From data to theory and back

Exercises

August 3rd, 2015

Exercise1 Linearized Riemann, Ricci and Einstein tensors

Using that the Christoffel symbols at linear level are

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h^\alpha_\nu + \partial_\nu h^\alpha_\mu - \partial^\alpha h_{\mu\nu}\right)$$

derive eqs.

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \frac{1}{2}\left(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}\right) , \\ R_{\mu\nu} &= \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) , \\ R &= \partial_\mu\partial_\nu h^{\mu\nu} - \Box h , \\ G_{\mu\nu} &= \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial_\nu\partial_\nu h^{\mu\nu} + \eta_{\mu\nu}\Box h\right) , \end{aligned} \tag{1}$$

Exercise2 Retarded Green function I

Show that the two representation of the retarded Green function given by

$$\begin{aligned} G_{ret}(t,\mathbf{x}) &= -\delta(t-r)\frac{1}{4\pi r} , \\ G_{adv}(t,\mathbf{x}) &= -\delta(t+r)\frac{1}{4\pi r} , \end{aligned}$$

and

$$G_{ret}(t,x) = -i\theta(t)\left(\Delta_+(t,x) - \Delta_-(t,x)\right) ,$$
$$G_{adv}(t,x) = i\theta(-t)\left(\Delta_+(t,x) - \Delta_-(t,x)\right) ,$$

where

$$\Delta_\pm(t,x) \equiv \int_{\mathbf{k}} e^{\mp ikt}\frac{e^{i\mathbf{kx}}}{2k}$$

are equivalent. Hint: use that

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx} = \delta(x) ,$$

and that

$$\theta(t)\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(t+r)}=0\quad\text{for } r\geq 0\,.$$

Exercise3 Retarded Green function II

Use the representation of the $G_{ret,adv}$ obtained in the previous exercise to show that

$$\begin{aligned}G_{ret}(t,\mathbf{x}) &= -\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{-i\omega t+i\mathbf{kx}}}{k^2-(\omega+i\epsilon)^2}\,,\\ G_{adv}(t,\mathbf{x}) &= -\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{-i\omega t+i\mathbf{kx}}}{k^2-(\omega-i\epsilon)^2}\,.\end{aligned}$$

where ϵ is an arbitrarily small positive quantity. Hint: use that

$$\theta(\pm t)=\mp\frac{1}{2\pi i}\int\frac{e^{-i\omega t}}{\omega\pm i\epsilon}\,.$$

Show that G_{ret} is real.

Exercise4 Feynman Green function I

Show that the G_F defined by

$$G_F(t,\mathbf{x})=-i\int_{\mathbf{k}}\frac{d\omega}{2\pi}\frac{e^{-i\omega t+i\mathbf{kx}}}{k^2-\omega^2-i\epsilon}$$

is equivalent to

$$G_F(t,\mathbf{x})=\theta(t)\Delta_+(t,\mathbf{x})+\theta(-t)\Delta_-(t,\mathbf{x})\,.$$

Derive the relationship

$$G_F(t,\mathbf{x})=\frac{i}{2}\left(G_{adv}(t,\mathbf{x})+G_{ret}(t,\mathbf{x})\right)+\frac{\Delta_+(t,\mathbf{x})+\Delta_-(t,\mathbf{x})}{2}\,.$$

Exercise5 Feynman Green function II

By integrating over $\mathbf{k}$ the G_F in the $\sim 1/(k^2-\omega^2)$ representation, show that G_F implements boundary conditions giving rise to field h behaving as

$$h(t,\mathbf{x})\sim\int d\omega e^{-i\omega t+i|\omega|r}\,,$$

corresponding to out-going (in-going) wave for $\omega>(<)0$.

Exercise6 TT gauge

Show that the projectors defined by

$$\begin{aligned}\Lambda_{ij,kl}(\hat{n}) &= \frac{1}{2}\left[P_{ik}P_{jl}+P_{il}P_{jk}-P_{ij}P_{kl}\right]\,,\\ P_{ij}(\hat{n}) &= \delta_{ij}-n_in_j\,,\end{aligned}$$

satisfies the relationships

$$
\begin{aligned}
P_{ij} P_{jk} &= P_{ik} \\
\Lambda_{ij,kl} \Lambda_{kl,mn} &= \Lambda_{ij,mn} \,,
\end{aligned}
$$

which characterize projectors operator.

Exercise7 Energy of circular orbits in a Schwarzschild metric

Consider the Schwarzschild metric

$$
ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2G_N M}{r}\right)} + r^2 d\Omega^2 \,. \tag{2}
$$

The dynamics of a point particle with mass m moving in such a background can be described by the action

$$
S = -m \int d\tau = -m \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}
$$

for any coordinate λ parametrizing the particle world-line. Using $S = \int d\lambda \, L$, we can write

$$
L = -m \left[\left(1 - \frac{2G_N M}{r}\right) \left(\frac{dt}{d\tau}\right)^2 - \frac{\left(\frac{dr}{d\tau}\right)^2}{\left(1 - \frac{2G_N M}{r}\right)} - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \right]^{1/2} .
$$

Verify that L has cyclic variables t and ϕ and derive the corresponding conserved momenta.
(Hint: use $g_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = -1$. Result: $E = m(dt/d\tau)(1 - 2G_N M/r) \equiv m\, e$ and $L = mr^2(d\phi/d\tau) \equiv m\, l$).
By expressing $dt/d\tau$ and $d\phi/d\tau$ in terms of e and l, derive the relationship

$$
e^2 = (1 - 2G_N M/r)\left(1 + l^2/r^2\right) + \left(\frac{dr}{d\tau}\right)^2 .
$$

From the circular orbit conditions ($\frac{de}{dr} = 0 = dr/d\tau = 0$), derive the relationship between l and r for circular orbits.
(Result: $l^2 = r^2/(\frac{r}{M} - 3)$).
Substitute into the energy function e and find the circular orbit energy

$$
e(x) = \frac{1 - 2x}{\sqrt{1 - 3x}} \,,
$$

where $x \equiv (G_N M \dot{\phi})^{2/3}$ is an observable quantity as it is related to the GW frequency f_{GW} by $x = (G_N M \pi f_{GW})^{2/3}$.
(Hint: Use

$$
\dot{\phi} = \frac{d\phi}{d\tau} \dot{\tau} = \frac{l}{r^2} \left[1 - \frac{2M}{r} - r^2 \dot{\phi}^2\right]^{1/2}
$$

to find that on circular orbits $\left(M\dot{\phi}\right)^2 = (G_N M/r)^3$, an overdot stands for derivative with respect to t.)
Derive the relationships for the Inner-most stable circular orbit

$$\begin{aligned} r_{ISCO} &= 6G_N M = 4.4\text{km}\left(\frac{M}{M_\odot}\right) \\ f_{ISCO} &= \frac{1}{6^{3/2}}\frac{1}{G_N M\pi} \simeq 8.8\text{kHz}\left(\frac{M}{M_\odot}\right)^{-1} \\ v_{ISCO} &= \frac{1}{\sqrt{6}} \simeq 0.41 \end{aligned}$$

Exercise8 Newtonian force exerced by GWs

Derive the equivalent Newtonian-like force

$$\ddot{\xi} = \frac{1}{2}\ddot{h}_{ij}\xi^j \,, \tag{3}$$

from the geodesic deviation equation

$$\frac{D^2\xi^i}{d\tau^2} = -R^i{}_{0j0}\xi^j\left(\frac{dx^0}{d\tau}\right)^2 \,.$$

Exercise9 World-line action

Derive the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0$$

from the world-line action

$$S_{wl} = \int dt d^3y\sqrt{-g_{\mu\nu}\dot{y}^\mu\dot{y}^\nu}\,\delta^{(3)}(y - x(t))$$