4/4 + Y^4/4 + Z^4/4 - tXYZ.$$

It has a $\mathbb{Z}(4) \otimes \mathbb{Z}(4)$ discrete symmetry. Using the rules of sect. 6, one finds

$$H = SO(3) \otimes U(2)^3 \otimes SO(2)^3 \otimes U(1)^3.$$

The part of the metric corresponding to the "abelian" part of H, $SO(2)^3 \otimes U(1)^3$, (corresponding to 12 chiral primary operators out of 27) decouple from the rest, and hence it is Toda. What is remarkable, is that the ground state metric for these 12 operators is a rational function of the metric for the theory with $W = X^3 - tX$.

8. The magic of the solutions

Up to now we have just discussed how equations take, for special models, the form of interesting differential systems of mathematical physics, typically Toda

equations. However, the real magic of the ground state geometry appears only when we consider the corresponding *solutions*.

In particular, we want to illustrate how the conditions we have already stated uniquely fix the metric. Basically, the requirement that g is a non-singular positive-definite metric will fix it uniquely. Thus, in particular, the boundary conditions for the differential equations are *predicted*. These boundary conditions correspond to the values of the ground state metric for the unperturbed conformal theory which is well understood. For the models of sect. 7, this implies that the absolute normalization of the OPE coefficients for, say, the minimal models can be deduced from our equations as the unique boundary condition allowed by regularity. This will be shown here and, in a more general class of examples, in sect. 9. On the other hand, the behaviour as $|t| \to \infty$ should be the semiclassical one, as described at the end of sect. 4. Thus the equations also encode in a beautiful way the geometry of solitons in the theory. Finally, the unique solution should also lead to the correct behaviour for the algebraic c-function.

8.1. THE MODEL $W = X^3/3 - tX$

Consider the first model in (7.1). The equation in this case is $\hat{A}_1$ Toda, i.e. the sinh-Gordon equation. We know that the metric is a function of $|t|$ only. Let $|t|^2 = x$ and $y(x) = \langle 1|1 \rangle$. Then the equation becomes

$$\frac{d}{dx}\left(x\frac{d}{dx}\log y\right) = y^2 - \frac{x}{y^2}.$$

Consistency requires that, as $t \to 0$, we get back the result for the A_2 minimal model, i.e.

$$y^2(t = 0) = \frac{\langle 1|1 \rangle}{\langle 0|0 \rangle} = 3^{2/3}\left[\frac{\Gamma(2/3)}{\Gamma(1/3)}\right]^2. \tag{8.1}$$

On the other hand, as $t \to \infty$, the two classical vacua at $X = \pm\sqrt{t}$ decouple. Denoting by $l_\pm$ the corresponding chiral primary operators (the "point" basis) we must have

$$\langle l_\pm | l_\pm \rangle = \frac{1}{2|t|^{1/2}} + \ldots,$$

$$\langle l_\pm | l_\mp \rangle = \frac{\beta}{2|t|^{1/2}}z^{-1/2}\exp[-2z] + \ldots$$

where

$$z = |W(\sqrt{t}) - W(-\sqrt{t})| = \tfrac{4}{3}|t|^{3/2} \equiv \tfrac{4}{3}x^{3/4},$$

and β is some numerical coefficient. β is real by "Weil symmetry". Its sign would be predicted by the "c-theorem". Since

$$1 = l_+ + l_-, \qquad X = \sqrt{t}\,(l_+ - l_-),$$

we get

$$y^2(x \sim \infty) \simeq \sqrt{x}\left[1 - 2\beta\sqrt{\tfrac{3}{4}}\,x^{-3/8}\exp\left(-\tfrac{8}{3}x^{3/4}\right) + \ldots\right].$$

We write

$$y^2(x) = \sqrt{x}\,Y^2\left(\tfrac{4}{3}x^{3/4}\right), \tag{8.2}$$

where $Y(z)$ satisfies *

$$Y'' = \frac{(Y')^2}{Y} - \frac{Y'}{z} + Y^3 - \frac{1}{Y}. \tag{8.3}$$

This is just the special third Painlevé transcendent equation (PIII). The general form of this equation is

$$Y'' = \frac{(Y')^2}{Y} - \frac{Y'}{z} + \frac{1}{z}(\alpha Y^2 + \beta) + \gamma Y^3 + \frac{\delta}{Y},$$

the special case corresponds to $\alpha = \beta = 0$, $\gamma = -\delta = 1$.

Our metric $y(x)$ should be regular, real and strictly positive on the positive real axis. The solutions to this equation without poles on the positive real axis are well known. Following ref. [15] we introduce the function

$$u(z) = 2 \log Y(z).$$

u is a solution to the self-similar sinh-Gordon equation

$$u_{zz} + \frac{u_z}{z} = 4 \sinh(u).$$

In ref. [15] it is shown that this equation arises from an isomonodromy problem. In fact, it turns out that the associated isomonodromy ($=$ zero-curvature) problem is nothing else than our linear problem (3.11), for the model at hand. Indeed, let

$$z = \tfrac{4}{3}x^{3/4}, \qquad \lambda = -\tfrac{3}{2}i\frac{1}{(\bar{t})^{3/2}},$$

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* It is assuming that this very same equation is satisfied by the spin–spin correlation functions of the 2d Ising model off criticality [47].

and make the "gauge transformation"

$$\psi \to \left(1/\sqrt{2}\,\right)\sigma_3(1 + i\sigma_2)\; \mathrm{e}^{\frac{1}{4}\sigma_3 \log t}\psi.$$

In the new variables, the linear problem becomes

$$\partial_z\psi = \partial_\lambda\psi = 0,$$

with

$$\partial_\lambda = \frac{\partial}{\partial\lambda} + \frac{zu'(z)}{4}\sigma_1 + i\frac{z^2}{\lambda^2}\sigma_3 \cosh u(z) - \frac{1}{\lambda^2}\sinh u(z),$$

$$\partial_z = \frac{\partial}{\partial z} + \tfrac{1}{2}u'(x)\sigma_1 + \tfrac{1}{2}iz\lambda\sigma_3,$$

which is the isomonodromy problem discussed in ref. [15]. The relevant monodromy which remains constant is precisely the monodromy of the period-map Π for the SQM vacuum wave-forms introduced in sect. 5. In fact, this is true for the general case. The linear problem (the generalized Gauss–Manin connection) is always an isomonodromy problem for the SQM period map Π. Exploiting this interpretation of the equation, one finds the properties of its solutions [15].

   The real solutions (for which the origin is not an accumulation point of poles *) are classified by their asymptotic behaviour as $z \to 0$

$$u(z) \simeq r \log z + s + \mathrm{O}(z^{2-|r|}) \quad \text{for } |r| < 2,$$

$$u(z) \simeq \pm 2 \log z \pm 2 \log\!\left[-\left(\log \tfrac{1}{2}z + C\right)\right] + \mathrm{O}(z^4 \log^2 z) \quad (r = \pm 2), \quad (8.4)$$

(C is the Euler constant). For each pair (r, s) with $|r| \leqslant 2$ there is a solution. A real solution is regular (no poles on the positive real axis) if and only if the two boundary data r and s are related by the equation

$$\mathrm{e}^{s/2} = \frac{1}{2^r}\frac{\Gamma\!\left(\tfrac{1}{2} - \tfrac{1}{4}r\right)}{\Gamma\!\left(\tfrac{1}{2} + \tfrac{1}{4}r\right)}. \tag{8.5}$$

So, requiring regularity fixes s as a function of r. Note that a regular solution $Y(s)$ has no zero on the positive real axis. Indeed, Y^{-1} is also a solution of eq. (8.3), with just the opposite signs for r and s. Since (8.5) is invariant under this change of signs, Y^{-1} has no poles and hence Y no zeros.

---

* By "pole" we mean a pole of the associated Painlevé transcendent of the third kind $Y(z)$.

The connection formula for PIII states that the asymptotic behaviour of these real solutions as $z \to \infty$ is

$$u(z) \sim \frac{\alpha(r)}{z^{1/2}} \exp[-2z], \quad z \to \infty, \tag{8.6}$$

where

$$\alpha(r) = -\frac{2}{\sqrt{\pi}} \sin\left(\frac{\pi r}{4}\right).$$

From eq. (8.2) one gets

$$u(z) = 2 \log\langle 1|1\rangle(z) - \tfrac{2}{3} \log(\tfrac{3}{4}z).$$

Since the ground state metric is regular and non-zero as $z \to 0$, we have

$$r = -\tfrac{2}{3},$$

$$s = 2 \log\langle 1|1\rangle|_{t=0} - \tfrac{2}{3} \log \tfrac{3}{4}.$$

Using the regularity condition (8.5) one gets

$$\frac{\langle 1|1\rangle}{\langle 0|0\rangle} = 3^{2/3} \left[\frac{\Gamma(\tfrac{2}{3})}{\Gamma(\tfrac{1}{3})}\right]^2 \left(1 + O(|t|^2)\right),$$

in agreement with eq. (8.1).

More generally, all the elements of g for the A_n minimal models can be obtained (in fact in many ways) from regularity constraints on the solutions of our equations.

On the other hand, the asymptotic behaviour predicted by eq. (8.6) precisely matches with that predicted by semiclassical arguments (cf. appendix B). The sign of the asymptotic behaviour of u may be surprising at first, since a naive classical picture might suggest the opposite one. In fact, the intuitive picture would apply to the leading semiclassical correction, which in this case just vanishes by supersymmetry. The sub-leading one has a sign which cannot be inferred by classical ideas. However, the sign is fixed from the point of view of the c-theorem. Let us work in the point basis, normalizing $l_{\pm}$ so that $\det g = 1$. Then the metric reads

$$g = \exp[-u(z)\sigma_3/2],$$

By the redefinition $X \to t^{3/2}X$, we put W in the standard form with an overall coupling $\lambda = t^{3/2}$. Then the charge matrix q introduced in sect. 4 becomes

$$q = \tfrac{1}{4}\sigma_3 z \frac{\partial u(z)}{\partial z}.$$

So the algebraic c-function is

$$c = -\tfrac{3}{2}z \frac{\partial u(z)}{\partial z},$$

as $z \to 0$, we get $c \to 1$, and as $z \to \infty$, $c \to 0$, as expected. The derivative of c with respect to the scale is

$$\frac{\partial c}{\partial z} = -6z \, \sinh(u).$$

c is stationary only if $z = 0$ or $u = 0$. $u = 0$ implies

$$\langle l_{\pm} \, | \, l_{\mp} \rangle = 0,$$

i.e. the "classical" theory. In between, c is obviously monotonic with the scale. Since for $z \to \infty$ we have $c = 0$, for large, but finite z, c should be a small *positive* number. Using the asymptotic expansion (8.6) we get

$$c \simeq \left(3/\sqrt{\pi}\right) z^{1/2} \exp[-2z] > 0.$$

If the leading behaviour of u had the opposite sign, c would be negative in this regime. Thus the c-theorem explains physically the peculiar sign of the "instanton" correction.

8.2. OTHER MODELS LEADING TO SPECIAL PIII

In the list of models discussed in sect. 7 there are other whose equations can be reduced to special PIII.

The first one is

$$W(X) = \frac{X^4}{4} - \frac{t}{2}X^2.$$

Again we put $x = |t|^2$ and $y(x) = (\langle 0 | \rangle)^{-1}$. Then this equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x \frac{\mathrm{d}}{\mathrm{d}x}\log y\right) = \frac{1}{4}\left(y^2 - \frac{x^2}{16}\frac{1}{y^2}\right). \tag{8.7}$$

By the redefinition

$$y = \sqrt{z}\, Y(z), \qquad z = \tfrac{1}{4}x$$

we reduce eq. (8.7) to the standard form of special PIII, (8.3).

For $t = 0$ we have

$$y^2(0) = \frac{\langle 2|2 \rangle}{\langle 0|0 \rangle} \bigg|_{t=0} = \left[2 \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right]^2.$$

The soliton mass is

$$2 | W(\sqrt{t}) - W(0) | = \tfrac{1}{2} |t|^2 \equiv 2z.$$

As in the above model we put

$$u(z) = 2 \log Y(z) = 2 \log y(z) - \log z.$$

So $u(z)$ is the solution to PIII with

$$r = -1,$$

$$s = 2 \log y(0) \equiv 2 \log 2 + 2 \log \Gamma(\tfrac{3}{4}) - 2 \log \Gamma(\tfrac{1}{4}).$$

These numbers satisfy the regularity condition (8.5) (i.e. $y^2(0)$ is predicted by regularity alone). The large-t expansion is

$$y(|t|') \simeq \frac{|t|}{2} \left(1 + \sqrt{\frac{2}{\pi}} \frac{2}{|t|} \exp\left[-|t|^2/2 \right] + \cdots \right)$$

in agreement with the semiclassical analysis.

By the same token as in the previous model, the c-function reads

$$c = -\tfrac{3}{2} z \frac{\partial}{\partial z} u(z). \tag{8.8}$$

In this case, as $z \to 0$ we get $c = 3/2$, as we should. The comments above on the sign of the "instanton" corrections apply to the present model as well.

Note that the boundary data r is (essentially) the central charge at the UV fixed point. That is, the UV central charge is a monodromy data (basically, the Stokes multiplier). The condition $|r| < 2$ is just

$$c < 3,$$

i.e. restricts to the minimal models! Then the PIII regularity condition (8.5) can be seen as saying that in order to have a regular solution $\exp[s]$ should be the OPE coefficient appropriate for the given central charge. These remarks will become clear in full generality in sect. 9.

Another model that can be reduced to special PIII is

$$W(X) = \frac{X^6}{6} - \frac{t}{2} X^2.$$

The matrix elements

$$\langle 1|1\rangle \quad \text{and} \quad \langle 3|3\rangle$$

can be obtained from the $X^3/3 - tX$ model by a change of variable

$$f: \quad X \to X^2. \tag{8.9}$$

Then there remains a single unknown function

$$y(x) = (\langle 0|0\rangle)^{-1},$$

which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\log y\right) = \tfrac{1}{4}y - \frac{x}{16}\frac{1}{y}.$$

At $t = 0$ we must have

$$y^2(0) = \frac{\langle 4|4\rangle}{\langle 0|0\rangle}\bigg|_{t=0} = \left[6^{2/3}\frac{\Gamma(\tfrac{5}{6})}{\Gamma(\tfrac{1}{6})}\right]^2.$$

Putting

$$y = \sqrt{\tfrac{1}{4}x}\,Y^2(z), \qquad z = \tfrac{1}{3}x^{3/4},$$

we get again special PIII for $Y(z)$. Then

$$u(z) = 2\log Y(z) = \log y - \tfrac{2}{3}\log z - \tfrac{2}{3}\log 3 + \log 2.$$

which gives

$$r = -\tfrac{2}{3},$$

$$s = \log y(0) - \tfrac{2}{3}\log 3 + \log 2.$$

Since r is as in the cubic model, $Y(s)$ – if regular – should be the same. Thus regularity implies an algebraic relation between the two independent elements of the ground state metric.

One has

$$|W(t^{1/4}) - W(0)| = \tfrac{1}{3}|t|^{3/2} \equiv z,$$

so the large-t behaviour is again the correct one.

By the same argument as above, we have

$$c = -3z\frac{\partial}{\partial z}u(z).$$

(The factor 2 with respect to eq. (8.8) is due to the fact that now $(\langle 0|0\rangle)^{-1}$ is proportional to $Y^2(z)$ rather than $Y(z)$). So, as a function of z the central charge is just twice that of the perturbed A_2 model which, pulled back by the map (8.9), gives the present model. In particular, for $t = 0$ we get $c = 2$, as we should.

There are other models whose equations can be reduced to special PIII. A very important class will be discussed in sect. 9. There are a few other models that we omit for brevity. We have explicitly checked that all these models satisfy the regularity and consistency criteria.

8.3. THE MODEL $W = X^4/4 - tX$

Next we consider the model leading to $\widehat{BC_1}$ Toda. Putting $y = \langle 2|2\rangle$ and $x = |t|^2$, we get

$$y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{y^2}{x} - \frac{1}{y},$$

which is again a special case of the third Painlevé equation, with $\alpha = -\delta = 1$, $\beta = \gamma = 0$. This is the so-called "degenerate" PIII. Putting

$$\tau = \tfrac{9}{16}x^{4/3}, \qquad \log y = u(\tau) + \tfrac{1}{4}\log(\tfrac{16}{9}\tau), \qquad (8.10)$$

we recast this equation in the form of the self-similar Bullough–Dodd equation

$$(\tau u_\tau)_\tau = e^u - e^{-2u}.$$

The properties of the asymptotically regular solutions were studied in ref. [16], again by the isomonodromic deformation method. It turns out that these solutions are parametrized by four complex numbers g_1, g_2, g_3, and s satisfying

$$g_1 + g_2(1 - s) + g_3 = 1, \qquad g_2^2 - g_1 g_3 = g_2,$$

so we have a two-dimensional manifold of solutions. From eq. (8.10) we see that regularity implies that, as $\tau \to 0$,

$$\exp[u] \sim \frac{\text{const.}}{\tau^{-1/4}}.$$

This selects $s = 1$. In this case, one has

$$\exp[u] \sim \tfrac{9}{8} r_1 \frac{C_2}{C_0} \tau^{-1/4} \quad \tau \to 0,$$

where (for $s = 1$)

$$\frac{C_2}{C_0} = \frac{8}{3^{3/2}} \frac{\Gamma\left(\tfrac{3}{4}\right)}{\Gamma\left(\tfrac{1}{4}\right)},$$

$$r_1 = g_3 - g_1 + (1 - i)(g_1 - g_2).$$

To fix the residual ambiguity of the solution, we require that, as $\tau \to \infty$, there are no exponentially growing terms (i.e. no negative-mass solitons). Then one gets

$$g_1 = g_2 = 0, \quad g_3 = 1 \quad \Rightarrow \quad r_1 = 1,$$

and the solution is uniquely fixed.

At this point, both the value of the metric at $t = 0$ and the strength of the "instanton" correction are *predicted*. One gets

$$\langle 2 | 2 \rangle |_{t=0} = 2 \frac{\Gamma\left(\tfrac{3}{4}\right)}{\Gamma\left(\tfrac{1}{4}\right)},$$

the expected value. The asymptotical expansion for $\tau \to \infty$ is

$$\exp[u(\tau)] \simeq 1 + \frac{1}{2} \sqrt{\frac{3}{\pi}} (3\tau)^{-1/4} e^{-2\sqrt{3\tau}} + \ldots .$$

This is the correct strong-coupling behaviour, because

$$z \equiv |W(t^{1/3}) - W(e^{2\pi i/3} t^{1/3})| = \frac{3\sqrt{3}}{4} |t|^{4/3} = \sqrt{3\tau}$$

and the coefficient in front of the exponential agrees with the soliton picture discussed in appendix B.

Again one has

$$c(\tau) = -3z \frac{\partial u}{\partial z} \equiv -6\tau \frac{\partial u(\tau)}{\partial \tau}.$$

As $\tau \to 0$, we get $c = 3/2$, the correct value. To the best of our knowledge, no mathematician has ever studied in detail the properties of the higher equations in sect. 7. However, we can easily work the other way around, namely, start from the

known physical properties of the metric and deduce the corresponding mathematical theorems, analogous to the above ones for the $\hat{A}_1$ and the $\widehat{BC}_1$ cases. In some sense, this is just what a mathematician would do. In fact, the known results are obtained by exploiting the isomonodromic method, which is somehow built-in the physical approach.

9. Models associated to Verlinde rings: the $SU(2)_k$ case

Recently Gepner [18] has shown that the Verlinde rings of some rational CFTs have the same algebraic structure as the chiral rings of the $N = 2$ LG models, namely they are polynomial rings modulo the ideal generated by the derivatives of a certain superpotential $W(X_i)$. This has been considered further recently [48,49]. The main case considered in ref. [18] is that of $SU(N)_k$ theories. From the $N = 2$ viewpoint, the corresponding superpotentials correspond to particular (relevant) perturbations of $N = 2$ coset models. Then it is natural to ask whether, for these special perturbations, the equations for g (as we vary the RG scale) are "solvable" in the sense that they can be reduced to Toda. The answer to this question is yes! Moreover, the trick to solve them is based on the interpretation of the corresponding $\mathcal{R}$'s as fusion rings. In particular, for the model associated to the $SU(N)_k$ Verlinde ring the ground state metric is written in terms of k linearly independent [*] solutions to the (self-similar) affine $SU(N)$ Toda equations.

In this section we discuss in detail the $SU(2)_k$ situation, the generalization to arbitrary N being discussed in sect. 10. In this case, the superpotentials are the Chebyshev polynomials [18]

$$W_K(X) = \lambda T_{k+1}(X), \quad \text{where} \quad T_m(\cos Y) = \cos(mY).$$

Rescaling the field X, we see that as the coupling $\lambda \to 0$ one gets back the minimal model A_k, which is equivalent to the grassmannian model at level 1

$$SU(k+1)_1/U(k).$$

The fact that one gets Chebyshev polynomials is remarkable, since for these polynomials the SQM Schrödinger equation is separable, and hence the ground state metric is computable *by brute force*. In fact, separability for the SQM Schrödinger equation (with one field) is equivalent to separability for the 2d Helmholtz equation (related in turn to SU(2) Toda). However, the corresponding wave functions are not very managable, so it is more convenient to use the information coming from separability to simplify our equations, rather than to compute g directly. It has not yet been shown, in the sense of having infinitely

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[*] However, the reality constraint gives non-linear algebraic relations between these solutions.

many conserved currents, that the Chebyshev perturbation of minimal models is in that class, but the fact that we find an affine Toda equation even for this case suggests that this must be true. In fact for the A_n model $W = X^{n+1}$ it has only been shown that X and X^2 perturbations are integrable [33,42], and it was suspected that perturbation by X^{n-1} is also integrable. Chebyshev perturbation to leading order (as $\lambda \to 0$) is of this type. So what we are finding is that this is, to leading order, integrable but to get it to be fully integrable it must be "dressed" by lower-dimension operators which make it become precisely the Chebyshev polynomial. It would be very interesting to verify this by studying perturbation theory near the conformal point.

The method we use for solving the Chebyshev models is again using the change of variables trick discussed in sect. 5. This will in fact allow to solve them all at once. We take

$$W = \lambda T_n(X),$$

$$f = \cos(Y/n) \equiv X,$$

$$W_f(Y) = \lambda \cos(Y). \qquad (9.1)$$

Then, if we are able to compute the ground-state metric for the $N = 2$ sine-Gordon model, $W_f(Y)$, we get all Chebyshev superpotentials at once by truncation to the operators $\phi_k \in \mathscr{R}_f$ of the form

$$\phi_k(Y) = P_k(\cos(Y/n)) \sin(Y/n),$$

where $P_k(X)$ are polynomials of degree $k \leqslant n - 2$.

9.1. $N = 2$ SINE-GORDON

For the sine-Gordon model we identify an element of $\mathscr{R}$ with the set of its values at the (non-singular) critical points (the "point" basis). For $W_f(X)$ the critical points are

$$X_r = \pi r, \quad r \in \mathbb{Z},$$

and we identify an element $\phi \in \mathscr{R}_f$ with the sequence

$$\{(\phi)_r \equiv \phi(\pi r), r \in \mathbb{Z}\}.$$

The ring operations act componentwise on ϕ. One has (using definition (2.10))

$$\mathrm{Res}[\phi] = \frac{1}{\lambda} \sum_{r \in \mathbb{Z}} (-1)^{r+1} (\phi)_r.$$

We choose as basis in $\mathscr{R}_f$ the elements a_k $(k \in \mathbb{Z})$ such that

$$(a_k)_r = \delta_{kr}. \tag{9.2}$$

In this basis we have

$$\eta_{kh} = (-1)^{(k+1)} \frac{1}{\lambda} \delta_{kh},$$

$$(C_\lambda)_k^h = (-1)^k \delta_k^h.$$

The superpotential (9.1) is invariant (up to phase) for

$$T: \quad Y \to Y + \pi,$$

$$P: \quad Y \to -Y.$$

Then, in our basis one has

$$g_{i+1,\overline{j+1}} = g_{i,j},$$

$$g_{-i,-j} = g_{i,j}. \tag{9.3}$$

Given an integer i, there is a unique decomposition

$$i = \langle i \rangle + 2\{i\}, \quad \text{with} \quad \langle i \rangle = 0, 1.$$

Using (9.3) we write

$$g_{ij} = g_{\langle i \rangle \langle j \rangle}(\{i\} - \{j\}),$$

and introduce its Fourier series

$$g_{\langle i \rangle \langle j \rangle}(\theta) = \sum_r e^{i\theta r} g_{\langle i \rangle \langle j \rangle}(r).$$

Next, we consider the 2×2 matrix $(0 \leqslant \theta \leqslant 2\pi)$

$$g(\theta) = \begin{pmatrix} g_{0\bar{0}}(\theta) & g_{0\bar{1}}(\theta) \\ g_{1\bar{0}}(\theta) & g_{1\bar{1}}(\theta) \end{pmatrix}.$$

Eq. (9.3) implies

$$g_{0\bar{0}}(\theta) = g_{1\bar{1}}(\theta),$$

$$g_{0\bar{1}}(\theta) = e^{i\theta} g_{1\bar{0}}(\theta),$$

and

$$g_{0\bar{0}}(\theta) = g_{0\bar{0}}(-\theta),$$

$$g_{1\bar{0}}(\theta) = e^{-i\theta} g_{1\bar{0}}(-\theta).$$

Then we can parametrize the metric as

$$g(\theta) = \begin{pmatrix} A(\theta) & e^{i\theta/2}B(\theta) \\ e^{-i\theta/2}B(\theta) & A(\theta) \end{pmatrix},$$

where

$$A(\theta) = A(-\theta), \qquad B(\theta) = B(-\theta).$$

The transpose and the conjugate of the ground state metric in terms of the 2×2 matrix $g(\theta)$ read

$$g^{\mathrm{T}}(\theta) = [g(-\theta)]^{\mathrm{T}}, \qquad g^*(\theta) = [g(-\theta)]^*.$$

Then

$$g^{\dagger}(\theta) = [g(\theta)]^{\dagger},$$

and $g(\theta)$ is hermitian in the 2×2 sense. Therefore

$$A(\theta) = A(\theta)^*, \qquad B(\theta) = B(\theta)^*.$$

Moreover, $A(\theta) > 0$, since the metric is positive.

Finally, we must impose the "real structure" constraint on $g(\theta)$, namely

$$\eta^{-1}(\theta)g(\theta)(\eta^*)^{-1}(\theta)g^*(\theta) = \mathbb{1}. \qquad (9.4)$$

In the 2×2 notation, one has

$$\eta(\theta) = \frac{1}{\lambda}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C_{\lambda}(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So eq. (9.4) reduces to

$$|\lambda|^2\left(A(\theta)^2 - B(\theta)^2\right) = 1.$$

Therefore, we can parametrize $g(\theta)$ in terms of a single function of $x(=|\lambda|^2)$

$$A(x, \theta) = (1/\sqrt{x}) \cosh[L(x, \theta)],$$

$$B(x, \theta) = (1/\sqrt{x}) \sinh[L(x, \theta)].$$

Putting everything together, we get

$$g(x, \theta) = (1/\sqrt{x})U(\theta) \exp[\sigma_1 L(x, \theta)]U(\theta)^{-1},$$

where

$$U(\theta) = \exp(\tfrac{1}{4}i\theta\sigma_3).$$

Now,

$$\partial_{\bar{\lambda}}[g \, \partial_\lambda g^{-1}](\theta) = -U\sigma_1 U^{-1} \partial_\lambda \, \partial_{\bar{\lambda}} L(x, \theta),$$

$$[G_\lambda, \overline{G}_\lambda](\theta) = -2U\sigma_1 U^{-1} \sinh[2L(x, \theta)],$$

and the final equation reads

$$\partial_\lambda \, \partial_{\bar{\lambda}} L(x, \theta) = 2 \sinh[2L(x, \theta)],$$

i.e. for each θ, $2L(x, \theta)$ is a self-similar solution to the sinh-Gordon equation and we are back with our old friend the special PIII. To put the equation in canonical form, let

$$2L(x, \theta) = u(z, \theta) \quad \text{where} \quad z = 2x^{1/2}.$$

For $z \to 0$ we have the asymptotics (cf. sect. 8)

$$u(z, \theta) \simeq r(\theta) \log z + s(\theta) + \ldots, \quad \text{with} \quad |r(\theta)| \leqslant 2,$$

that is

$$L(x, \theta) \simeq \tfrac{1}{4}r(\theta) \log x + \tfrac{1}{2}[s(\theta) + r(\theta) \log 2] + \ldots,$$

whereas for $x \to \infty$ we get (cf. (8.6))

$$L(x, \theta) \simeq \frac{\alpha(\theta)}{2\sqrt{2}} \frac{\exp(-4x^{1/2})}{x^{1/4}} \equiv \frac{\alpha(\theta)}{2\sqrt{2} \, |\lambda|^{1/2}} e^{-4|\lambda|}.$$

Notice that the exponent is precisely the soliton mass

$$2|\Delta W| = 2|\lambda| \, |\cos(k\pi) - \cos((k+1)\pi)| = 4|\lambda|,$$

in agreement with the semiclassical picture.

To specify completely the metric for $N = 2$ sine-Gordon, it remains only to fix the boundary conditions, i.e. the function $r(\theta)$. This will be done below.

In terms of $L(x, \theta)$, the point-basis metric reads

$$g_{kj}(x) = \frac{1}{4\pi\sqrt{x}} \int_0^{2\pi} d\theta \; e^{i\theta(j-k)/2} \left\{ \exp[L(x, \theta)] + (-1)^{(j-k)} \exp[-L(x, \theta)] \right\}.$$

$$(9.5)$$

Since $g(\theta)$ is periodic with period 2π, one has

$$L(x, \theta + 2\pi) = -L(x, \theta),$$

$$L(x, -\theta) = L(x, \theta).$$

In particular,

$$L(x, \pi) = 0.$$

9.2. BACK TO CHEBYSHEV

Now we return to the original Chebyshev superpotentials,

$$W = \lambda T_n(X).$$

The critical points are

$$X_r = \cos\left(\frac{r\pi}{n}\right) \quad r = 1, \ldots, n - 1.$$

Again we work in the point basis. We denote by l_r the chiral field with value 1 at the rth critical point and zero elsewhere. From each l_r, by pull-back, we get a chiral primary operator in the sine-Gordon theory. Taking into account the jacobian, we get ($j = 1, \ldots, n - 1$)

$$f^* l_j = -\frac{1}{n} \sin\left(\frac{\pi}{n} j\right) \sum_{r \in \mathbb{Z}} [a_{2nr+j} - a_{2nr-j}],$$

where a_k is as in (9.2).

Then eq. (9.5) gives

$$2\delta(0)\langle l_j^* l_k \rangle = \frac{1}{n^2} \sin\left(\frac{\pi}{n} j\right) \sin\left(\frac{\pi}{n} k\right)$$

$$\times \sum_{r,s \in \mathbb{Z}} [g_{2nr+j,\overline{2ns+k}} - g_{2nr-j,\overline{2ns+k}} - g_{2nr+j,\overline{2ns-k}} + g_{2nr-j,\overline{2ns-k}}],$$

where $2\delta(0)$ is the degree of the cover. The sums in the r.h.s. can be computed via the Poisson formula

$$\sum_{r,s \in \mathbb{Z}} g_{2nr+j, \overline{2ns+k}} = \frac{1}{2n\sqrt{x}} \delta(0) \sum_{r=0}^{n-1} e^{i\pi r(k-j)/n} \left\{ \exp\left[L\left(x, \frac{2\pi r}{n}\right)\right] \right.$$

$$\left. + (-1)^{(k-j)} \exp\left[-L\left(x, \frac{2\pi r}{n}\right)\right] \right\}.$$

Putting everything together, we get the ground state metric for the model $W = \lambda T_n(X)$,

$$\langle l_j^* l_k \rangle = \frac{1}{n^3 \sqrt{x}} \sin\left[\frac{\pi}{n} j\right] \sin\left[\frac{\pi}{n} k\right]$$

$$\times \sum_{r=1}^{n-1} \sin\left(\frac{\pi}{n} rk\right) \sin\left(\frac{\pi}{n} rj\right) \left[e^{L(x, 2\pi r/n)} + (-1)^{(k-j)} e^{-L(x, 2\pi r/n)}\right], \quad (9.6)$$

which expresses the metric as a combination of a finite number of solutions to special PIII. All these solutions are bounded for $x \to \infty$ and regular on the positive real axis. Taking into account that

$$L(x, 2\pi - \alpha) = -L(x, \alpha),$$

we see that the metric for the T_n-model involves $[(n-1)/2]$ independent solutions to PIII. In particular, for $n = 2$ we have just elementary functions, and for $n = 3, 4$ we have a single Painlevé transcendent. This is in full agreement with previous work, since T_2 is equivalent to the free theory, $W = X^2/2$, T_3 is equivalent to $W = X^3/3 - tX$, and T_4 to $W = X^4/4 - tX^2/2$. These last two models have already been solved in sect. 8 in terms of a single Painlevé transcendent. In fact, by going through the field redefinitions needed to put these superpotentials in the standard form (paying attention to the "anomalous" jacobian) one checks that for $n = 2, 3, 4$ the above results reproduce the results of sects. 7 and 8. For brevity, we omit the details of this check.

9.3. REGULARITY VERSUS BOUNDARY CONDITIONS

As in sect. 8, the boundary condition $r(\theta)$ is fixed by requiring that the metric is finite and non-zero as $\lambda \to 0$. Then the value of $s(\theta)$ is predicted by the condition of no pole on the positive real axis. We recall that for $W = Y^n$ the ground state metric reads

$$\langle Y^k | Y^k \rangle = \Gamma\left(\frac{k+1}{n}\right) \bigg/ n\Gamma\left(1 - \frac{k+1}{n}\right) \quad (k = 0, \ldots, n-2). \quad (9.7)$$

One has

$$T_n(X) = 2^{n-1}X^n + k_{n-2}X^{n-2} + \dots.$$

The field redefinition

$$Y \equiv 2(\lambda/2)^{1/n} X,$$

puts the superpotential in the form

$$W = \lambda T_n(X) = Y^n + O(\lambda^{2/n}).$$

Consistency requires that, as $\lambda \to 0$, the Chebyshev metric reproduces (9.7).

The critical points for $T_n(X)$ are $X_k = \cos(k\pi/n)$. Then in the point basis the monomials $X^k \in \mathcal{R}$ read as

$$X^k = \sum_{r=1}^{n-1} \left[\cos\left(\frac{r\pi}{n}\right) \right]^k l_r \qquad (k = 0, 1, \dots, n-2).$$

Taking into account the jacobian, one has

$$\langle Y^k | Y^h \rangle = \left[2(\bar{\lambda}/2)^{1/n} \right]^{k+1} \left[2(\lambda/2)^{1/n} \right]^{h+1} \langle X^k | X^h \rangle.$$

Let us define the sums

$$A_{k,t} = \sum_{r=1}^{n-1} \cos^k\left(\frac{\pi}{n}r\right) \sin\left(\frac{\pi}{n}r\right) \sin\left(\frac{\pi}{n}rt\right),$$

$$B_{k,t} = \sum_{r=1}^{n-1} (-1)^r \cos^k\left(\frac{\pi}{n}r\right) \sin\left(\frac{\pi}{n}r\right) \sin\left(\frac{\pi}{n}rt\right).$$

Explicitly, one has

$$A_{0,t} = \tfrac{1}{2}n\left[\delta_{(t)_n,1} - \delta_{(t)_n,2n-1} \right],$$

and, for $k \neq 0$

$$A_{k,t} = \frac{n}{2^{k+2}}\left[1 - (-1)^{(k+t)} \right] \left\{ \left(\begin{array}{c} k \\ \tfrac{1}{2}(k+t-1)_n \end{array} \right) - \left(\begin{array}{c} k \\ \tfrac{1}{2}(k+t+1)_n \end{array} \right) \right\},$$

where $(a)_n$ is a short-hand notation for the unique number $0 \leqslant (a)_n < 2n$, which is congruent to a modulo $2n$. Moreover,

$$B_{k,t} = A_{k,t+n}.$$

Putting everything together, the metric in the monomial basis reads

$$\langle Y^k \mid Y^h \rangle = \frac{1}{n^3 \mid \lambda \mid} \left[2(\bar{\lambda}/2)^{1/n} \right]^{k+1} \left[2(\lambda/2)^{1/n} \right]^{(h+1)}$$

$$\times \sum_{t=1}^{n-1} \left[A_{k,t} A_{h,t} \, e^{[L(x,2\pi t/n)]} + B_{k,t} B_{h,t} \, e^{-[L(x,2\pi t/n)]} \right].$$

The coefficients $A_{k,t}$, $B_{k,t}$ satisfy the "selection rules" (for $0 < t < n$)

$$\begin{aligned} A_{k,t} &= 0 \quad \text{for} \quad t > k + 1, \\ B_{k,t} &= 0 \quad \text{for} \quad t < n - 1 - k, \end{aligned} \tag{9.8}$$

The first non-vanishing coefficients are

$$A_{k,k+1} = -B_{k,n-1-k} = n/2^{k+1}. \tag{9.9}$$

A consequence of the selection rules is that $\langle 1 \mid 1 \rangle$ is equal to (up to trivial factors) $\exp[L(x, 2\pi/n)]$, i.e. it is expressed in terms of a single Painlevé transcendent. More generally, the matrix element $\langle Y^k \mid Y^h \rangle$ involves, at most, $\min(k + 1, h + 1)$ transcendents.

The asymptotic behaviour of the diagonal elements of the metric as $\lambda \to 0$ is

$$\langle Y^k \mid Y^k \rangle \simeq \left(1/n^3 \right) \left(2^{1-1/2} \right)^{2k+2} \mid \lambda \mid^{((2k+2)/n)-1}$$

$$\times \left[\sum_{t=1}^{n-1} A_{k,t}^2 (2 \mid \lambda \mid)^{r(2\pi t/n)/2} \, e^{s(2\pi t/n)/2} \right.$$

$$\left. + \sum_{t=1}^{n-1} B_{k,t}^2 (2 \mid \lambda \mid)^{-r(2\pi t/n)/2} \, e^{-s(2\pi t/n)/2} \right].$$

Using the selection rules, the requirement that the r.h.s. has a finite non-zero limit, gives

$$r\left(\frac{2\pi}{n} t \right) = 2\left(1 - \frac{2t}{n} \right) \quad (t = 1, \dots, n - 1). \tag{9.10}$$

Note that in particular $\mid r \mid < 2$, as required by regularity. Assuming that the solutions are regular, we get (8.5)

$$\exp\left[\tfrac{1}{2} s\left(\frac{2\pi}{n} t \right) \right] = 2^{(4t-2n)/n} \Gamma\left(\frac{t}{n} \right) \Big/ \Gamma\left(1 - \frac{t}{n} \right).$$

This, using (9.9), implies

$$\langle Y^k \mid Y^k \rangle|_{\lambda \downarrow 0} = (1/n^3) 2^{(n-1)(2k+2)/n} \left[A_{k,k+1}^2 + B_{k,n-k-1}^2 \right] 2^{(n-2k-2)/n} \, e^{[s(2\pi(k+1)/n)/2]}$$

$$= (1/n) \left(\Gamma \left(\frac{k+1}{n} \right) \bigg/ \Gamma \left(1 - \frac{k+1}{n} \right) \right),$$

in full agreement with eq. (9.7). Moreover, the off-diagonal elements

$$\langle Y^k \mid Y^h \rangle \quad k \neq h,$$

go to zero in this limit, as they should. Therefore regularity implies the correct boundary conditions for Chebyshev superpotentials. It is amusing that *all the normalization coefficients of the A_m minimal models can be deduced from regularity theorems on Painlevé transcendents of third kind and vice versa.*

It remains to specify the boundary conditions for the solution of the $N = 2$ sine-Gordon model. We assume that $r(\theta)$ is a continuous (albeit not smooth) function of θ. From eq. (9.10) we know it at all rational values of θ/π. Then it should be

$$r(\theta) = 2 \left(1 - \frac{\theta}{\pi} \right) \quad \text{for} \quad 0 \leqslant \theta \leqslant 2\pi.$$

Outside this interval, the function is obtained by using

$$r(\theta) = -r(\theta + 2\pi), \quad r(\theta) = r(-\theta).$$

Then the bound-state metric for the $N + 2$ sine-Gordon is completely determined. Note that $|r(\theta)| \leqslant 2$, and that *all* the regular solutions to special PIII appear in the metric for the $N + 2$ sine–Gordon model. The points $\theta = 2\pi k$ where $|r(\theta)| = 2$ coincide with the points where $r(\theta)$ is not smooth. These are also the points where $L(x, \theta)$ even if continuous in θ changes its asymptotic behaviour for $\lambda \to 0$ (cf. sect. 8). At the se points one has "logarithmic violations of scaling". This is precisely the boundary condition satisfied by the Ising model correlation functions [47].

9.4. STRONG-COUPLING LIMIT

Let us take the limit $\lambda \to \infty$. in this limit the various vacua at different critical points, $X_k = \cos(\pi k/n)$, decouple (up to exponentially small corrections corresponding to soliton corrections). Then we must have

$$\langle l_j^* l_k \rangle \simeq \frac{\delta_{jk}}{|W''(X_k)|} + \frac{\alpha_{jk}}{\sqrt{|W''(X_j)W''(X_k)|}} \frac{1}{\sqrt{2|\lambda|}} e^{-4|\lambda|},$$

for certain constants α_{jk}. Since

$$W''(X_k) = \lambda n^2 \frac{(-1)^{k+1}}{\left[\sin\left(\frac{\pi}{n}k\right)\right]^2},$$

we must get

$$\langle l_j^* l_k \rangle = \frac{\sin(\pi k/n) \sin(\pi j/n)}{n^2 |\lambda|} \left(\delta_{jk} + \alpha_{jk} \frac{1}{\sqrt{2|\lambda|}} e^{-4|\lambda|} + \ldots \right). \quad (9.11)$$

Using the asymptotics of $u(z, \theta)$, eq. (8.6), and the identity (valid for j, $k = 1, \ldots, n - 1$)

$$\frac{1}{n}\left[1 + (-1)^{(k-j)}\right] \sum_{r=1}^{n-1} \sin\left(\frac{\pi}{n}rk\right) \sin\left(\frac{\pi}{n}rj\right) = \delta_{k,j},$$

the r.h.s. of eq. (9.6) for large λ has the behaviour of eq. (9.11) with

$$\alpha_{j,k} = \frac{1}{2n}\left[1 - (-1)^{(k-j)}\right] \sum_{s=1}^{n-1} \sin\left(\frac{\pi}{n}sk\right) \sin\left(\frac{\pi}{n}sj\right) \alpha\left(\theta = \frac{2\pi s}{n}\right)$$

$$= -\frac{1}{2n\sqrt{\pi}}\left[1 - (-1)^{(k-j)}\right] \sum_{s=0}^{2n-1} \sin\left(\frac{\pi}{n}sk\right) \sin\left(\frac{\pi}{n}sj\right) \cos\left(\frac{\pi}{n}s\right)$$

$$= -\frac{1}{2\sqrt{\pi}}\left(\delta_{j,k+1} + \delta_{k,j+1}\right),$$

in agreement with the results of sect. 8 and appendix B.

9.5. THE c-FUNCTION

Next we consider the c-function. By the same agreement as in sect. 8, for the T_n model we have ($z \equiv 2|\lambda|$)

$$c(z) = \tfrac{3}{2}z\frac{\partial}{\partial z}u(z, 2\pi/n),$$

(in particular for $n = 2$, c is identically zero, and for $n = 3, 4$ it is just what we got in sect. 8). This follows from the fact that the Ramond operator associated to 1 is the one with lowest charge. As z goes to 0, we get for the UV central charge

$$c_{\text{uv}} = \tfrac{3}{2}r\left(\frac{2\pi}{n}\right) = 3\left(1 - \frac{2}{n}\right),$$

which is the well known result for the A_{n-1} minimal model. The leading correction to this result is of order $|\lambda|^{4/n}$, i.e. the modulus square of the perturbation.

The "running" U(1) charges of the Ramond ground state are

$$q_k(z) = \tfrac{1}{4} z \frac{\partial}{\partial z} u(z, 2\pi k/n) \quad (k = 1, 2, \ldots, n-1).$$

As $z \to 0$, we get back the result of the A_{n-1} minimal model, whereas as $z \to \infty$ they all go to zero, as they should since the IR fixed point is trivial.

For the $N = 2$ sine-Gordon theory itself, we have

$$c(z) = \tfrac{3}{2} z \frac{\partial}{\partial z} u(z, 0),$$

which in the UV limit gives $c = 3$. However, now the corrections are logarithmic,

$$c(z) \simeq 3 + \frac{3}{2(\log z + C)}, \quad z \to 0.$$

It is tempting to speculate about the relation of this logarithmic scaling violation with the ones appearing in 2d gravity at $c = 1$. This is in particular tempting in view of the conjecture of Li [9] about the relation of topological $N = 2$ minimal models with 2d quantum gravity.

All the discussion in sect. 8 on the properties of these c-functions applies word-for-word to the present general case.

9.6. VARIATIONS ON THE THEME

One interesting aspect of the equations for g is that they have a tendency to reproduce nice field equations. For example, above we got the equations of 2d sinh-Gordon. There are other models leading to even more suggestive equations. As a divertissement we present a class of model which lead to 3d chiral models.

We consider the multicritical sine-Gordon models. By this we mean a model which has the same critical points as the sine-Gordon one, but with a multiplicity $\mu > 1$. All the critical points are assumed to have the same multiplicity μ. For simplicity, we assume μ to even $(= 2m)$. Then the superpotential is

$$W(X) = \lambda \int \sin^{2m} X \, dX$$

$$= \frac{\lambda}{2^{2m}} \frac{(2m)!}{(m!)^2} X + \frac{\lambda}{2^{2m}} \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{(m-k)} \binom{2m}{k} \sin[2(m-k)X],$$

which has the pseudosymmetries

$$X \to X + k\pi, \qquad X \to -X. \tag{9.12}$$

An element $\phi \in F$ is uniquely specified by its $(2m - 1)$-jets at the critical points, i.e. by the set of data

$$\left\{ \phi(k\pi), \partial\phi(k\pi), \frac{1}{2!}\partial^2\phi(k\pi), \ldots, \frac{1}{(2m-1)!}\partial^{(2m-1)}\phi(k\pi) \mid k \in \mathbb{Z} \right\},$$

(the "point" basis). Then F is identified with this set of numbers written as a two-index object

$$\phi \equiv (\phi)_{k,r} \quad k \in \mathbb{Z}, \, r = 1, \ldots, 2m.$$

In this notation the ring product reads

$$(\phi\psi)_{k,r} = \sum_{s=1}^{2m} (\phi)_{k,s}(\psi)_{k,r-s}.$$

Consider the ground state metric in such a basis $g_{i,r;\bar{j},s}$. From (9.12) we have

$$g_{i+1,r;\overline{j+1,s}} = g_{i,r;\overline{j,s}},$$

$$g_{-i,r;\overline{-j,s}} = (-1)^{(r+s)} g_{i,r;\overline{j,s}}.$$

As above we introduce the Fourier transform

$$g_{i,r;\overline{j,s}} \equiv g_{r,\bar{s}}(i - j), \qquad g_{r\bar{s}}(\theta) = \sum_k e^{ik\theta} g_{r\bar{s}}(k).$$

The $2m \times 2m$ matrix $g(\theta)$ satisfies

$$g(-\theta) = \Sigma_3 g(\theta) \Sigma_3, \tag{9.13}$$

where

$$\Sigma_3 = \mathrm{diag}(1, -1, 1, -1, \ldots, 1, -1).$$

In this notation, the residue pairing is

$$\eta(\theta) = (1/\lambda)\Sigma_1,$$

where

$$(\Sigma_1)_{ij} = \delta_{i+j,2m+1}.$$

As in the sine-Gordon case, we have

$$[g(\theta)]^\dagger = g(\theta).$$

The reality structure constraint reads

$$\Sigma_1 g(\theta) \Sigma_1 g(-\theta)^* = \frac{1}{|\lambda|^2} \, \mathbb{1}$$

Let $\mathscr{G}(\theta) = |\lambda| \, g(\theta)$. Then the above equation becomes

$$\mathscr{G}(\theta) \Sigma_1 \mathscr{G}(-\theta)^T = \Sigma_1,$$

or, using eq. (9.13),

$$\mathscr{G}(\theta) \Omega \mathscr{G}(\theta)^T = \Omega,$$

where

$$\Omega = -\Sigma_1 \Sigma_3$$

is a symplectic matrix. Hence

$$\mathscr{G}(\theta) \in \mathrm{Sp}(2m).$$

The matrix C_λ reads

$$(C_\lambda)_{k,r}^{h,s} = \delta_k^h \frac{(2n)!}{2^{2m}(m!)^2} \pi k,$$

or, in the θ basis,

$$C_\lambda = -i\pi \frac{(2m)!}{2^{2m}(m!)^2} \frac{\mathrm{d}}{\mathrm{d}\theta}.$$

To save print we put

$$z = \lambda \frac{(2m)!}{2^{2m}(m!)^2}.$$

Then the equations become

$$\partial_{\bar{z}}\left[\mathscr{G}(\theta) \, \partial_z \mathscr{G}(\theta)^{-1} \right] = -\pi^2 \frac{\mathrm{d}}{\mathrm{d}\theta} \left[\mathscr{G}(\theta) \frac{\mathrm{d}}{\mathrm{d}\theta} \mathscr{G}(\theta)^{-1} \right].$$

Putting

$$x_1 = \mathrm{Re}\; z, \qquad x_2 = \mathrm{Im}\; z, \qquad x_3 = \theta/2\pi,$$

and using the fact that $\mathscr{G}$ is invariant under rotations in the $(1, 2)$ plane, this equation is rewritten as ($\mu = 1, 2, 3$)

$$\partial_\mu \left[\mathscr{G}\, \partial_\mu \mathscr{G}^{-1} \right] = 0, \qquad \mathscr{G} \in \mathrm{Sp}(2m),$$

which are the field equations of the (complexified) $\mathrm{Sp}(2m)$ principal chiral model in three dimensions. This is the model corresponding to the lagrangian

$$\mathscr{L} = \mathrm{Tr}\left[\partial_\mu \mathscr{G}\, \partial_\mu \mathscr{G}^{-1} \right].$$

Of course, the metric is a very special solution to these field equations. $\mathscr{G}$ should be a positive hermitian matrix, invariant under rotations in the $(1, 2)$ plane, periodic with respect to translations in the orthogonal direction, and such that

$$\mathscr{G}(x_1, x_2, -x_3) = \Sigma_3 \mathscr{G}(x_1, x_2, x_3)\Sigma_3.$$

Nevertheless, it is amusing that we get a formal "unification" of the coupling constant λ with θ which labels the different critical points!

10. Generalization to SU(N)$_k$

In this section we generalize the results of sect. 9 to arbitrary $\mathrm{SU}(N)_k$. The ground state metric of the associated models will be expressed as a finite combination of (self-similar) solutions to $\hat{A}_{N-1}$ Toda theory.

10.1. N CHEBYSHEV POLYNOMIALS

We start by describing the superpotentials corresponding to $\mathrm{SU}(N)_k$ Verlinde rings, i.e. the generalization of Chebyshev polynomials to arbitrary N. These superpotentials are closely related to those for the grassmanian cost models of sect. 7, and indeed reduce to them in the UV limit.

Following Gepner [18], we introduce the variables q_i $(i = 1, \ldots, N)$. These variables are subject to the constraint

$$\prod_{i=1}^{N} q_i = 1. \tag{10.1}$$

As in sect. 7, we denote by $\sigma_r(q_i)$ the rth elementary symmetric function of the q_i. Obviously, $\sigma_N(q_i) = 1$.

The superpotential corresponding to the $SU(N)_k$ Verlinde ring

$$W_{N,k}(X_1, X_2, \ldots, X_{N-1})$$

is the unique polynomial such that

$$W_{N+k}(\sigma_1(q), \sigma_2(q), \ldots, \sigma_{N-1}(q)) = \frac{\lambda}{N+k} \sum_{i=1}^{N} q_i^{N+k},$$

the only difference with respect to the grassmannian case being the constraint (10.1). Of course, this is a major difference since it spoils quasi-homogeneity. These polynomials are mutually orthogonal with respect to the L^2-measure defines [*] by the weight $\sqrt{\Delta(q_i)}$ and obey the recursion relation

$$(m+N)W_{m+N}(X_j) + \sum_{i=1}^{N-1} (-1)^i X_i(m+N-i)W_{m+N-1}(X_j) + (-1)^N mW_m(X_j) = 0.$$

Let us parametrize q_i as $(m = N + k)$

$$q_i = \exp\left[\frac{1}{m}(\phi_i - \phi_{i-1})\right] \quad i = 1, 2, \ldots, N,$$

with the understanding that

$$\phi_0 = \phi_N = 0.$$

Let $f_{(m)}$ be the map

$$X_r = \left(f_{(m)}(\phi_j)\right)_r \equiv \sigma_r\left(\exp\left[(\phi_j - \phi_{j-1})/m\right]\right).$$

Then,

$$f_{(m)}^* W_m = \frac{\lambda}{m}\left[e^{\phi_1} + \sum_{i=1}^{N-2} e^{(\phi_{i+1} - \phi_i)} + e^{-\phi_{N-1}}\right],$$

which, up to an obvious field redefinition, is just the $N = 2$ $SU(N)$ Toda superpotential. Then, by a change of variables, to solve the problem for $W_{N,k}(X_i)$ it is enough to compute the ground state for the supersymmetric Toda models. The jacobian is again $\Delta(q_j)$, the Vandermonde determinant.

___

[*] As in sect. 7, $\Delta(q_i)$ is the Vandermonde determinant.

10.2. $N = 2$ TODA THEORIES

We are reduced to compute the ground state metric for the $N = 2$ SU(N) Toda theories,

$$W(\phi_1, \phi_2, \ldots, \phi_{n-1}) = \frac{\lambda}{N}\left[e^{\phi_1} + \sum_{i=1}^{N-2} e^{(\phi_{i+1} - \phi_i)} + e^{-\phi_{N-1}}\right].$$

This model has two symmetries:

$$\phi_r \to \phi_r + i\frac{2\pi}{N}rk + 2\pi il_r, \quad \text{with} \quad k = 0, 1, \ldots, N-1, \quad l_r \in \mathbb{Z},$$

and

$$\phi_j \to \phi_{N-j}.$$

The critical points correspond to the orbit of the origin with respect to the first symmetry. Then a critical point is labelled by the numbers

$$(k, l_1, l_2, \ldots, l_{N-1}), \quad k = 0, 1, \ldots, N-1, \quad l_r \in \mathbb{Z}.$$

As usual, we denote by $a_{(k,l_r)}$ the chiral operator with value 1 at the given critical point and zero elsewhere. The value of W at the critical point (k, l_r) is

$$W_{(k,l_r)} = e^{2\pi ik/N}.$$

So,

$$(C_\lambda)_{(k,l_r)}^{(h,m_s)} = e^{2\pi ik/N}\delta_{(k,l_r)}^{(h,m)},$$

and the residue pairing is

$$\eta_{(k,l_r);(h,m_s)} = C_N\, e^{2\pi ik/N}\delta_{(k,l_r);(h,m_s)}.$$

Here C_N is a numerical constant depending on N only

$$(C_N)^{-1} = (1/N)^{N-1}\det[C_{ab}],$$

where C_{ab} is the SU(N) Cartan matrix.

The above symmetries imply the following conditions on the metric

$$\langle(k, l_r)|(h, m_s)\rangle = \langle(k, l_r + a_r)|(h, m_s + a_s)\rangle, \quad a_r \in \mathbb{Z},$$

$$\langle(k, l_r)|(h, m_s)\rangle = \left\langle\left(\{k+p\}, l_r + \left[\frac{k+p}{N}\right]r\right)\middle|\left(\{h+p\}, m_s + \left[\frac{k+p}{N}\right]s\right)\right\rangle$$

($p = 0, \ldots, N - 1$), where $\{a\}$ is the unique number between 0 and $N - 1$ which is congruent to a modulo N. Moreover,

$$\langle (k, l_r) | (h, m_s) \rangle = \langle (k, -k - l_{N-r}) | (h, -k - m_{N-s}) \rangle.$$

The first property allows us to introduce the Fourier transform

$$g_{k\bar{h}}(\theta_1, \ldots, \theta_{N-1}) = \sum_{l_r \in \mathbb{Z}} \exp\left(i \sum_r l_r \theta_r\right) \langle (h, 0) | (k, l_r) \rangle.$$

Then the other two properties read

$$g_{k+1,\overline{h+1}}(\boldsymbol{\theta}) = g_{k\bar{h}}(\boldsymbol{\theta}) \quad (\text{for } 0 \leqslant k, h \leqslant N - 2),$$

$$g_{k+1,\bar{0}}(\boldsymbol{\theta}) = \exp\left[-i \sum_r r\theta_r\right] g_{k,\overline{N-1}}(\boldsymbol{\theta}),$$

$$g_{0,\overline{h+1}}(\boldsymbol{\theta}) = \exp\left[i \sum_r r\theta_r\right] g_{N-1,\bar{h}}(\boldsymbol{\theta}),$$

$$g_{k\bar{h}}(\theta_1, \theta_2, \ldots, \theta_{N-1}) = \exp\left[-i(h - k) \sum_r \theta_r\right] g_{k\bar{h}}(-\theta_{N-1}, -\theta_{N-2}, \ldots, -\theta_1).$$

To put the equations in the Toda form, we have to diagonalize the $N \times N$ matrix $g(\boldsymbol{\theta})$. It has the structure

$$g_{k\bar{h}}(\boldsymbol{\theta}) = A_{(h-k)}(\boldsymbol{\theta}) + \exp\left[-i \sum_r r\theta_r\right] A_{(N+h-k)}(\boldsymbol{\theta}),$$

where

$$A_h(\boldsymbol{\theta}) = \begin{cases} g_{0\bar{h}}(\boldsymbol{\theta}) & \text{for } h = 0, 1, \ldots, N - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Given the peculiar structure of $g(\boldsymbol{\theta})$, its diagonalization is elementary. We introduce a new basis in $\mathscr{R}$ ($k = 0, \ldots, N - 1$)

$$\psi_k(\boldsymbol{\theta}) = \sum_{r=0}^{N-1} \exp\left[\frac{ir}{N}\left(2\pi k + \sum_s s\theta_s\right)\right] a_r(\boldsymbol{\theta}),$$

$$a_r(\boldsymbol{\theta}) = \sum_{l_s \in \mathbb{Z}} \exp\left[i \sum_s l_s \theta_s\right] a_{(r,l_s)}.$$

In this basis, the ground state metric is diagonal, indeed

$$\left\langle \psi_k(\theta) \middle| \psi_h(\theta') \right\rangle = \delta(\theta - \theta')\delta_{h,k} N \sum_{l=0}^{N-1} \exp\left[\frac{-il}{N}\left(2\pi k + \sum_s s\theta_s \right)\right] A_l(\theta).$$

In the new basis,

$$W\psi_k(\theta) = \psi_{\{k+1\}}(\theta),$$

i.e.

$$\left(C_\psi\right)_k^h(\theta) = \delta_{\{k+1\}}^h.$$

Therefore, for each value of $\theta_1, \dots, \theta_{N-1}$ the ground state metric $\mathscr{G}_k(\theta)$,

$$\left\langle \psi_h(\theta) \middle| \psi_k(\theta') \right\rangle = \delta(\theta' - \theta)\delta_{kh}\mathscr{G}_k(\theta),$$

satisfies the $\hat{A}_{N-1}$ Toda equation,

$$-\partial_{\bar\lambda}\partial_\lambda \log \mathscr{G}_k(\theta) = \frac{\mathscr{G}_{\{k+1\}}(\theta)}{\mathscr{G}_k(\theta)} - \frac{\mathscr{G}_k(\theta)}{\mathscr{G}_{\{k-1\}}(\theta)}.$$

However, in this basis the residue pairing is rather involved,

$$\mathrm{Res}\left[\psi_k(\theta)\psi_h(\theta')\right] = \delta(\theta' - \theta)C_N \sum_{r=0}^{N-1} \exp\left\{ \frac{ir}{N}\left[2\pi(k+h+1) + 2\sum_s s\theta_s \right]\right\},$$

so the reality constraint is not as simple as in sect. 9. Notice that – contrary to the SU(2) case – the reality constraint gives $\mathscr{G}(-\theta)$ in terms of $\mathscr{G}(\theta)$ instead of putting a condition on the metric for fixed θ.

This completes the argument showing that for $N = 2$ *quantum* SU(N) affine Toda, associated to SU(N)$_k$ Verlinde rings, the ground state metric can be written as a finite combination of solutions to the *classical* $\hat{A}_{N-1}$ (self-similar) affine Toda equation. Here we see the group SU(N) in operation in three seemingly unrelated ways!

11. Conclusions

We have seen that the metric on the space of ground state vacua of $N = 2$ QFTs can in principle be determined by solving certain interesting differential equations which express the flatness of certain holomorphic and antiholomorphic connections for the vacuum bundle over the parameter space. Not surprisingly, this

flatness condition reduces in special cases to well known systems of equations of mathematical physics (of the Toda type) which are expressible in the Lax form. In examples which lead to equations which had been studied by mathematicians, we were able to reproduce some of their results, derived from isomonodromic deformation techniques, from a purely $N = 2$ QFT point of view. The generalizations that this $N = 2$ point of view would naturally lead to, are yet to be verified using the isomonodromic deformation techniques.

The system of equations that we have used does not distinguish a "preferred" direction of perturbation, and in a sense treats all the directions on the same footing. This is partly a surprise, because only very special directions are integrable QFT's in the sense of having infinitely many conserved current *. It is precisely in these cases that our equations reduce to equations of the Toda type. Nevertheless it is natural to study the full space of perturbations. In particular it should be possible to flow from one conformal theory to another conformal theory and see how the OPE of the two theories are *predicted* by self-consistency, and in particular by the absence of singularity in the solution to the differential equations. The examples leading to affine Toda are always massive at the IR, and unfortunately do not provide any examples of this type.

We have seen that some examples of $N = 2$ theories whose rings are the same as the rings of RCFT $(SU(N)_k)$ lead to affine Toda equations. Is this a general property? Is it true that each case where Verlinde ring of a RCFT can be represented by the chiral ring of an $N = 2$ theory the equations we get are integrable and lead to Toda equations? Is it true that each time our equations are of the toda type we can interpret the ring as that of a RCFT? These are mysterious links between a *conformal* theory (RCFT) and a *massive* $N = 2$ theory, which deserve a serious study. Could it be that $N = 2$ theories lead to knot invariants in three dimensions through this link? (if this were true singularity theory might be connected to knot invariants). Do the $N = 2$ theories admit a direct three-dimensional interpretation?

We have seen that the affine Toda equations that characterize the metric encode a lot of the information about the solitons in the theory. Can one *derive* the soliton scattering amplitudes from this viewpoint using the techniques of thermodynamic Bethe ansatz [51]? The discussion in appendix B points in this direction.

Many of our constructions work for Donaldson theory and is worth investigating. This might lead to a simpler derivations of Ward identities in the context of $N = 2$ supersymmetric Yang–Mills theories [52]. This would be interesting to study.

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* It would be interesting to see if one can imbed this in an integrable setup by infinitely extending the number of couplings, similar to what one has in matrix models [50]. We would like to thank authors of the first reference in [50] for discussions on this point.

It is our distinct feeling that we have only found the tip of an iceberg. There are too many different things being related in too many seemingly accidental ways for there not to be a bigger story. We hope that this will motivate further study to find this bigger story.

Appendix A. The ground state metric in the critical regime

At a conformal point W is quasi-homogeneous. In this case one can give explicit representations of the metric in terms of integrals of holomorphic forms. Basically, this is the generalization of Gepner's correspondence for minimal models: at criticality an $N = 2$ model is related to a σ-model and thus can be studied by complex geometry techniques. There are three (equivalent) formulations of these integral representations:

(i) In terms of the integrals ($\{\phi_k\}$ a holomorphic basis of $\mathscr{R}$)

$$\varpi_k{}^j \equiv \int_{\gamma_j^+} e^{-W} \phi_k \, dX_1 \wedge \ldots \wedge dX_n. \tag{A.1}$$

(ii) In terms of the period integrals for the pure (p, q) components of the groups

$$H^{n-2}(E_1) \otimes H^{n-1}(E_2),$$

where E_i are the (weighted) projective manifolds

$$\begin{aligned} E_1: &\quad W(X_i) = 0, \\ E_2: &\ W(X_i) + X_{n+1}^d = 0. \end{aligned} \tag{A.2}$$

(iii) For marginal operators the ground state metric is Kähler. The Kähler potential has the representation

$$e^{-K} = \int d^n X \, d^n \overline{X} \, \exp\left[W(X) - \overline{W}(\overline{X}) \right], \tag{A.3}$$

which can be rewritten as a bilinear form in the integrals of point i) as explained in sect. 4.

To simplify the arguments notice that (without loss of generality) we can assume W to be homogeneous. Indeed let the fields X_i have U(1) charge $q_i = r_i/d$. Then make the change of variables

$$X_i = Y_i^{r_i}.$$

In terms of the new fields W is homogeneous, and the original ground state metric is related to the new one as in sect. 5.

A part of the above statements is elementary, Indeed, we known that (for marginal deformations) the metric is Kähler. Then it is elementary to show that

$$e^{-K(t,\bar{t})} = \sum_{k,h=1}^{\Delta} I_{kh}\chi_k(t_a)\left[\tilde{\chi}_h(t_b)\right]^*,$$

where I_{kh} is the intersection matrix and $\chi_k(t_a)$, $\tilde{\chi}_k(t_a)$ are holomorphic. In fact (cf. sect. 4) $\exp[-K] = \langle 0|0\rangle$, and (sect. 5)

$$\langle 0|0\rangle = \sum_{h,k} \rho_{hk}\left[\int_{\gamma_k^-} e^{W+\overline{W}}\omega_0\right]\left[\int_{\gamma_h^+} e^{-W-\overline{W}} * \omega_0\right]^*. \tag{A.4}$$

Then it remains to show that

$$\int_{\gamma_k^-} e^{W+\overline{W}}\omega_0, \qquad \int_{\gamma_h^+} e^{-W-\overline{W}} * \omega_0, \tag{A.5}$$

are holomorphic. Indeed

$$0 = \tilde{\partial}_a \int_{\gamma_k^+} e^{-W-\bar{W}} * \omega_0 = \bar{\partial}_a \int_{\gamma_k^+} e^{-W-\overline{W}} * \omega_0,$$

since $(\overline{C}_a)_0^k = 0$ by charge conservation. The same argument (using the dual connection $\bar{\partial}'$) works for the other integral in (A.5).

According to the discussion in sect. 4, to prove eq. (A.3) it remains to show that in (A.4) one can replace the integrals of the vacuum wave-forms with those of the corresponding holomorphic forms. The proofs are hidden in ref. [14]. Here we try to present them in a more "physical" form. We have already mentioned that the basic flatness equations

$$\partial\Pi = \bar{\partial}\Pi = 0, \tag{A.6}$$

have the same general structure as Toda's. In the case of (quasi) homogeneous W they are analogous to the non-affine Toda, and hence can be solved by the usual

Leznov–Saveliev method [39]. One starts from the Gauss decomposition of Π,

$$\Pi = e^D AB$$

(here B is an upper-triangular * matrix, A is a lower-triangular one and D is block-diagonal). In terms of B one gets simpler equations

$$\bar{\partial} B = \bar{\partial} D = 0,$$

$$(\partial + e^{-D} C\, e^D) B = 0. \tag{A.7}$$

The crucial point of the method is that, once we are given an upper-triangular matrix B satisfying (A.7) (for some D), we can reconstruct the full solution by Lie-algebraic techniques.

A direct computation gives

$$(\partial_a + C_a) \varpi = L_a \varpi, \tag{A.8}$$

with L_a zero above the diagonal. Now, consider the Gauss decomposition of ϖ,

$$\varpi = e^{\tilde{D}} \tilde{A} \tilde{B}.$$

Eq. (A.8) implies that $\tilde{B}$ is a solution to eq. (A.7) (with $D = \tilde{D}$). Thus, out of the periods ϖ we can reconstruct a solution to our equations. The hard part of the argument is to show that this solution coincides with the one given by the SQM "period map" Π. We postpone the discussion of this point to the end.

Then one has

$$\Pi = e^F \mathcal{N} \varpi, \tag{A.9}$$

with F block-diagonal and holomorphic and $\mathcal{N}$ *strictly* lower-triangular, i.e. $\mathcal{N} = 1 + Z$, with Z decreasing the charge by one or more units. The first component of (A.9) gives

$$\int_{\gamma_k^+} e^{-W - \bar{W}} * \omega_0 = \exp\big(F_0^0(t) \big) \int_{\gamma_k^+} e^{-W} \, dX_1 \wedge \ldots \wedge dX_n.$$

Analogously,

$$\int_{\gamma_k^-} e^{W + \bar{W}} \omega_0 = \exp\big(\hat{F}_0^0(t) \big) \int_{\gamma_k^-} e^{W} \, dX_1 \wedge \ldots \wedge dX_n.$$

---

* By upper-(lower-)triangular matrix we mean the identity plus the matrix of an operator which increases (decreases) the U(1) charge. It is actually block-triangular.

Then

$$
e^{-K} = \exp\left(\left(F_0^0\right)^* + \hat{F}_0^0\right) \sum_{h,k} \rho_{kh} \int_{\gamma_h^-} e^W \, dX_1 \wedge \ldots
$$

$$
\wedge \, dX_n \left[\int_{\gamma_k^+} e^{-W} \, dX_1 \wedge \ldots \wedge dX_n\right]^*.
\tag{A.10}
$$

This, together with the discussion in sect. 4 shows property (iii). (The factor in front of the sum can be re-absorbed by a Kähler gauge transformation). That ρ can be identified with the inverse intersection matrix C^{ij} can be seen by the same argument used in appendix C to show eq. (4.4).

A slight generalization of this argument leads to eq. (4.1). Let $\phi_j(X)$ be the *relevant* chiral operators with U(1) charges r_j/d ($0 < r_j < d$). Consider the auxiliary superpotential

$$
W_{\text{aux}}(X_k, Y; t_a, s_i) = W(X_k, t_a) + Y^d + \sum_j s_j \phi_j(X) Y^{d-r_j}.
$$

W_{aux} is quasi-homogeneous and the couplings s_j are moduli. So the above analysis applies. As $s_j \to 0$ the field Y decouples and then

$$
\left\langle \overline{\phi_j Y^{d-r_j}} \phi_i Y^{d-r_i} \right\rangle_{\text{aux}}\Big|_{s_l=0} = \left\langle \overline{\phi_j} \phi_i \right\rangle \left\langle \overline{Y^{d-r_j}} Y^{d-r_i} \right\rangle_d,
\tag{A.11}
$$

where $\langle \ldots \rangle_d$ denotes the metric for the A_{d-1} minimal model. On the other hand, the l.h.s. of eq. (A.11) is equal to

$$
-\langle 0|0 \rangle_{\text{aux}} \, \partial_{\bar{s}_j} \, \partial_{s_i} \, \log\langle 0|0 \rangle_{\text{aux}}\Big|_{s_l=0}.
$$

Replacing the integral representation (A.3) for $\langle 0|0 \rangle_{\text{aux}}$ and neglecting terms which vanish by symmetry reasons, we get

$$
g_{ij} = \left\langle \overline{\phi_j} \phi_i \right\rangle = \int \prod dX_l \, d\overline{X}_l \, \phi_i(X_k) \overline{\phi}_j(\overline{X}_k) \exp\left[W(X) - \overline{W}(\overline{X})\right].
\tag{A.12}
$$

In this form the equality holds only for *relevant* operators. Let us explain why the irrelevant ones are different. First of all, it would be contradictory to assume eq. (A.12) to be true for all fields. In fact, $\langle \overline{\phi}_j \phi_i \rangle = 0$ if $q_i \neq q_j$, whereas the r.h.s. of eq. (A.12) does not vanish for $q_i - q_j$ *integral*. In other words, the bilinear form in the r.h.s. mixes operators with charges differing by an integral amount. More precisely, an operator ϕ_i of charge q_i gets mixed with operators of *lower* charge $q_i - 1$, $q_i - 2$, ... Only the relevant operators are well defined, whereas the marginal ones can mix only with the identity. In this last case, the problem is

solved by taking the "connected" part of the integral in eq. (A.12), i.e. one takes the *logarithm* of the integral as Kähler potential. The fundamental reason behind this mixing is the dependence on the choice of a particular representative for the classes in $\mathscr{R}$. Under a change of representatives (preserving their U(1) charges)

$$\phi_j \, \mathrm{d} X_1 \wedge \ldots \wedge \mathrm{d} X_n \to \phi_j \, \mathrm{d} X_1 \wedge \ldots \wedge \mathrm{d} X_n + \mathrm{D} w \wedge \alpha_j,$$

the periods ϖ change as

$$\varpi \to \varpi + \mathscr{Z}\varpi,$$

where the matrix $\mathscr{Z}$ decreases the charge by an integral amount. Then mixing in unavoidable unless we have a preferred representative to start with. Instead the SQM period Π is unambiguous since it is defined in terms of given forms. A change of representatives is compensated in eq. (A.9) by a change in the matrix $\mathscr{N}$. Restricting to operators with $0 \leqslant q \leqslant 1$, in eq. (A.9) we can replace $\mathscr{N}$ by 1 and hence effectively identify the period ϖ with the SQM periods Π (F is absorbed in the conventions). This explain why for relevant/marginal operators we get nice formulae and why they do not hold for $q > 1$. In fact in the general case the metric can still be written in terms of ϖ though not so explicitly [*]. The mixing above has deep mathematical meaning. Some aspects are discussed in ref. [14]. To do better than this one has to leave the elementary methods. Luckily the mixing – which at the elementary level is a nuisance – at a more sophisticated level turns into a welcome simplification.

We just sketch the idea of how one can compute the metric for irrelevant operators out of the periods ϖ. More details can be found in ref. [14]. Basically, one has to reconstruct the complete solution of the linear problem (A.6) from its triangular part $\mathrm{e}^D B$. In the Toda case this is done by Lie-theoretical methods [39]. The same applies here, but since in our case H is not abelian (in general) the reconstruction is a bit less elementary. It is convenient to present the tricks in a slightly more abstract language than in the abelian case. From sct. 6 we know that $\psi(t, \bar{t})$ is an element of the group G. So it can be seen as a map from coupling-constant space to the group G. However, it is more convenient to project it to a map ϱ into the coset space [**] G/H. G/H is an open domain in $G_\mathbb{C}/B$ where B is the group of lower triangular matrices (in our sense). This space is obviously a homogeneous complex manifold. In fact, it is the classifying space for complex flags of given type. Over $G\mathbb{C}/B$ we have universal tautological bundles corresponding to these flags. They are homogeneous with respect to the action of $G_\mathbb{C}$ and holomorphic. They have a unique hermitian metric $\langle \cdot \mid \cdot \rangle$ which is homogeneous

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[*] However, for operators with $\hat{c} - 1 \leqslant q \leqslant \hat{c}$ one also has nice expressions. Indeed they can be connected to the relevant ones by the reality constraint. Then for $\hat{c} \leqslant 2$ elementary methods suffice to get all g.

[**] H is assumed to act on the left.

and such that G acts by isometries. Correspondingly there is a unique universal connection which can be constructed by Lie-group techniques. Embedding G/H into G_C/B enlarges the "gauge group" from H to B. Then ψ and its triangular part are related by a gauge transformation, i.e. define the same map *

$$\varrho: \quad \text{couplings} \to G_C/B.$$

In the triangular gauge ψ is holomorphic. Hence the map ϱ is holomorphic. Now, the crucial point is that the ground state metric is precisely the pullback of the universal one *via* the map ϱ. This is a consequence of the fact that the group G acts homogeneously on the ground state metric and hence g must correspond to the unique homogeneous one **. Since the universal one is known, we can reconstruct the full g out of the map ϱ. But the triangular part of ψ is sufficient to specify the map.

   In fact ϱ is not just a holomorphic map, it is also horizontal. By this we mean that it satisfies eq. (A.7). Horizontal maps are very rigid. Then in various situations we have uniqueness theorems for the metric g. Using these results one can show, e.g. that the map ϱ is the direct sum of the periods maps for the projective manifolds E_1 and E_2 defined in (A.2) [14]. Here we want to exploit them to prove that the map ϱ defined by the SQM period map coincides (at criticality) with the one defined by the periods ϖ. A typical rigidity theorem for horizontal maps [24,40] states that two such maps are equal if: (i) they transform the same way under modular transformations and (ii) they agree at a single point in moduli space.

   Then everything is proven if we can show that: (1) under a modular transformation the chiral primary fields transform as the periods ϖ (equivalently, as the periods for the projective manifolds E_i) and (2) that at a particular point in moduli space we have equality between the ground state metric and the metric computed out of the above integrals. Point (1) has been discussed in detail for $\hat{c} = 1$ in ref. [53]. The general proof is very easy. It is enough to check the equality of the monodromy action in the topological theory. In the topological case one can indeed identify the chiral operators with the integrals ϖ (see appendix C). Hence the equality is manifest. To show (2), we assume W to be homogeneous of degree d. Then we consider the family

$$\mathscr{W}(X_i, t; s) = sW(X_i, t) + (1 - s)\sum_i X_i^d.$$

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* ϱ is the period map in the Griffiths sense [40].
** The reader may wonder about the overall normalization of the metric. It is also fixed. Indeed, we know already that, restricting to marginal deformations, the metric is the curvature of a certain line bundle. Then its overall scale is fixed topologically.

It is enough to check equality at $s = 1$. In this case we end up with a bunch of decoupled A_{d-1} minimal models. For the A-series the equality was explicitly checked in ref. [14].

Appendix B. Semiclassical considerations

In this appendix we discuss the leading semiclassical corrections and show the result quoted in eq. (4.7). So we are interested in the limit where the superpotential λW has simple critical points which are very far from each other (in the limit of large λ) and to leading order decouple from one another.

On general grounds one can argue that the leading off-diagonal semiclassical correction to the metric, which to leading order is diagonal in the basis of critical points is a "universal" function of the mass of the soliton interpolating between critical points (in units of inverse length of the cylinder) if there is a soliton connecting the two points. The mass of the soliton has simple dependence on the superpotential and is given by

$$m = 2 \, |\lambda| \, |\Delta W|.$$

In the case of just one field, which we will mainly concentrate on, a precise statement of this universality is as follows [*]. Assume there is a convex domain $\tilde{\Omega} \subset \mathbb{C}$ containing only two (distinct) critical values $W(X_j)$ and $W(X_k)$. Suppose that there is a simply connected domain $\Omega \subset \mathbb{C}$ containing only two critical points ($\equiv$ classical vacua), X_j and X_k, such the $W(\Omega) = \tilde{\Omega}$. Finally, assume that the two Milnor vanishing classes associated with these critical points have an intersection number ± 1 (i.e. in the Dynkin diagram of the polynomial $W(X)$ the two points corresponding to X_j and X_k are connected by a single link). These conditions imply in particular that there exists a soliton connecting the critical points. As before let $|l_j\rangle$ label the critical point basis of chiral fields, i.e. up to topologically trivial terms they are eigenstates of X with eigenvalue X_j. Then, as $\lambda \to \infty$

$$|\lambda| \left[-W''(X_j)\overline{W}''(X_k) \right]^{1/2} \langle l_k | l_j \rangle$$

$$= U\big(2 \, |\lambda| \, |W(X_k) - W(X_j)|\big) + O(\exp[-\mu |\lambda|]), \qquad (B.1)$$

where

$$\mu = \min \left\{ 4 \inf_{W \in \partial \tilde{\Omega}} |W - W(X_k)|, \, 4 \inf_{W \in \partial \tilde{\Omega}} |W - W(X_j)| \right\},$$

_____

[*] More general arguments are available but, unfortunately, they do not give more detailed results.

and $U(m)$ is an universal function. Comparing with the known $W = X^3 - X$ case, we get

$$U(m) = -\int_{-\infty}^{\infty} \frac{dp}{2\pi\sqrt{p^2 + m^2}} \exp\left(-\sqrt{p^2 + m^2}\right) = -\frac{1}{\pi} K_0(m). \qquad \text{(B.2)}$$

Then as $m \to \infty$ we have the asymptotical expansion

$$U(m) \sim -\frac{e^{-m}}{\sqrt{2\pi m}}\left\{1 + \sum_{k=1}^{} \frac{(-1)^k}{k!}\left[(2k-1)!!\right]^2 \frac{1}{(8m)^k}\right\}.$$

Since $\hbar \sim \lambda^{-1}$, the various terms in this expansion can be seen as loop corrections to the one-instanton (soliton) process. It is remarkable that all the perturbative corrections are universal.

So stated, universality can be proven in many ways. We will concentrate on three different ways: The first, and the most direct way, is to use our equation (3.9) in the asymptotic region. The second, is to use WKB approximation to write down the overlap of wave functions based at different critical points – this can be done both in the path-integral language as an instanton sum or in the Schrödinger equation. The third one is not as rigorous, but has the advantage of giving the overall normalization in a simple way and suggesting a physical picture of how the corrections to the metric might be related to a kind of partition function in the soliton subsector *. This is very much in the spirit of the thermodynamic Bethe ansatz [54]. We will discuss these three different view points in turn. At the end of this appendix, as an example, we discuss t he leading correction of the metric for $W = x^{n+1}/(n+1) - x$ in the asymptotic region.

We first show how this universality property can be shown starting from our basic equations (3.9). We present the details of the argument since it can be easily extended to prove more general "universality theorems" for multi-instanton processes. Assume that all the zeros of W' are simple. In this point basis, we rewrite the metric as

$$g = n \exp[\gamma] n^{\dagger},$$

where

$$n_k^h = \frac{\delta_k^h}{\sqrt{\lambda W''(X_k)}}.$$

At the classical level $\gamma = 0$. As $\lambda \to \infty$, γ is dominated by the (leading) 1-instanton contribution. Neglecting terms exponentially suppressed with respect to the lead-

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*   We wish to thank A.B. Zamolodchikov for encouraging us to take this interpretation seriously.

ing instanton, we can work to first order in γ. In this approximation (3.9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\,|\,\lambda\,|^{\,2}}\left(\,|\,\lambda\,|^{\,2}\frac{\mathrm{d}}{\mathrm{d}\,|\,\lambda\,|^{\,2}}\gamma_{jk}\right) = |\,W(X_j) - W(X_i)\,|^{\,2}\gamma_{jk}.$$

Putting

$$\gamma_{jk} = \gamma_{jk}(z_{jk}),$$

$$z_{jk} = 2\,|\,\lambda\,|\,|\,W(X_j) - W(X_k)\,|,$$

one gets

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}z}\gamma(z)\right) = z\gamma(z). \tag{B.3}$$

The general solution to this equation (vanishing as $z \to \infty$) is

$$\gamma_{jk} = \beta_{jk}K_0(z_{jk}), \tag{B.4}$$

and universality is proven up to an overall constant β_{jk}. That this argument does not fix the overall constant was to be expected. In particular, in this argument we did not use the fact that there is a soliton connecting the critical points. If there were no solitons connecting the two critical points, the corresponding β_{jk} would have to vanish. However, in case there exists a soliton connecting the two critical points we would still like to determine the overall constant and show its universality. We accomplish this by showing that in such a case the constant β_{jk} is the same we got for the $X^3 - X$ model (which does have a soliton connecting the critical points).

Consider the auxiliary superpotential

$$W(X; s) = \mu_{kj}W_{kj}(X) + s\big[W(X) - \mu_{kj}W_{kj}(X)\big],$$

where

$$W_{kj}(X) = \tfrac{1}{3}X^3 - \tfrac{1}{2}(X_j + X_k)X^2 + (X_jX_k)X,$$

$$\mu_{kj} = \big[W(X_k) - W(X_j)\big]/\big[W_{kj}(X_k) - W_{kj}(X_j)\big].$$

As $s \to 1$ we get back the original superpotential $W(X)$, whereas for $s \to 0$ we get a cubic one [*]. Note that for this superpotential the mass of the soliton $2\,|\,\Delta W(X; s)\,|$ is independent of s.

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[*] The limit $s \to 0$ is not smooth in general (the Witten index jumps). However, the limit is smooth for the quantities of interest here.

Assume that $W(X)$ is such that, for λ large enough, we can consistently use the linearized approximation in the whole range $0 \leqslant s \leqslant 1$ (this in particular means that there is a soliton in the original theory at $s = 1$). Then the linearized equations read

$$\partial_{\bar{s}} \, \partial_\lambda \gamma_{jk} = \partial_{\bar{\lambda}} \, \partial_s \gamma_{jk} = 0,$$

or, using eq. (B.4)

$$\partial_{\bar{s}} \beta_{jk} = \partial_s \beta_{jk} = 0.$$

Since β_{jk} is independent of s, it takes the same value as in the cubic case, namely $\beta_{jk} = -1/\pi$. It is easy to check this universality result in the models explicitly solved in the main body of the paper.

The second method uses WKB approximation. We first sketch the proof using SQM, omitting technicalities. One writes the restrictions to Ω of the wave functions associated to the states $|l_j\rangle$ as

$$\psi_j = \frac{1}{\sqrt{\lambda W''(X_j)}} f_j^* \psi_0 + \delta \psi_j,$$

where ψ_0 is a certain universal function and f_j is a model-dependent field-redefinition. $\delta \psi_j$ is the deviation with respect to exact universality. Then one uses residue-like techniques to rewrite

$$\| \delta \psi_j \|_B^2 \equiv \int_B | \delta \psi_j |^2 \quad (\text{B any domain in } \Omega),$$

in terms of the value of the wave function on the *boundary* of B. To evaluate the error one makes by replacing the true wave function ψ_j by its universal counterpart, we can use domains B such that their boundaries remain at a finite distance from the critical points. Then go to the semiclassical limit, $\lambda \to \infty$. We know that the WKB approximation to the wave functions is reliable in this limit only as long as we are away from the critical points. one cannot compute $\langle l_j | l_k \rangle$ directly by WKB methods, since there is a non-negligible contribution to this quantity from regions of radius $O(\sqrt{\hbar})$ around the critical points where WKB is totally unreliable. However, the tricks above guarantee that we can evaluate the error with respect to the universal answer using only the values of ψ_j away from the critical points. Therefore in the formula for the error we can use the WKB wave functions. In this way we get the result stated above. We will now investigate WKB approximation in more detail from a slightly different viewpoint and show why the leading semiclassical correction is of order

$$O\left(\frac{1}{\sqrt{2 \, | \lambda | \, | \Delta W |}} \exp[-2 \, | \lambda | \, | \Delta W |] \right)$$

(unfortunately, we are not able to get the numerical coefficient in front by this method). This is a tricky point. Indeed at a first glance one would rather expect a vanishing result for $\langle l_j | l_k \rangle$ $(j \neq k)$. In fact, from the topological–anti-topological fusion point of view, ignoring the two hemispheres at the two ends and concentrating on the infinitely long intermediate cylinder with circumference β, one would (naively) identify $\langle l_j | l_k \rangle$ with

$$\mathrm{Tr}_{(j,k)}(-1)^F \exp[-\beta H],$$

the trace being over the soliton sector corresponding to the path integral with boundary conditions

$$X(+\infty) = X_j, \qquad X(-\infty) = X_k.$$

In the soliton subsector all state appear in supersymmetry multiplets (see e.g. ref. [33]) and due to the $(-1)^F$ in the above expression we seem to be getting zero. So it seems with this naive interpretation of the topological–anti-topological fusion we are getting a paradox.

The point is that the identification of $|l_j\rangle$ with the vacuum $|X_j\rangle$, corresponding to the boundary condition $X(\tau = -\infty) = X_j$, is correct only at $\hbar = 0$. Indeed, the "point" basis, which the topological theory gives, is defined as the one which diagonalizes $\mathscr{R}$, i.e. for any holomorphic function f

$$F(X)|l_j\rangle = f(X_j)|l_j\rangle + Q^+ |\text{something}\rangle.$$

There is also an anti-point basis, obtained from the anti-topological theory, which diagonalizes the Q^--cohomology ring

$$F(\overline{X})\big|\tilde{l}_j\big\rangle = f\big(\overline{X}_j\big)\big|\tilde{l}_j\big\rangle + Q^- |\text{something}\rangle.$$

For $\hbar \neq 0$ $|\tilde{l}_j\rangle \neq |l_j\rangle$ because the chiral and anti-chiral rings cannot be diagonalized simultaneously. Instead, the definition of the vacua $|X_j\rangle$ is symmetric between Q^+- and Q^--cohomology and hence it is real with respect the real structure M. In other words, the state $|X_j\rangle$ is a "real" admixture of topological and anti-topological states. The correct identification has the general form (using results of sect. 5)

$$2|X_j\rangle = \sqrt{\lambda W''(X_j)} \, |l_j\rangle + \left(\sqrt{\lambda W''(X_j)}\right)^* m_j^k |l_k\rangle$$

$$+ \text{sub-leading instanton corrections.} \tag{B.5}$$

Susy predicts $\langle X_j | X_k \rangle = 0$ for $j \neq k$. This is consistent with eq. (B.5). Indeed

$$\langle X_j | X_k \rangle = \tfrac{1}{4}\left[(e^\gamma)_{kj} + (e^\gamma)_{jk} + 2\delta_{jk}\right] + \ldots = \delta_{jk} + \mathrm{O}(\gamma^2),$$

and hence (at least at the one-instanton level) there is no tunnelling between distinct classical vacua $|X_j\rangle$. Therefore $\langle l_j | l_k \rangle$ is non-vanishing not because there is a "physical" tunnelling process but because the topological states $|l_j\rangle$ are combinations of different classical vacua.

Despite the fact that $\langle l_j | l_k \rangle$ is not an instanton tunnelling amplitude in an obvious sense its evaluation is quite reminiscent of an instanton computation. We will now make this connection a little more clear. Our finding supports the idea that *loop* corrections in an instanton background is responsible for the leading semiclassical correction to the metric. For the sake of comparison, we recall what we would have found in an actual instanton computation. We would get a factor $\exp[-2|\lambda| \, |\Delta W|]$ from the classical action, a factor $\sqrt{4\pi|\lambda|}\, |\Delta W|$ from the integration over the position of the center of the instanton, no determinant factor (by susy) and, unless we soak them up, a factor 0 from the Fermi zero-modes.

For definiteness we consider the model $W = (X^3/3 - X)$, and compute $\langle l_1 | l_2 \rangle$ as $\lambda \to \infty$. There are two (equivalent) techniques available, one can use WKB either in the path integral or in the Schrödinger equation. We choose the second one since using explicit wave functions the identification of the various vacuum states in simpler. In this framework, $\langle l_1 | l_2 \rangle$ is just the overlap integral for the two vacua. However as mentioned above there is a difficulty. In SQM we compute such overlaps by residue techniques. This requires only the knowledge of the leading behaviour of the wave functions at the critical points of W. But these are precisely the points where the WKB approximation breaks down! In other words, for the vacuum wave functions the limits $X \to X_j$ and $\hbar \to 0$ do *not* commute. This is why making reliable semiclassical computations is very hard. Of course, we can try to compute the overlap by integrating the WKB wave functions in the region where they can be trusted but, as we shall see, this will give us only a rough estimate of the amplitude.

We parametrize the wave form corresponding to l_1 as

$$\omega_1 = \frac{1}{\sqrt{2\pi}} \frac{e^{-2|\lambda| \, |W(X) - W(1)|}}{\sqrt{2|\lambda| \, |W(X) - W(1)|}} \left[\lambda \phi_1(X) \, dW + \bar{\lambda} \phi_2(X) \, d\overline{W} \right].$$

From the Schrödinger equation we know that the functions $\phi_i(X)$ have the properties

$$\phi_1(1 + \epsilon \, e^{i\theta}) \simeq e^{-i \arg[W''(1)]/2} \, e^{-i\theta} + \ldots,$$

$$\phi_2(1 + \epsilon \, e^{i\theta}) = - \left[\phi_1(1 + \epsilon \, e^{i\theta}) \right]^* + \ldots.$$

Moreover, WKB methods give

$$|\phi_i| = 1 + O(1/|\lambda|) \tag{B.6}$$

both near the critical point $X = 1$ and in the region where

$$| \lambda | \, | W(X) - W(1) | \gg 1,$$

provided we are away from the other critical point by at least $O(1/|\lambda|)$. It is crucial that the $1/|\lambda|$ corrections in eq. (B.6) cannot vanish identically.

The wave form for $|l_2\rangle$ is

$$\omega_2 = \frac{1}{\sqrt{2\pi}} \frac{e^{-2|\lambda||W(X)+W(1)|}}{\sqrt{2|\lambda||W(X)+W(1)|}} \left[\lambda \tilde{\phi}_1(X) \, dW + \bar{\lambda} \tilde{\phi}_2(X) \, d\overline{W} \right].$$

by "functoriality"

$$\tilde{\phi}_1(X) = i\phi_1(-X), \qquad \tilde{\phi}_2(X) = -i\phi_2(-X).$$

The idea is to evaluate the overlap by integrating only over the intermediate region between the two critical points where (apart for points very near the critical ones) the WKB functions are reliable enough. This region dominates the integral. We must compute

$$\int * \, \bar{\omega}_1 \wedge \omega_2 = \text{const.} \, |\lambda|$$

$$\times \int \left[\phi_1^* \tilde{\phi}_1 + \phi_2^* \tilde{\phi}_2 \right]$$

$$\times \frac{\exp\left[-2|\lambda|(|W(X) - W(1)| + |W(X) + W(1)|) \right]}{\sqrt{|W(X)^2 - W(1)^2|}} \, d^2W.$$

The argument of the exponential is of order λ. Since we are interested in $\lambda \to \infty$, we can evaluate this integral by saddle-point methods. In other words, the integral is dominated by the minima of the "action". It is convenient to work in the W-plane. In this plane the "action" at a given point is the sum of the distances from the points $W(1)$ and $-W(1)$, and hence it is minimal along the segment connecting these two critical values. Then, in doing the d^2W integral, we integrate in $d(\text{Re } W)$ between $-W(1)$ and $W(1)$, whereas we use the gaussian approximation for the integral in $d(\text{Im } W)$. To quadratic order in $\text{Im } W$ the exponential is

$$\exp\left[-4|\lambda||W(1)| - 2|\lambda||W(1)| \frac{(\text{Im } W)^2}{W(1)^2 - (\text{Re } W)^2} \right].$$

Integrating over $d(\text{Im } W)$ we get

$$\text{const.} \sqrt{|\lambda|} \, e^{-4|\lambda||W(1)|} \int_{-W(1)}^{W(1)} \left[\phi_1^* \tilde{\phi} + \phi_2^* \tilde{\phi}_2 \right] \, d(\text{Re } W).$$

This formula is consistent with instanton physics. Apart for the factor involving the ϕ's (related to the fermionic part of the wave function and the sub-leading WKB corrections) this is what we expect: a factor $\exp[-S]$ from the classical action and a factor $\sim \sqrt{|\lambda|}$ from the integration over the collective coordinate. Moreover, the computation realizes manifestly the idea [33] that the soliton is the segment in the W-plane connecting the two critical values. The phases of the ϕ's are such that on this segment one has

$$\phi_1^* \tilde{\phi}_1 + \phi_2^* \tilde{\phi}_2 = O(1/|\lambda|).$$

The fact that to leading order this vanishes just reflects the presence of Fermi zero-modes. However, the sub-leading terms need not vanish (in fact, the Schrödinger equation suggests they are not zero). Then we get

$$\langle l_1 | l_2 \rangle = O\left(\frac{1}{\sqrt{|\lambda|}} e^{-4|\lambda||W(1)|} \right), \tag{B.7}$$

as claimed. The constant in front cannot be computed by these methods both because the sub-leading corrections are poorly understood and because regions where WKB fails may also give contributions of this magnitude. Anyhow, this constant is predicted by our differential equations.

The third idea in getting this universal result is suggested by the form (B.2) that we wrote the universal correction to the metric in. Indeed $U(m)$ is related to the contribution of a single particle of mass m in two space-time dimensions to $\mathrm{Tr} \exp(-\beta H)$ (where we fix a point in space in taking the trace) [*], where m is the mass of the soliton connecting the two critical points and we have set $\beta = 1$. Note that in particular the normalization (up to the phase) is easily predicted in this way. So this means that the naive picture of soliton partition function, which led to the paradox mentioned above, is essentially right, but with taking the contribution of one soliton from each supersymmetry multiplet to $\mathrm{Tr}[(-1)^F \exp(-\beta H)]$ to avoid vanishing. Somehow the loop corrections to the instantons are responsible for giving this "effective" soliton description. It would be worthwhile understanding this connection more clearly. In particular this may allow one to compute the scattering matrices of solitons from solutions to our equations using the thermodynamic Bethe ansatz. In fact the asymptotic solution to PIII equation, given in the second reference in [47] can presumably be interpreted as giving an *exact* multi-soliton contribution to the $\mathrm{Tr} \exp(-\beta H)$ for the $\lambda(X^3/3 - X)$ model (and similarly for the Chebyshev case). In particular the quantity defined in eq. (1.4a) of that

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[*]   We would like to thank P. Fendley and K. Intriligator for a discussion on this point.

reference which is simply related to our functions can be viewed as computing the contribution of soliton in the form

$$G = \sum_{n=0}^{\infty} g_{2n+1}, \tag{B.8}$$

where g_{2n+1} (after specializing to our case and a suggestive redefinition of variables) takes the form

$$g_{2n+1} = \int \prod_{i=1}^{2n+1} \frac{\mathrm{d}p_i \, \exp\left(-\sqrt{p_i^2 + m^2}\right)}{2\pi\sqrt{p_i^2 + m^2}} \left[\prod_{j=1}^{2n} \left(\sqrt{p_j^2 + m^2} + \sqrt{p_{j+1}^2 + m^2}\right)^{-1} \prod_{j=1}^{n} \left(p_{2j}^2\right) \right],$$

which should clearly have the interpretation of the contribution of $2n + 1$ solitons whose contribution to the partition function has been modified from the free case by the presence of "interaction" encoded in the above equation by the term inside [...]. It would be interesting to connect this to the S matrix of the $N = 2$ theories computed in ref. [33], using ideas similar to thermodynamic Bethe ansatz.

As another example let us consider

$$W = \frac{x^{n+1}}{n+1} - x$$

considered in this paper. Let $|\tilde{l}_r\rangle$ denote the critical points of W as r runs from 0 to $n - 1$ with an appropriate phase factor to cancel the hessian term appearing in eq. (B.1). Let $|x^r\rangle$ denote the usual chiral basis for the vacua. Let $\omega = \exp(2\pi i/n)$. We have

$$|x^s\rangle = \frac{1}{\sqrt{n}} \sum_{r=0}^{n-1} \omega^{r(s+1/2)} |\tilde{l}_r\rangle.$$

Using eq. (B.1) we see that the phase of the leading correction to $\langle \tilde{l}_r | \tilde{l}_{r+1}\rangle$ is i, and its absolute value is $\exp(-m)/\sqrt{2\pi m}$, where m is the mass of the soliton connecting the nearest critical points

$$m = 2|\lambda(W(r+1) - W(r))| = 4|\lambda|\sin\pi/n.$$

Computing q_i defined in sect. 7, as logarithm of $\langle x^i | x^i\rangle$, we see from the above that (for $n > 2$)

$$q_i \sim \frac{-2\sin\left[\frac{2\pi}{n}\left(i + \tfrac{1}{2}\right)\right]\exp\left(-4|\lambda|\sin\frac{\pi}{n}\right)}{\sqrt{8\pi|\lambda|\sin\frac{\pi}{n}}}.$$

It is easy to check that to leading order this satisfies eq. (7.4), where z defined there is the same as λ here.

Appendix C. Special coordinates and all that

In this paper we used a coordinate-independent formulation of generalized special geometry. However, in the physics literature it is more usual to formulate this geometry using some special coordinates in which the formulae look quite simpler. The only drawback of these coordinates is that one has to work hard just to *define* them. In this appendix we describe the construction of such coordinates in our framework and use them to simplify the proof of some technical results we claimed in the main body of the paper. To avoid all misunderstandings, we use Greek letters to label the various chiral fields in the model.

The basic formula, arising from SQM perturbation theory, is (cf. subsect. 9.1 of ref. [5])

$$D_a \phi_k = \partial_\alpha \sigma_{ak}^\alpha + T_{ak}^h \phi_h, \tag{C.1}$$

where

$$\sigma_{ak}^\alpha \, \partial_\alpha W = \partial_a W \phi_k - C_{ak}^h \phi_h,$$

and T_a is the "torsion". The two terms in the r.h.s. of eq. (C.1) have very different origins. The first is the true variation of the topological operator whereas the torsion arises because of the special representatives of BRST-classes one needs to use in order to get the actual vacuum states [*].

T_a has the form

$$T_a = [Z, C_a],$$

with

$$\partial_{\bar{a}} Z = -\overline{C}_a, \qquad Z\eta = \eta Z^{\mathrm{T}}.$$

Hence,

$$T_a \eta = -\eta T_a^{\mathrm{T}}, \qquad \partial_{\bar{a}} T_b = -\left[\overline{C}_a, C_b\right]. \tag{C.2}$$

The first of eqs. (C.2) justifies the name torsion for T_a: It is the antisymmetric part

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[*] Here the tricky point is that, since Q^+ depends on t_a, the derivative of a Q^+-exact state is not Q^+-exact in general. Then computing the derivatives the actual representatives matter. In the definition of D_a they are uniquely fixed by the vacua. This is why a torsion appears.

(with respect to η) of the connection. The second one shows that our curvature originates from the torsion. In fact

$$[\partial_{\bar{a}}, D_b]\phi_k = \partial_{\bar{a}}(D_b\phi_k) = (\partial_{\bar{a}}T_b)_k{}^h\phi_h.$$

Now, consider the connection [*]

$$\mathscr{D}_a = D_a - T_a \quad (= \partial_a - \mathscr{A}_a).$$

With respect to g, $\mathscr{D}_a$ is not metric any longer. But it is still metric for η. This was to be expected since from a purely topological point of view the two connections differ only by a gauge transformation. Next we consider a "curved" basis for $\mathscr{R}$, i.e. of the form

$$\phi_a = \partial_a W.$$

Then one has

$$\sigma_{ab}^\alpha\partial_\alpha W = \partial_a W \partial_b W - C_{ab}{}^c\partial_c W, \qquad \text{(C.3)}$$

thus $\sigma_{ab}^\alpha = \sigma_{ba}^\alpha$, or

$$\mathscr{D}_a\phi_b = \mathscr{D}_b\phi_a.$$

Moreover,

$$\mathscr{D}_a\phi_b = \partial_a\partial_b W - \mathscr{A}_{ab}{}^c\partial_c W,$$

which gives

$$\mathscr{A}_{ab}{}^c = \mathscr{A}_{ba}{}^c.$$

Thus $\mathscr{A}$ is torsionless. Then it is the Christoffel connection of η. Let us compute its Riemann curvature. One has

$$[\mathscr{D}_a, \mathscr{D}_b]\phi_c = \partial_\alpha(\mathscr{D}_a\sigma_{bc}^\alpha - \mathscr{D}_b\sigma_{ac}^\alpha). \qquad \text{(C.4)}$$

From eq. (C.3) one has

$$(\mathscr{D}_a\sigma_{bc}^\alpha - \mathscr{D}_b\sigma_{ac}^\alpha)\partial_\alpha W = \partial_\alpha\big[(\phi_b\sigma_{ac}^\alpha + C_{ac}{}^d\sigma_{bd}^\alpha - (b \leftrightarrow a)\big],$$

$$(\phi_b\sigma_{ac}^\alpha + C_{ac}{}^d\sigma_{bd}^\alpha)\partial_\alpha W = \phi_a\phi_b\phi_c - (C_aC_b)_c{}^e\phi_e.$$

[*] In ref. [5] it was shown that $\mathscr{D}_a$ is the Gauss–Manin connection in the sense of versal deformations of a given singularity.

Then the r.h.s. of eq. (C.4) is in the jacobian ideal, and hence the curvature vanishes. Then we can find (local) coordinates t_a such that

$$\eta = \text{const.} \qquad \mathscr{A} = 0.$$

This result is a standard mathematical fact [55]. These are the so-called special coordinates. They are characterized by

$$\partial_a \partial_b W = \partial_\alpha \sigma_{ab}^\alpha \qquad (C.5)$$

with σ_{ab}^α as in eq. (C.3). Before going to more useful characterizations, let us show that for $n = 1$ this formula reproduces the results obtained in ref. [10] by KdV flows considerations.

In the one-field case

$$\sigma_{ab} W' = \partial_a W \partial_b W - C_{ab}^c \partial_c W,$$

or

$$\sigma_{ab} = \left(\frac{\partial_a W \partial_b W}{W'} \right)_+,$$

where $(\dots)_+$ means the non-negative part. Then eq. (C.5) becomes

$$\partial_a \phi_b = \partial_X \left(\frac{\phi_a \phi_b}{W'} \right)_+$$

which is equivalent to eq. (4.45) in ref. [10].

Put

$$\varpi_{aj}^\pm = \int_{\gamma_j^\mp} e^{\pm W} \partial_a W \, dX_1 \wedge \dots \wedge dX_n. \qquad (C.6)$$

Using eq. (C.5) we find

$$\partial_a \varpi_{bj}^\pm = \pm C_{ab}^c \varpi_{cj}^\pm. \qquad (C.7)$$

This is a characterization of special coordinates which is more convenient for computations. Since $\det[\varpi^\pm] \neq 0$, we can define the matrix C^{jk} by

$$C = (\varpi^+)^{-1} \eta \left[(\varpi^-)^{\mathrm{T}} \right]^{-1}.$$

Then from eq. (C.7)

$$\partial_a C = (\varpi^+)^{-1} \left[\eta C_a^{\mathrm{T}} - C_a \eta \right] \left[(\varpi^-)^{\mathrm{T}} \right]^{-1} = 0.$$

Then we have the general formula for the residue pairing

$$\eta_{ab} = \varpi_{aj}^{+} C^{jk} \varpi_{bk}^{-} \tag{C.8}$$

with C^{jk} a constant matrix. Now we can show that this matrix is precisely the intersection discussed in sect. 4. In fact, we show it for the "good" cases, where in the UV limit we get a non-degenerate quasi-homogeneous W, although it is plausibly true in general. Since C^{jk} does not depend on λ, we can limit ourselves to quasi-homogeneous W, and hence to homogeneous ones. Then we consider the homogeneous superpotential

$$\mathscr{W}(X, t; s) = sW(X, t) + (1-s) \sum_i X_i^d.$$

C^{jk} is independent of s. So we can compute it for $s = 0$, i.e. it is enough to show our statement for Fermat W's. In this case our periods factorize into the product of A_{d-1} minimal model periods. That in this last case C^{jk} is the inverse intersection matrix can be seen by a direct computation.

We end this appendix by showing that our "perturbative" characterization of the special coordinates agrees with the mathematical one [28,55]. Indeed, define

$$u_{kj}(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{0} \mathrm{d}g\, g^{\lambda} \varpi_{kj}(g),$$

where

$$\varpi_{kj}(g) = \int_{\gamma_j(g)} e^{gW} \phi_k \, \mathrm{d}X_1 \wedge \dots \wedge \mathrm{d}X_n.$$

$(\varpi^{\pm} \equiv \varpi(\pm))$. Eq. (C.7) generalizes to

$$\partial_a \varpi_b(g) = g C_{ab}^{\ c} \varpi_c(g).$$

Taking the Mellin transform, in terms of $u_{kj}(\lambda)$ this becomes eq. (55) of ref. [28].

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