$\hbar \neq 0$ Harmonic oscillator + squeezing

(see e.g. Allen gr-qc/9604033)

For $\hbar = 0$ the Universe would be driven (if inflation lasts long enough) to a completely homogeneous + flat Universe

We want to claim: all the structure in our Universe is created by quantum fluctuations! WOW!

We do not need initial conditions, which in a classical theory will always be there. QM does the job: we are making predictions about initial conditions

It seems the first to think about this possibility was Dirac

A first example: as always an harmonic oscillator

$$\omega_i \longrightarrow \omega_f$$
$$\sqrt{\frac{g}{l_i}} \longrightarrow \sqrt{\frac{g}{l_f}}$$

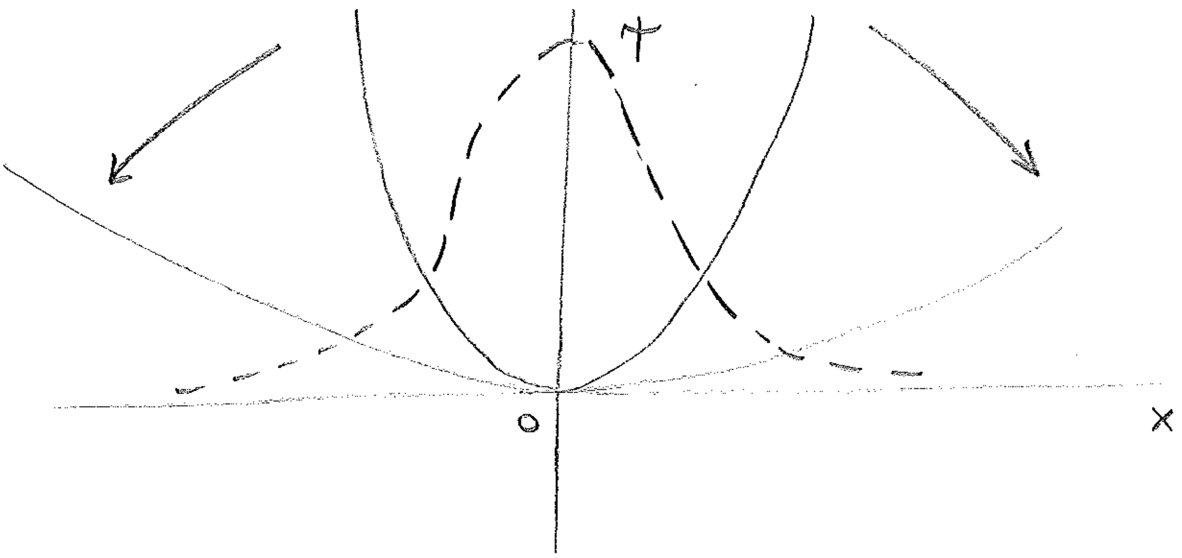
- Classically nothing happens (if I start from the vacuum)
- QM. Two possible behaviours

$\longrightarrow T \gg \omega_i^{-1}; \omega_f^{-1}$ adiabatic limit

The system follows the vacuum at every step and you end up in the final vacuum at $t = t_f$

$\longrightarrow T \ll \omega_i^{-1}; \omega_f^{-1}$ $\left(\frac{\dot{l}}{l} \gg \omega \text{ at every moment}\right)$

## Diabotic limit




The potential varies but there is no time to adjust so $\psi$ remains in the old vacuum, which is an excited state of the new harmonic oscillator

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{1}{2}m\omega^2 x^2\psi = E_n\psi$$

$$-\frac{\hbar}{2m\omega}\nabla^2\psi + \frac{1}{2}\frac{m\omega}{\hbar}x^2\psi = \frac{E_n}{\hbar\omega}\psi$$

The equation is just a function of $\sqrt{\alpha}x$

$$\psi_n(x) = N_n H_n(\sqrt{\alpha}x)\, e^{-\frac{\alpha}{2}x^2} \qquad \alpha = \frac{m\omega}{\hbar}$$

Old vacuum: $\psi(x) = \left(\frac{\alpha_i}{\pi}\right)^{1/4} e^{-\frac{\alpha_i}{2}x^2}$

Decomposed wrt new eigenstates : $\psi(x) = \sum c_n \psi_n(x)$

$$c_n = 0 \qquad n \text{ odd}$$
$$c_{2n} = (\alpha_i\alpha_f)^{1/4}\sqrt{\frac{2(2n)!}{\alpha_i+\alpha_f}}\,\frac{1}{n!}\left[\frac{\omega_f-\omega_i}{2(\omega_f+\omega_i)}\right]^n$$

$$E_f = \sum_{n=0}^{\infty}\left(2n+\frac{1}{2}\right)\hbar\omega_f\,|c_{2n}|^2 = \hbar\omega_f\left(\frac{1}{2}+\frac{(\omega_f-\omega_i)^2}{4\omega_f\omega_i}\right)$$

For $\omega_f \ll \omega_i$ $\qquad E_f = \frac{1}{4}\hbar\omega_i \qquad N_f = \frac{1}{4}\frac{\omega_i}{\omega_f} \gg 1$

( $\hat{a}_{NEW}$ and $\hat{a}^{\dagger}_{NEW}$ satisfy $[\hat{a} ; \hat{a}^{\dagger}] = 1$ check! )

$$| \xi \rangle = S(\xi) | 0 \rangle$$ is a <u>squeezed state</u>

(Not a coherent state!)

In particular the old vacuum is a squeezed state. $|0\rangle_{old}$ must be annihilated by $\hat{a}_{old}$, a linear combination of $\hat{a}_{new}$ and $\hat{a}^{\dagger}_{new}$: $S \hat{a}_{NEW} S^{\dagger}$

$$S \hat{a}_{NEW} S^{\dagger} | 0_{old} \rangle = 0 \qquad S \hat{a}_{NEW} S^{\dagger} \underbrace{S | 0_{NEW} \rangle}_{|0_{old}\rangle}$$

What are the features of this new state?

The <u>uncertainty is squeezed in one direction</u>

$$Y_1 + i Y_2 \equiv (X_1 + i X_2) e^{-i\theta/2} \qquad [Y_1 ; Y_2] = i$$

$$S^{\dagger} (Y_1 + i Y_2) S = Y_1 e^{-r} + i Y_2 e^{r}$$

$$\langle (\Delta Y_1)^2 \rangle = \frac{1}{2} e^{-2r} \qquad \langle (\Delta Y_2)^2 \rangle = \frac{1}{2} e^{2r}$$

$$\langle \Delta Y_1 \Delta Y_2 \rangle = \frac{1}{2}$$

Expectation values in the squeezed state can be calculated as

$$\langle 0 | S^{\dagger} \text{ Operator } S | 0 \rangle$$

The uncertainty is squeezed in one direction and enlarged in the other

You end up in a very excited state

$E_f = \frac{1}{4}\hbar\omega_i$     same $\Psi$ with potential energy $\to 0$

Energy is equally spent potential / kinetic so

I have $\frac{1}{2}$ of the initial energy

Usual definition of $\hat{a}$ and $\hat{a}^+$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}\right) \equiv (\hat{x}_1 + i\hat{x}_2)/\sqrt{2}$$

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} - i\frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}\right) \equiv (\hat{x}_1 - i\hat{x}_2)/\sqrt{2}$$

If I change $\omega$, I change the linear combination that defines $\hat{a}$ and $\hat{a}^+$. In particular the old vacuum is annihilated by $\hat{a}_{old}$, a linear combination of $\hat{a}_{NEW}$ and $\hat{a}^+_{NEW}$

I am interested in a transformation mixing $\hat{a}$ and $\hat{a}^+$, such that the commutator $[\hat{a};\hat{a}^+] = 1$ remains the same

Squeeze operator:

$$S(\xi) = e^{\frac{1}{2}\xi^*\hat{a}^2 - \frac{1}{2}\xi\hat{a}^{+2}}$$

$$\xi = r e^{i\theta}$$

$$S^+(\xi) = S^{-1}(\xi) = S(-\xi)$$

Using

$$e^{A}Be^{-A} = B + [A;B] + \frac{1}{2}[A[AB]] + \dots$$

$$S^+(\xi)\,\hat{a}\,S(\xi) = \hat{a}\,\mathrm{ch}\,r - \hat{a}^+ e^{i\theta}\,\mathrm{sh}\,r$$

$$S^+(\xi)\,\hat{a}^+ S(\xi) = \hat{a}^+\mathrm{ch}\,r - \hat{a}\,e^{-i\theta}\,\mathrm{sh}\,r$$

Bogoliubov transformation

check that $Y_1$ and $Y_2$ are squeezed

$$X_1 = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^+\right)$$

$$X_2 = \frac{1}{\sqrt{2}}\,\frac{1}{i}\left(\hat{a} - \hat{a}^+\right)$$

$$Y_1 = \sqrt{2}\,\hat{a}\,e^{-i\theta/2} + h.c.$$

$$\xrightarrow{S} \sqrt{2}\left[\hat{a}\,ch\,r\,e^{-i\theta/2} - \hat{a}^+ sh\,r\,e^{i\theta/2} + \hat{a}^+ ch\,r\,e^{i\theta/2} - \hat{a}\,sh\,r\,e^{-i\theta/2}\right]$$

$$= e^{-r}\sqrt{2}\left[\hat{a}\,e^{-i\theta/2} + h.c.\right]$$

Time evolution of the squeezing parameter

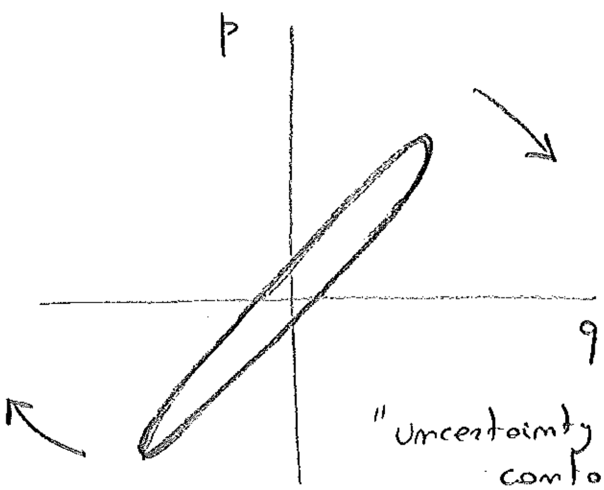
$$e^{-iHt} S|0\rangle \qquad\qquad e^{-iHt} S\, e^{+iHt}$$

Writing $S$ as an exponential series, I simply get the evolution of each of the $\hat{a}$ and $\hat{a}^+$

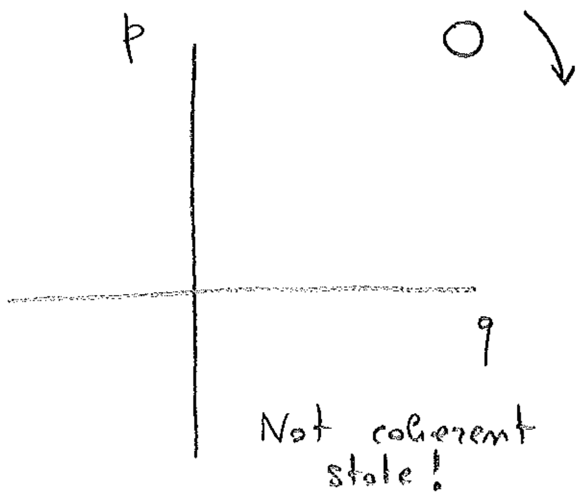
$$S = e^{\frac{1}{2}\xi^* \hat{a}^2 - \frac{1}{2}\xi \hat{a}^{+2}} \qquad \xi = \xi_0\, e^{-2i\omega t}$$

The state remains squeezed, but the uncertainty rotates clockwise as in the classical evolution

(Notice the time evolution would have $e^{-iHt}$ on the right, but we are interested in the evolution of $S|0\rangle$ ...)




"Uncertainty contours"




Not coherent state!

Decoherence without decoherence :

in the limit in which I can forget about the decaying mode
I have a <u>classical stochastic variable</u>, I can assign a probability
to its amplitude
It is not a mathematic property of squeezed states (I can
always define observables such that the state is fully quantum)
but a reasonable assumption about decoherence :
any measurement will be sensitive only to a coarse grained
description of the modes in which the decaying mode can be
neglected

# Massless scalar in an expanding Universe

We want to see that each Fourier mode behaves quite similarly to the example of a single HO

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \right] =$$

$$= \int d^4x \, a^2 \left[ \frac{1}{2} (\partial_\eta \phi)^2 - \frac{1}{2} (\partial_i \phi)^2 \right]$$

To make the Lagrangian more similar to the case of an HO where only the frequency is changing I define

$$y = a\phi$$

(Polarski, Starobinsky gr-qc/9504030)

$$S = \int d^4x \; \frac{1}{2} \left( \partial_\eta y - \frac{a'}{a} y \right)^2 - \frac{1}{2} (\partial_i y)^2 =$$

$$= \int d\eta \int \frac{d^3k}{(2\pi)^3} \; \frac{1}{2} \left[ y'(k) - \frac{a'}{a} y(k) \right] \left[ y'(-k) - \frac{a'}{a} y(-k) \right] - \frac{k^2}{2} y(k) y(-k)$$

$$p(k) = \frac{\partial \mathcal{L}}{\partial y'(k)} = y'(k)^\dagger - \frac{a'}{a} y(k)^\dagger$$

The field is real...

$$\mathcal{H} = \int_{\text{Half space}} \frac{d^3k}{(2\pi)^3} \; \frac{1}{2} p(k) p(k)^\dagger + \frac{1}{2} k^2 y(k) y^\dagger(k) + \frac{1}{2} \frac{a'}{a} \left[ y(k) p^\dagger(k) + p(k) y^\dagger(k) \right]$$

$$\hat{b}(k) = \frac{1}{\sqrt{2}} \left( \sqrt{k} \, y(k) + \frac{i}{\sqrt{k}} p(k) \right)$$

$$y(\vec{k}) = \frac{\hat{b}(k) + \hat{b}^\dagger(-k)}{\sqrt{2k}} \qquad p(\vec{k}) = -i \sqrt{\frac{k}{2}} \left( \hat{b}(k) - \hat{b}^\dagger(-k) \right)$$

(Notice I want $\hat{b}_k e^{i\vec{k}\vec{x} - i\omega t}$ and $\hat{b}^\dagger_k e^{-i\vec{k}\vec{x} + i\omega t}$, so actually I have some $-\vec{k}$ in $\hat{b}^\dagger$)

$$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \frac{k}{2} \left[ \hat{b}(k)\, \hat{b}^+(k) + \hat{b}^+(k)\, \hat{b}(k) \right] + i \frac{a'}{a} \left[ \hat{b}^+(k)\, \hat{b}^+(-k) - \hat{b}(k)\, \hat{b}(-k) \right]$$

$$\begin{pmatrix} \hat{b}(k) \\ \hat{b}^+(-k) \end{pmatrix}' = \begin{pmatrix} -ik & \frac{a'}{a} \\ \frac{a'}{a} & ik \end{pmatrix} \begin{pmatrix} \hat{b}(k) \\ \hat{b}^+(-k) \end{pmatrix}$$

- The evolution mixes $\hat{b}$ and $\hat{b}^+$, you'll <u>end up in a</u> <u>squeezed state</u>

- Adiabatic / diabatic = inside / outside $H^{-1}$

$$k \gtrless \frac{a'}{a} = aH$$

- In the limit of small $k$ we have

$$\begin{pmatrix} \ \end{pmatrix}' = \frac{a'}{a} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \ \end{pmatrix}$$

$$y(\vec{k}) \propto a \qquad \phi = \text{const}$$
$$p(\vec{k}) \propto a^{-1}$$

Staying constant for $\phi$ describes the squeezing

- The <u>squeezing ratio</u> is the ratio of the scale factors between when the mode goes out and comes back in $H^{-1}$; more than 60 e-folds

- Each mode behaves as a classical stochastic variable at large squeezing

# 2-point function in de Sitter

$$a = a_0 e^{Ht} \qquad \eta = \int \frac{dt}{a_0 e^{Ht}} = -\frac{1}{a_0 H} e^{-Ht} = -\frac{1}{aH}$$

$$\eta \in (-\infty; 0)$$

$$S = \int d^4x \, \frac{1}{2H^2\eta^2} \left[ (\partial_\eta \phi)^2 - (\partial_i \phi)^2 \right]$$

$$\phi(\eta; \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \phi_k(\eta) \, e^{i\vec{k}\cdot\vec{x}}$$

EOM: $\partial_\eta \left[ \eta^{-2} \partial_\eta \phi \right] + \eta^{-2} k^2 \phi = 0$

$$\boxed{\phi_k^{cl} = \frac{H}{\sqrt{2k^3}} (1 - ik\eta) \, e^{ik\eta}}$$

$$\phi_k(\eta) = \phi_k^{cl}(\eta) \, \hat{a}_k^\dagger + \phi_k^{cl}(\eta)^* \, \hat{a}_{-k}$$

- $\phi_k^{cl}$ is a solution of the EOM

$$\partial_\eta \left[ \eta^{-2} \left( (\cancel{1} - ik\eta) ik \cancel{- ik} \right) e^{ik\eta} \right] + k^2 \eta^{-2} (1 - ik\eta) \, e^{ik\eta} = 0$$

- Minkowski normalization

$$\phi_k^{cl} \xrightarrow[k \to +\infty]{} \frac{H}{\sqrt{2k^3}} (-ik\eta) \, e^{ik\eta_0} \cdot e^{ik/a \, \Delta t}$$

$$= \frac{1}{\sqrt{2k/a}} \, a^{-3/2} \, e^{ik/a \cdot \Delta t}$$

The factor of $a^{-3/2}$ comes from the different normalization of $\hat{a}$ and $\hat{a}^\dagger$ in comoving coordinates, and the integral over $k$ comoving

$$\left[ \hat{a}_k ; \hat{a}_{k'}^\dagger \right] = (2\pi)^3 \delta(\vec{k} + \vec{k}')$$

$$\left[ \hat{a}_p ; \hat{a}_{p'}^\dagger \right] = (2\pi)^3 \delta(\vec{p} + \vec{p}') \qquad \hat{a}_p = a^{3/2} \hat{a}_k$$

$$\int d^3p\ \hat{a}_p = \int d^3k\ a^{-3}\ \hat{a}_k\ a^{3/2} \quad \checkmark$$

$$\langle \phi_{\vec{k}}(\eta)\ \phi_{\vec{k}'}(\eta) \rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}')\ |\phi_k^{cl}|^2 =$$

$$= (2\pi)^3 \delta(\vec{k}+\vec{k}')\ \frac{H^2}{2k^3}\left(1+k^2\eta^2\right) \simeq (2\pi)^3 \delta(\vec{k}+\vec{k}')\ \frac{H^2}{2k^3}$$

large scales

In Minkowski I would get

$$\langle \phi_{\vec{p}}(t)\ \phi_{\vec{p}'}(t) \rangle = (2\pi)^3 \delta(\vec{p}+\vec{p}')\ \frac{1}{2p}$$

$$\boxed{\langle \phi_k(\eta)\ \phi_{\vec{k}'}(\eta) \rangle \cong (2\pi)^3 \delta(\vec{k}+\vec{k}')\ \frac{H^2}{2k^3}}$$

Celebrated scale invariant result

In real space

$$\langle \phi(\vec{x};t)\ \phi(\vec{y};t) \rangle \simeq -\frac{H^2}{(2\pi)^2} \log \frac{|x-x'|}{L}$$

$$ds^2 = \frac{1}{H^2\eta^2}\left(-d\eta^2 + d\vec{x}^2\right) \qquad \eta \longrightarrow \lambda\eta \quad \vec{x} \longrightarrow \lambda\vec{x}$$

$$\varphi_{\vec{k}} \longrightarrow \lambda^3 \varphi_{\vec{k}/\lambda} \qquad \langle \varphi_{\vec{k}_1}, \varphi_{\vec{k}_2} \rangle = (2\pi)^3 \delta(\vec{k}_1+\vec{k}_2)\ \frac{1}{k_1^3} F(k\eta)$$

As fields become time-independent I get the $k^{-3}$ behaviour

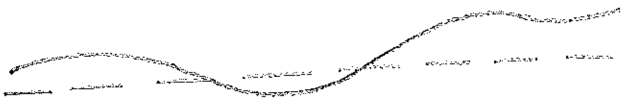
Independent of the Lagrangian!

Getting the results, we will justify them later

We are going through the full construction later as it connects with NG. Now we are sloppy

You have to buy two things

Spatially flat gauge: $g_{ij} = a^2(t)\,\delta_{ij}$ (assume no GWs)

$\varphi$  $t = \text{const}$

$$\phi(t;\vec{x}) = \phi_0(t) + \varphi(t;\vec{x})$$

Buy 1 — The quadratic action for $\varphi$ is the one of a massless scalar in dS up to slow-roll corrections

(Indeed the mass is slow-roll suppressed, dS up to $\epsilon$ and when I solve the constraints I will get $\epsilon;\eta$ suppression)

I am interested in the metric on constant inflaton surfaces as these will become constant T surfaces in RD, while $\varphi$ won't have any meaning

$$\delta t = -\frac{\varphi(t;\vec{x})}{\dot{\phi}}$$

Now this is the $t = \text{const}$ surface

The spatial metric in the new coordinates will be

$$a^2(t+\delta t)\,\delta_{ij} = a^2(t)\underbrace{\left[1 - 2\frac{H\varphi}{\dot{\phi}}\right]}_{1+2\zeta(\vec{x};t)}\delta_{ij}$$

Constant inflaton surfaces ( and later on T = const surfaces) are not flat any more. Non-flat induced metric

$$^{(3)}R = -\frac{4}{a^2}\nabla^2 \zeta$$

(Buy 2) <u>$\zeta$ is constant outside $H^{-1}$</u> (up to corrections $\frac{k^2}{a^2H^2}$)

This result will be true non-linearly and it is valid without slow-roll approximation

I am interested in the spectrum of $\zeta$ when it is comfortably out of $H^{-1}$

$$\langle \zeta_{\vec{k}}(\eta)\, \zeta_{\vec{k}'}(\eta')\rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}') \underbrace{\frac{H^2}{2k^3}}_{\text{free scalar in dS}} \underbrace{\frac{H^2}{\dot{\phi}^2}}_{\text{to go to } \zeta \text{ variable}} \Bigg|_{\text{horizon crossing}}$$

This quantity is quite close to observations. E.g. in the large scale limit (Sachs - Wolfe)

$$\frac{\delta T}{T} = -\frac{1}{5}\zeta$$

## Normalization of the spectrum

Experimental results are given in terms of $\frac{k^3 P_\zeta}{2\pi^2}$

where $\langle \zeta \zeta \rangle = (2\pi)^3 \delta(\quad) P_\zeta$

(This describes the contribution to the variance per logarithmic k-interval

$$\int \frac{d^3k}{(2\pi)^3} P_\zeta = \int \frac{dk}{k} \frac{P_\zeta(k)\, k^3}{2\pi^2}$$
)

$$\frac{k^3 P_\zeta}{2\pi^2} = (2.4 \pm 0.1)\, 10^{-9}$$

WMAP 7 + others (the old COBE normaliz)

Typical amplitude of $\left(\frac{\delta T}{T}\right)^2$ in the CMB

Roughly: $\langle \zeta \zeta \rangle \simeq \frac{H^4}{\dot\phi^2}$

Amplitude $\sim \frac{H^2}{\dot\phi}$ Ratio between quantum perturbations $(H)$ and classical motion in one Hubble time $\dot\phi / H$

It can also be written as: $\sim \frac{H}{M_P} \frac{1}{\sqrt{\epsilon}}$

(What not to do to estimate the spectrum

$$\frac{\delta\rho}{\rho} \simeq \frac{V'}{V}\delta\phi \simeq \frac{V'}{V} H \simeq \sqrt{\epsilon}\, \frac{H}{M_P}$$

WRONG!

Think of the inflaton as a clock, not something that carries energy)

E.g. $\quad V = \frac{1}{2} m^2 \phi^2$

$$\epsilon = \frac{1}{2} M_P^2 \left(\frac{V'}{V}\right)^2 \qquad \eta = M_P^2 \frac{V''}{V}$$

$$\phi_{end} \simeq M_P$$

$$d\phi \simeq -\frac{V'}{3H} dt = -\frac{V'}{3H^2} dN = -\frac{V'}{V} M_P^2 \, dN$$

$$N(\phi) = M_P^{-2} \int_{\phi_{end}}^{\phi} \frac{V}{V'} d\phi \underset{V=\frac{1}{2}m^2\phi^2}{=} M_P^{-2} \int \frac{\phi}{2} d\phi = M_P^{-2} \left[\frac{\phi^2}{4} - \frac{\phi_{end}^2}{4}\right]$$

$\phi \simeq 2 M_P \sqrt{N} \qquad \phi_{end}$ is irrelevant

$$\frac{H^4}{(2\pi)^2 \dot\phi^2} : \quad \frac{H^2}{2\pi\dot\phi} \simeq \frac{\frac{1}{3M_P^2} \frac{1}{2} m^2 \phi^2}{2\pi \sqrt{\frac{2}{3}} \, m M_P} = \frac{1}{4\pi\sqrt{6}} \frac{m M_P^2 4N}{M_P^3}$$

$$= \frac{1}{\pi\sqrt{6}} \frac{m}{M_P} N \quad \Longrightarrow \quad m \simeq 1.3 \; 10^{13} \text{ GeV}$$

<u>Scale dependence</u> : As $H$ and $\dot\phi$ are not exactly constant I get a small deviation from scale invariance

$$k^{-3+(n_s-1)}$$

$$n_s - 1 = \frac{d}{d\log k} \log \frac{H^4}{\dot\phi^2}\Big|_{\text{crossing}} = H^{-1}\frac{d}{dt}\log\frac{H^6}{V'^2}$$

$k \sim aH \qquad a \sim e^{Ht}$

$d\log k = H dt$

$$= \frac{6\dot H}{H^2} - 2\frac{V''}{V'}\frac{\dot\phi}{H} = -6\epsilon + 2\eta$$

$$\frac{\dot\phi}{H} = -\frac{V'}{3H^2}$$

Red spectrum $n_s - 1 < 0$ More power at low k

Blue spectrum $n_s - 1 > 0$ " at high k

E.g. $V = \frac{1}{2}m^2\phi^2 \qquad \epsilon = \eta = \frac{1}{2N}$

$$n_s - 1 = -6\epsilon + 2\eta = -8\frac{M_P^2}{\phi^2} = -\frac{2}{N}$$

Indeed we saw $P_\zeta \propto N^2$

$$n_s - 1 = -\frac{d}{dN}\log N^2 = -\frac{1}{N}\frac{d}{d\log N}\log N^2 = -\frac{2}{N}$$

$\simeq 0.97$ with $N = 60$

WMAP 7 : $n_s = 0.963 \pm 0.012$

Reheating enters in the predictions

A <u>non-trivial prediction of inflation</u> :

we are in dS, but only approximately!

Typically (?) : $|n_s - 1| \sim 1/N$

We may object: we did the calculation at leading order in slow-roll and we got the tilt that is slow-roll suppressed. Did I lose some piece?

$$\langle \zeta \zeta \rangle \sim \frac{H^4}{\dot{\phi}^2} \left( 1 + O(\epsilon; \eta) + O\left(\frac{k^2}{a^2 H^2}\right) \right)$$

slow-roll corrections in the action that I neglected

$\zeta$ becomes constant out of $H^{-1}$

These corrections will also be slowly varying with $k$ as $H^4/\dot{\phi}^2$ does. So they will give small corrections to the spectrum <u>and to the tilt</u>

(Alternatively I could switch to $\zeta$ for all the modes at the end of inflation. But in this case $\varphi$ does evolve out of $H^{-1}$:

slow-roll evolution $\times$ $N$ e-folds can be large

I should take into account its mass, the massless de Sitter approx. is not enough)

## Gravitational waves

Any light field $\gamma$ ($m \ll H$) gets excited during inflation.
Tensor modes behave quite similarly to a scalar field

$$S_{(2)} = \frac{M_P^2}{8} \int d\eta\, d^3x \; a^2 \left[ \gamma_{ij}'^2 - (\partial_l \gamma_{ij})^2 \right]$$

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \epsilon^s_{ij}(k)\, \gamma^s_{\vec{k}}(\eta)\, e^{i\vec{k}\vec{x}}$$

Polarization tensors are transverse, traceless $\epsilon_{ii} = k^i \epsilon_{ij} = 0$
and normalized $\epsilon^s_{ij}(k)\, \epsilon^{s'}_{ij} = 2\delta_{ss'}$

$$\langle \gamma^s_{\vec{k}}\, \gamma^{s'}_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') \frac{1}{2k^3} \frac{2H^2}{M_P^2} \delta_{ss'}$$

Notice a factor 2 comes from the normaliz. of $\epsilon_{ij}$

$$\frac{k^3 P_T}{2\pi^2} = 4 \cdot \frac{1}{2\pi^2} \frac{H^2}{M_P^2} = \frac{2}{\pi^2} \frac{H^2}{M_P^2}$$

2 polarizations, each normalized to 2

Notice we have no $\frac{1}{\epsilon}$ enhancement

$$r \equiv \frac{P_T}{P_\zeta} = \frac{8H^2/M_P^2}{H^4/\dot{\phi}^2} = -16 \frac{\dot{H}}{H^2} = 16\epsilon$$

E.g. $\quad V = \frac{1}{2} m^2 \phi^2 \implies r = 16\epsilon = \frac{8}{N} \simeq 0.13$

$$V^{1/4} \simeq \left( \frac{r}{0.01} \right)^{1/4} 10^{16}\,\text{GeV}$$

Gravitational waves are very robust: they just probe $H$ during inflation. They do not care about one field, two fields ...

A real probe of inflation!

WMAP 7: $r < 0.24$ $2\sigma$

Lyth's bound

$$r = \frac{8}{M_P^2}\frac{\dot{\phi}^2}{H^2} = \frac{8}{M_P^2}\left(\frac{d\phi}{dN}\right)^2$$

$$\frac{\Delta\phi}{M_P} = \int_{N_{end}}^{N_{CMB}} dN \sqrt{\frac{r}{8}} \Longrightarrow \frac{\Delta\phi}{M_P} = O(1)\left(\frac{r}{0.01}\right)^{1/2}$$

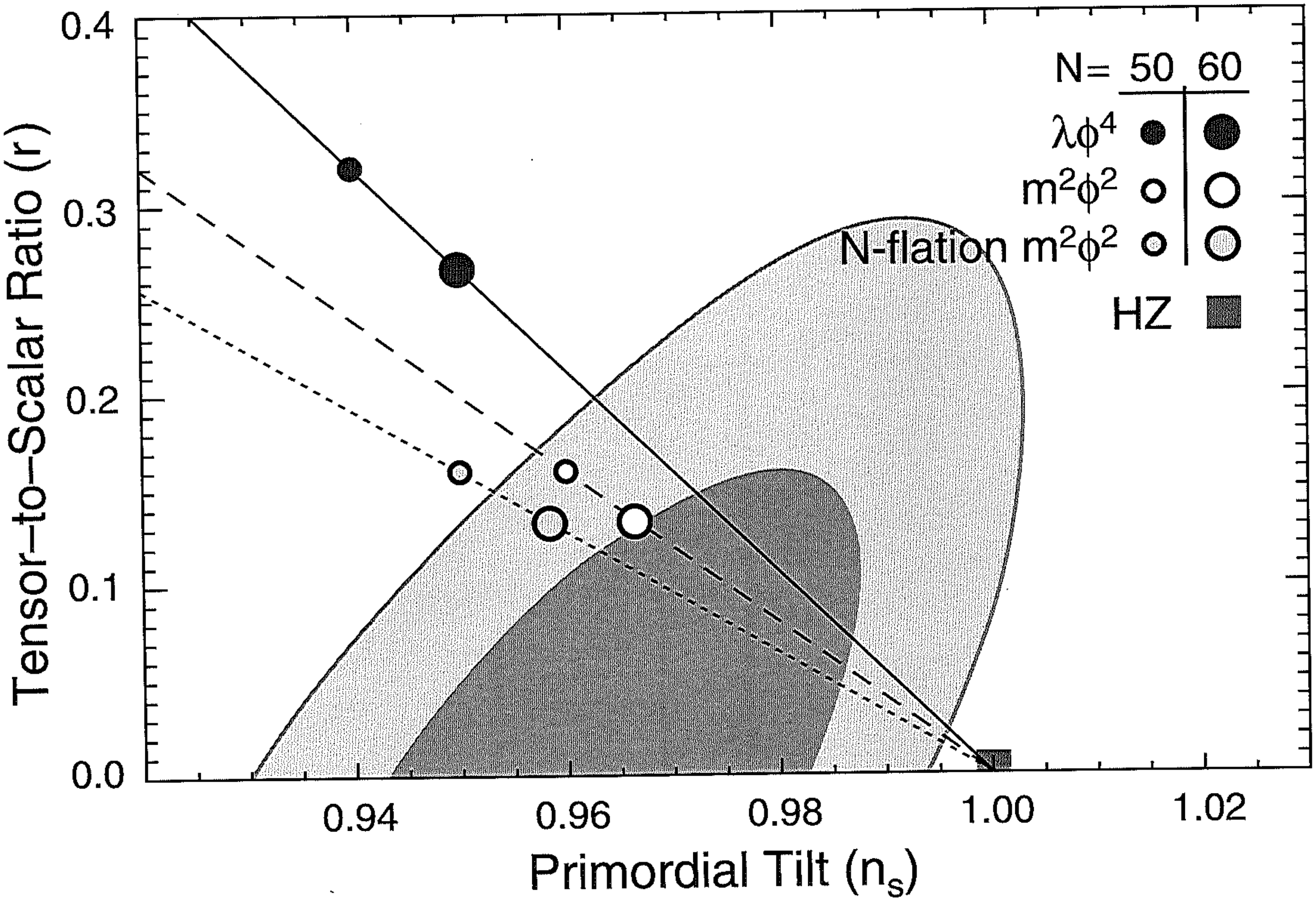
Detection of GWs $\Longrightarrow$ Super Planckian displacement

Tilt of the GW spectrum + consistency relation

$$n_t = \frac{d \log H^2}{d \log k} = -2\epsilon \qquad \text{Always red as } \dot{H} < 0$$

This is fixed by $r = 16\epsilon$ : consistency relation

Quite challenging to probe!


N= 50 60
λφ4
m2φ2
N-flation m2φ2
HZ
0.4
0.3
0.2
0.1
0.0
0.94
0.96
0.98
1.00
1.02
Tensor–to–Scalar Ratio (r)
Primordial Tilt (ns)

# Perturbations: helicity decomposition + gauges

- The background is homogeneous $\longrightarrow$ Conservation of 3d momentum

  E.g. No mixing of $\vec{k}$ at linear order

(Notice we do not have Poincaré invariance: I cannot boost, only rotate. We are looking for the irreducible representations of rotations)

- Helicity: I can classify states according to the way they transform under rotation around $\vec{k}$

  Scalar, vector and tensors. At first order there is no mixing

$$ds^2 = a^2(\eta)\Big\{-(1+2\phi)d\eta^2 + 2(\partial_i B - S_i)\,d\eta\, dx^i$$
$$+\left[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + 2\partial_i F_j + \gamma_{ij}\right]dx^i dx^j\Big\}$$

All vectors are transverse $\partial^i S_i = \partial^i F_i = 0$

$\gamma$ is transverse and traceless: $\gamma^i{}_i = \partial^i \gamma_{ij} = 0$

- In the absence of sources we would have only tensors, but the perturbations of the scalar will mix with the other scalar perturbations
  We can forget about vectors

- Gauge transformations

$$\tilde{\eta} = \eta + \xi^0 \qquad \tilde{x}^i = x^i + \partial^i \xi + \bar{\xi}^i$$
$$\partial_i \bar{\xi}^i = 0$$

I have two scalar transformations and one vector

E.g. I can choose $\delta\varphi = 0 \quad E = 0$    $\zeta$-gauge

$\Psi = E = 0$    Spatially flat gauge

$B = E = 0$    Newtonian gauge

- I have 5 scalar perturbations (inflaton + 4 metric) with 2 gauge freedoms and (we will see) two constraint equations. Left with 1 final scalar mode

- 2 vectors, 1 gauge and 1 constraint equation. Nothing left

## 2.1 Spectrum of perturbations in spatially flat gauge

Let us complete the calculation of the spectrum of scalar and tensor perturbations in the spatially flat gauge. We will see that in this gauge the action for scalar perturbations is the one of a massless scalar field at leading order in slow-roll, as the mixing with gravity only induces corrections which are explicitly slow-roll suppressed. We start from the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_P^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] . \tag{2.1}$$

It is useful to write the metric in the so-called ADM form

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) . \tag{2.2}$$

We can now rewrite the action using the Gauss-Codazzi relation, $R = {}^{(3)}R + K_{ij}K^{ij} - K^2$, which relates the 4d Ricci scalar $R$ with the 3d one ${}^{(3)}R$ and the extrinsic curvature of surfaces of constant time $K_{ij}$ defined as

$$K_{ij} \equiv N^{-1} E_{ij} \qquad E_{ij} \equiv \frac{1}{2}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i) , \tag{2.3}$$

where the covariant derivatives are with respect to the 3d induced metric. The action takes the form

Inverse of ADM metric

$$\begin{aligned} S = \int d^4x \sqrt{h} \Big\{ & \frac{M_P^2}{2} \left[ N\, {}^{(3)}R + N^{-1}(E_{ij}E^{ij} - E^2) \right] \\ & + \frac{1}{2} N^{-1} (\dot{\phi} - N^i \partial_i \phi)^2 - \frac{1}{2} N h^{ij} \partial_i \phi \partial_j \phi - NV \Big\} , \end{aligned} \tag{2.4}$$

where we have used $\sqrt{-g} = N\sqrt{h}$. It is time to choose the gauge. We choose to set to zero the 2 scalar degrees of freedom in the spatial part of the metric: this is called spatially flat gauge. Actually the spatial metric is not flat, but perturbed only by transverse and traceless perturbations, i.e. gravitational waves. Perturbations are therefore of the form

$$h_{ij} = a^2(t)(\delta_{ij} + \gamma_{ij}) \qquad \partial_i \gamma_{ij} = 0 \quad \gamma_{ii} = 0 \tag{2.5}$$

$$\phi = \phi_0(t) + \varphi(\vec{x}, t) \tag{2.6}$$

This choice of gauge does not affect the ADM variables $N$ and $N^i$. Notice that these variables appear in the action (2.4) without time derivatives. This means that these degrees of freedom do not propagate and act as Lagrange multipliers, whose equations of motion are constraint equations. These constraint equations will allow us to relate $N^i$ and $N$ to the scalar degree of freedom $\varphi$: plugging the solutions back into the action (2.4), we will get an action where only the physical propagating degrees of freedom appear: $\varphi$ and the tensor modes $\gamma_{ij}$. We need to express $N$ and $N^i$ in terms of $\varphi$ at first order in perturbations: a second order term would go to multiply $\partial S/\partial N$ and $\partial S/\partial N^i$ at zeroth order, i.e. on the unperturbed solution. But the unperturbed solution solves the constraint equations, so these terms vanish. From now on we take $\gamma = 0$ as at linear order there is no mixing between tensor and scalar modes.

With these simplifications the equations obtained by varying with respect to $N^i$ and $N$ are:

$$M_P^2 \partial_i \left[N^{-1}(E^i_j - \delta^i_j E)\right] - \partial_j \varphi \, \dot\phi_0 = 0 \tag{2.7}$$

$$\frac{M_P^2}{2N^2}(E_{ij}E^{ij} - E^2) + N^{-2}\frac{\dot\phi^2}{2} + V = 0 \tag{2.8}$$

One can write $N^i = \partial_i \alpha + N^i_T$, with $N^i_T$ transverse $\partial_i N^i_T = 0$. The vector mode $N^i_T$ is not sourced by the scalar $\varphi$ in eq. (2.7) and can thus be set to zero: vector and scalar perturbations cannot mix at linear order. Using that $E_{ij} = a\dot a \delta_{ij}$ for the unperturbed solution, and $E^{(1)i}_j = -\partial^i \partial_j \alpha$ for the linear perturbation, eq. (2.7) takes the form

$\frac{\dot\phi^2}{H^2 M_P^2} \simeq \epsilon$

$$\partial_i(2HM_P^2 \delta N - \dot\phi_0 \varphi) = 0 \,. \tag{2.9}$$

The other unknown $\alpha$ dropped from this equation so one can solve for the perturbation in the lapse

$$\delta N = \frac{\dot\phi_0}{2HM_P^2}\varphi \,. \tag{2.10}$$

$\delta N \simeq \frac{H}{M_P}\sqrt{\epsilon}$

Equation (2.8) takes the form

$$\frac{M_P^2}{2}\left[2\delta N \cdot 6H^2 + \frac{4}{a^2}H\partial^2\alpha\right] - \delta N \dot\phi_0^2 + \dot\phi_0 \dot\varphi + V'\varphi = 0 \tag{2.11}$$

which gives using (2.10)

$$\partial^2 \alpha = -\frac{a^2}{2HM_P^2}\left[\left(\frac{\dot\phi_0 V}{HM_P^2} + V'\right)\varphi + \dot\phi_0 \dot\varphi\right] \tag{2.12}$$

$\sim \frac{H}{M_P} \quad \frac{\dot\phi}{H M_P}$

Now we see what we were looking for: (2.10) and (2.12) imply that $\delta N$ and $\alpha$ are slow-roll suppressed with respect to $\varphi$. This implies that plugging these solutions back into the action will give corrections to the action for $\varphi$ which are slow-roll suppressed. Let us see this explicitly. The quadratic terms that we get expanding (2.4) are of the form

$$\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2a^2}(\partial\varphi)^2 - \frac{1}{2}V''\varphi^2$$
$$- 2HM_P^2\partial^2\alpha\delta N - 3H^2M_P^2\delta N^2 - \dot{\phi}_0\delta N\dot{\varphi} + \dot{\phi}_0\partial^2\alpha\varphi - \delta NV'\varphi \,, \tag{2.13}$$

where we have integrated by parts to obtain the term before the last in the second line. Let us concentrate on the second line, which comes from the mixing of $\phi$ with gravity. The contributions with $\alpha$ cancel using the expression for $\delta N$ and we are left with

$$-3H^2M_P^2\frac{\dot{\phi}_0^2}{4H^2M_P^4}\varphi^2 - \frac{\dot{\phi}_0^2}{2HM_P^2}\varphi\dot{\varphi} - \frac{\dot{\phi}_0}{2HM_P^2}V'\varphi^2 \,. \tag{2.14}$$

The second term can be integrated by parts using $\sqrt{h} = a^3$ and its dominant contribution in slow-roll cancels with the first term. Rewriting the last term using the slow-roll parameter $\epsilon$, we are thus led (neglecting terms further suppressed in slow-roll) to the action for $\varphi$

$$S = \int d^4x a^3 \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}V''\varphi^2 + 3\epsilon H^2\varphi^2\right] \,. \tag{2.15}$$

At leading order in slow-roll the action for $\varphi$ is the one of a massless scalar: the mixing with gravity just induces an $\epsilon$ suppressed mass term.[1] For reference, the exact action at all orders in slow-roll is

$$S = \int d^4x a^3 \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}V''\varphi^2 + \frac{1}{a^3}\frac{d}{dt}\left(\frac{a^3\dot{H}}{H}\right)\varphi^2\right] \,. \tag{2.16}$$

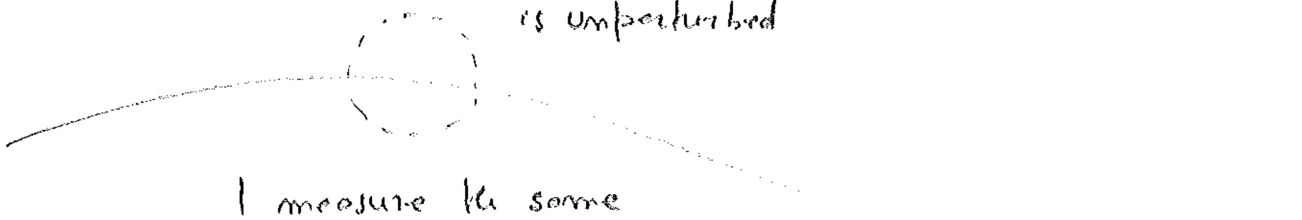
Of course this procedure induces cubic and higher-order terms in the action

[1] One can calculate the tilt of the spectrum of $\zeta$ also starting from the spectrum of $\varphi$ and converting to $\zeta$ at the end. This requires keeping the mass terms in the action (2.16) and the deviation of the metric from exact de Sitter.

# Conservation of $\zeta$

The intuitive idea behind it is the one of separate Universes

Locally the Universe
is unperturbed



I measure the same
$\rho_i(H)$ as in the unperturbed
Universe

No relative perturbations that can be locally measured

$\zeta(x)$ is just an unobservable rescaling of the spatial coordinates which matters only when the modes come back in

(Notice that this picture is useful also when there are more fields, only the local perturbation of the fields matters and not the gradients : in this case we have different homogeneous evolutions we will discuss about it later)

We are in $\zeta$-gauge ($\varphi = 0$, $E = 0$). In general this gauge can be defined as velocity orthogonal, the velocity of the fluid is $\perp$ constant time surfaces

Indeed $U^\mu \propto \partial^\mu \phi$

$$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^{\,2}$$

$\vec{x}' = \vec{x}\, e^{-\lambda(t)}$ is an unfixed diff. at $k=0$

I am not touching the slicing, $\delta\varphi = 0$ remains and I am not inducing $g_{ij} \supset \partial_i \partial_j E$

Is the $k \to 0$ limit of a physical mode?

I love to check whether some equation disappears in the $k \to 0$ limit

3d covariant derivative (which is the 3d projection of the 4d covariant derivative)

$$D^a \left( K_{ab} - K^c_c\, h_{ab} \right) = +8\pi G \left( h_b^c\, T_{ca}\, n^a \right)$$

$$\partial_b \left( H\, \delta N - \dot{\zeta} \right) \qquad\qquad 0$$

because $U^\mu$ is perpendicular to $t = \text{const}$

But $N = 1$ as we are not redefining the slicing so

$$\zeta = \text{const}$$

- This statement <u>does not depend on the equation of state</u> or any details of the unperturbed evolution

- Notice what happens if I look at the momentum constraint in the diff induced solution

$$\vec{x}' = e^{-\lambda(t)}\, \vec{x}$$

$$t' = t$$

$$N = 1 \qquad N_i = g_{0i} = \frac{\partial x^\mu}{\partial x^0} \frac{\partial x^\nu}{\partial x^i} g_{\mu\nu} = \dot{\lambda}\, e^{2\lambda} a^2 x^i$$

Spatial metric: $a^2(t)\, e^{2\lambda(t)} = (a e^\lambda)^2$

$$E_{ij} = \frac{1}{2}\left( \dot{h}_{ij} - \nabla_i N_j - \nabla_i N_j \right) = \frac{1}{2}\Big( 2 a e^\lambda \left( \dot{a} e^\lambda + \dot{\lambda} a e^\lambda \right) \delta_{ij}$$

$$- 2\dot{\lambda} e^{2\lambda} a^2 \delta_{ij} \Big) = e^{2\lambda} a^2 \delta_{ij} \left( H + \cancel{\dot{\lambda}} \right) - \cancel{\dot{\lambda} e^{2\lambda} a^2 \delta_{ij}}$$

So of course it is satisfied but in a very $\neq$ way that finite $K$ solution, where $\partial_i N_j$ cancel

- Non-linearly

$$\nabla_i \left[ N^{-1} \left( E^i_j - \delta^i_j E \right) \right] = 0$$

$N = 1$ For scalar shift $N^i = \partial^i \alpha$

$$\partial_i \left( - \partial_i \partial_j \alpha + \delta^i_j \partial^2 \alpha \right) = 0$$

(This assuming $N^i_T$ vanish at large scales)

As $E_{ij} = e^{2\lambda} a^2 (H + \dot\lambda)$ The gauge solution has $N^i_T = 0$

we have $\dot\zeta = 0$ $\nabla^2 N^i_T$

- In the presence of more fields

$$h^c_b T_{ca} n^a \neq 0$$

We are velocity orthogonal to one of them, but others give other surfaces. This solution always exists but there are more.

- Notice also the Hamiltonian constraint is subtle though it does not contain an explicit derivative

$$(3H^2 + \dot H) \delta N + H \partial_i N^i = - \frac{\nabla^2}{a^2} \zeta + 3H\dot\zeta$$

This cancels $\dot\zeta$ in the gauge mode, but in a physical solution is suppressed as $k \to 0$

- What goes wrong if I take $\lambda(t)$ and make it slightly $\vec x$ dependent? Helmholtz decomposition fails
- Genesis case : $\begin{cases} H\delta N \approx \dot\zeta & H \propto t^{-3} \quad \delta N \propto t \ldots \\ \text{it is not a perfect fluid: RHS} \end{cases}$