Exercise 1  Linearized Riemann, Ricci and Einstein tensors

Using that the Christoffel symbols at linear level are
\[ \Gamma^\alpha_\mu\nu = \frac{1}{2} \left( \partial_\mu h^\alpha_\nu + \partial_\nu h^\alpha_\mu - \partial^\alpha h_{\mu\nu} \right) \]

derive eqs.
\[ R_{\mu\nu\rho\sigma} = \frac{1}{2} \left( \partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\nu \partial_\sigma h_{\mu\rho} \right), \]
\[ R_{\mu\nu} = \frac{1}{2} \left( \partial_\mu \partial_\rho h^\rho_\nu + \partial_\nu \partial_\rho h^\rho_\mu - \Box h_{\mu\nu} - \partial^\rho h_{\mu\nu} \right), \]
\[ R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h, \]
\[ G_{\mu\nu} = \frac{1}{2} \left( \partial_\mu \partial_\nu h^\rho_\rho + \partial_\rho \partial_\nu h^\rho_\mu - \Box h_{\mu\nu} - \partial^\rho h_{\mu\nu} - \eta_{\mu\nu} \partial_\sigma \partial_\nu h^{\mu\nu} + \eta_{\mu\nu} \Box h \right), \]

(1)

Exercise 2  Retarded Green function I

Show that the two representation of the retarded Green function given by
\[ G_{\text{ret}}(t, x) = -\delta(t - r) \frac{1}{4\pi r}, \]
\[ G_{\text{adv}}(t, x) = -\delta(t + r) \frac{1}{4\pi r}, \]

and
\[ G_{\text{ret}}(t, x) = -i\theta(t) \left( \Delta_+(t, x) - \Delta_-(t, x) \right), \]
\[ G_{\text{adv}}(t, x) = i\theta(-t) \left( \Delta_+(t, x) - \Delta_-(t, x) \right), \]

where
\[ \Delta_\pm(t, x) \equiv \int \frac{dk}{2k} e^{ikx} \frac{e^{ikt}}{2k} \]
are equivalent. Hint: use that
\[ \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} = \delta(x), \]
and that
\[ \theta(t) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(t+r)} = 0 \quad \text{for} \quad r \geq 0. \]

**Exercise 3 Retarded Green function II**

Use the representation of the $G_{ret,adv}$ obtained in the previous exercise to show that
\[ G_{ret}(t, x) = -\int_k \frac{d\omega}{2\pi} e^{-i\omega t + ikx} \left( k^2 - (\omega + i\epsilon)^2 \right) \]
\[ G_{adv}(t, x) = -\int_k \frac{d\omega}{2\pi} e^{-i\omega t + ikx} \left( k^2 - (\omega - i\epsilon)^2 \right) \]

where $\epsilon$ is an arbitrarily small positive quantity. Hint: use that
\[ \theta(\pm t) = \mp \frac{1}{2\pi i} \int \frac{e^{-i\omega t}}{\omega \pm i\epsilon}. \]

Show that $G_{ret}$ is real.

**Exercise 4 Feynman Green function I**

Show that the $G_F$ defined by
\[ G_F(t, x) = -i \int_k \frac{d\omega}{2\pi} e^{-i\omega t + ikx} \]
is equivalent to
\[ G_F(t, x) = \theta(t)\Delta_+(t, x) + \theta(-t)\Delta_-(t, x). \]

Derive the relationship
\[ G_F(t, x) = \frac{i}{2} (G_{adv}(t, x) + G_{ret}(t, x)) + \frac{\Delta_+(t, x) + \Delta_-(t, x)}{2}. \]

**Exercise 5 Feynman Green function II**

By integrating over $k$ the $G_F$ in the $\sim 1/(k^2 - \omega^2)$ representation, show that $G_F$ implements boundary conditions giving rise to field $h$ behaving as
\[ h(t, x) \sim \int d\omega e^{-i\omega t + i|\omega| r}, \]
corresponding to out-going (in-going) wave for $\omega > (\omega <) 0$.

**Exercise 6 TT gauge**

Show that the projectors defined by
\[ \Lambda_{ij,kl}(\hat{n}) = \frac{1}{2} [P_{ij}P_{kl} + P_{kl}P_{ij} - P_{ij}P_{kl}], \]
\[ P_{ij}(\hat{n}) = \delta_{ij} - n_in_j, \]

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satisfies the relationships
\[ P_{ij} P_{jk} = P_{ik} \]
\[ \Lambda_{ij,kl} \Lambda_{kl,mn} = \Lambda_{ij,mn} , \]
which characterize projectors operator.

**Exercise 7  Energy of circular orbits in a Schwarzschild metric**

Consider the Schwarzschild metric
\[ ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega^2 . \] (2)

The dynamics of a point particle with mass \( m \) moving in such a background can be described by the action
\[ S = -m \int d\tau = -m \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} } \]
for any coordinate \( \lambda \) parametrizing the particle world-line. Using \( S = \int d\lambda L \), we can write
\[ L = -m \left[ \left(1 - \frac{2GM}{r}\right) \left( \frac{dt}{d\tau} \right)^2 - \left( \frac{dr}{d\tau} \right)^2 \right]^{1/2} . \]

Verify that \( L \) has cyclic variables \( t \) and \( \phi \) and derive the corresponding conserved momenta.

(Hint: use \( g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1 \). Result: \( E = m \left(dt/d\tau\right) \equiv m e \) and \( L = mr^2 \left(d\phi/d\tau\right) \equiv ml \).

By expressing \( dt/d\tau \) and \( d\phi/d\tau \) in terms of \( e \) and \( l \), derive the relationship
\[ e^2 = \left(1 - \frac{2GM}{r}\right) \left(1 + l^2/r^2\right) + \left( \frac{dr}{d\tau} \right)^2 . \]

From the circular orbit conditions \( \frac{de}{dr} = 0 = \frac{d\phi}{d\tau} \), derive the relationship between \( l \) and \( r \) for circular orbits.

(Result: \( l^2 = r^2 \left(1 - \frac{3}{x} \right) \).

Substitute into the energy function \( e \) and find the circular orbit energy
\[ e(x) = \frac{1 - 2x}{\sqrt{1 - 3x}} , \]
where \( x \equiv (GM\dot{\phi})^{2/3} \) is an observable quantity as it is related to the GW frequency \( f_{GW} \) by \( x = (GM\pi f_{GW})^{2/3} \).

(Hint: Use \( \dot{\phi} = \frac{d\phi}{d\tau} \equiv \frac{l}{r^2} \left[1 - \frac{2M}{r} - r^2 \dot{\phi}^2\right]^{1/2} \).
to find that on circular orbits \( (M \dot{\phi})^2 = (G_N M/r)^3 \), an overdot stands for derivative with respect to \( t \).

Derive the relationships for the Inner-most stable circular orbit

\[
\begin{align*}
    r_{\text{ISCO}} &= 6G_N M = 4.4 \text{km} \left( \frac{M}{M_\odot} \right) \\
    f_{\text{ISCO}} &= \frac{1}{6^{3/2}} \frac{1}{G_N M \pi} \simeq 8.8 \text{kHz} \left( \frac{M}{M_\odot} \right)^{-1} \\
    v_{\text{ISCO}} &= \frac{1}{\sqrt{6}} \simeq 0.41
\end{align*}
\]

Exercise 8  Newtonian force exerted by GWs

Derive the equivalent Newtonian-like force

\[
\ddot{\xi} = \frac{1}{2} h_{ij} \xi^j ,
\]

from the geodesic deviation equation

\[
\frac{D^2 \xi^i}{d\tau^2} = -R^i_{0j0} \xi^j \left( \frac{dx^0}{d\tau} \right)^2 .
\]

Exercise 9  World-line action

Derive the geodesic equation

\[
\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0
\]

from the world-line action

\[
S_{\text{wl}} = \int dt d^3y \sqrt{-g_{\mu\nu} \dot{y}^\mu \dot{y}^\nu} \delta^{(3)}(y - x(t))
\]