

IV Southern-Summer School on Mathematical Biology

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Lecture II

São Paulo, January 2015



Outline

1 Interacting Species



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2 Predation



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- 1 Interacting Species
- 2 Predation
- 3 Lotka-Volterra



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- 4 Beyond Lotka-Volterra

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- 5 Glory and Misery of the Lotka-Volterra Equations

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- 6 Further beyond the Lotka-Volterra equations



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- 4 Beyond Lotka-Volterra
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- 6 Further beyond the Lotka-Volterra equations
- 7 Final comments



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Types of interactions



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- We define **three** types of basic interactions between species:
- **Predation**: the presence of a species **(A)** is detrimental for species **(B)**, but the presence of **(B)** favours **(A)**. Species **(A)** is the predator, and **(B)** is its prey^a.
- **Competition**: the presence of **(A)** is detrimental for **(B)** and vice-versa.
- **Mutualism**: the presence of **(A)** favours **(B)** and vice-versa.

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Nota bene

There is also the **amensalism** (negative for one species, neutral for the other) and the **comensalism** (positive for one species and neutral for the other). Not to speak of **neutralism**.

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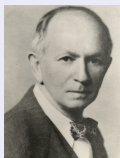
- **Predation** is a widespread interaction between species.
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- This is known as the *Lotka-Volterra* model.

Lotka and Volterra

Muoiono gl'imperi, ma i teoremi d'Euclide conservano eterna giovinezza (Volterra)



Vito Volterra (1860-1940), an Italian mathematician, proposed the equation now known as the Lotka-Volterra one to understand a problem proposed by his future son-in-law, Umberto d'Ancona, who tried to explain oscillations in the quantity of predator fishes captured at the certain ports of the Adriatic sea.



Alfred Lotka (1880-1949), was an USA mathematician and chemist, born in Ukraine, who tried to transpose the principles of physical-chemistry to biology. He published his results in a book called "Elements of Physical Biology", dedicated to the memory of Poynting. His results are independent from the work of Volterra.

The Lotka-Volterra equations

Let

- $N(t)$ be the number of predators,
- $V(t)$ the number of preys.

In what follows, a , b , c e d are **positive constants**

The Lotka-Volterra equations

0 number of prey will increase when there are no predators:

$$\frac{dV}{dt} = aV$$

The Lotka-Volterra equations

But the presence of predators should decrease the growth rate of prey:

$$\frac{dV}{dt} = V(a - bP)$$

The Lotka-Volterra equations

On the other hand the population of predators should decrease in the absence of prey :

$$\frac{dV}{dt} = V(a - bP)$$

$$\frac{dP}{dt} = -dP$$

The Lotka-Volterra equations

and presence of prey will increase the number of predators:

$$\frac{dV}{dt} = V(a - bP)$$

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The Lotka-Volterra equations

These two coupled equations are known as
The Lotka-Volterra equations

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Let's study them!

Lotka-Volterra: analysis

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- So that:

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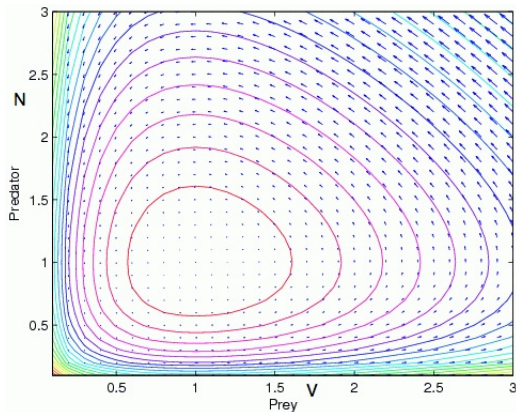
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Phase trajectories



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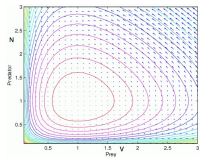
$$\frac{dP}{dt} = P(cV - d)$$

The phase trajectories of the Lotka-Volterra equations, with $a = b = c = d = 1$. Each curve corresponds to a given value of H . The curves obey: $c\mathbf{V}(\mathbf{t}) - b\mathbf{P}(\mathbf{t}) + a \ln \mathbf{P}(\mathbf{t}) + d \ln \mathbf{V}(\mathbf{t}) = H$

Lotka-Volterra: oscillations

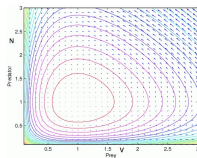


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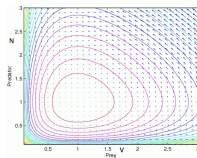
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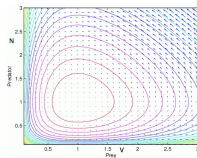
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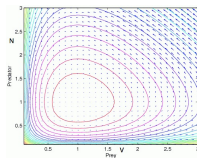
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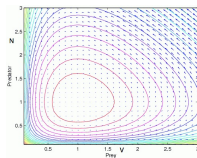
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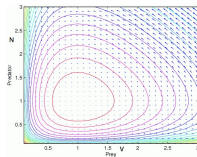
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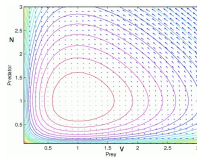
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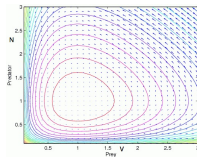
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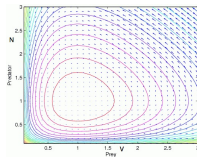
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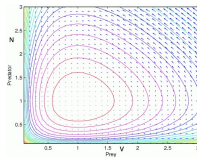
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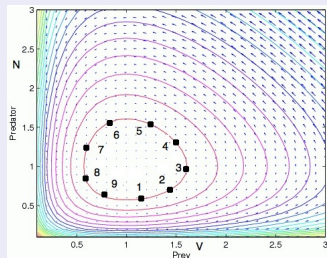
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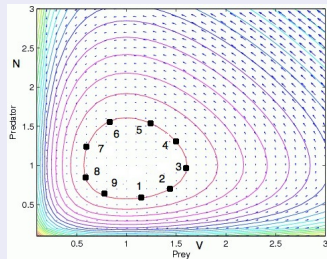
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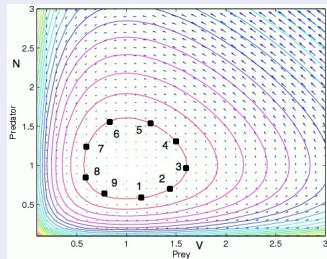
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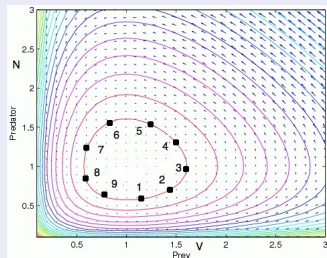
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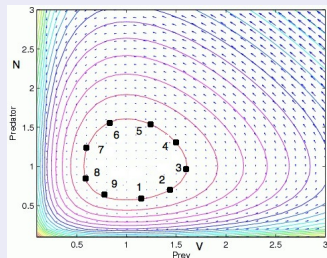
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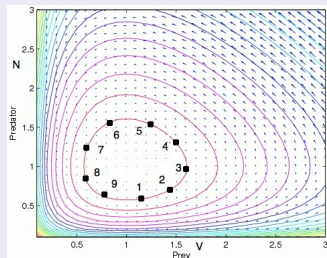
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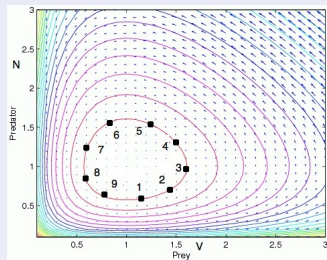
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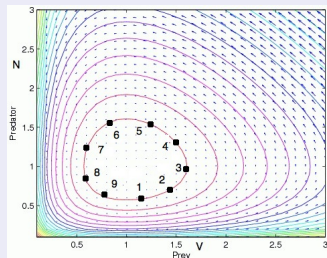


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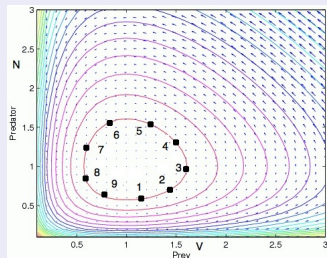


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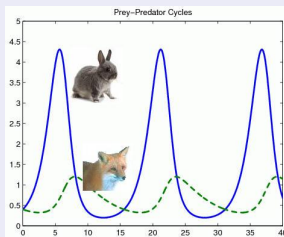
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In words...

- Lotka-Volterra equations tell us that:
 - ▶ Given a **small** number of predators and a certain number (not small) of prey ;
 - ▶ The availability of prey makes the population of predators **grow**;
 - ▶ And therefore the prey population will grow slower. After a certain amount of time, it will begin to **decrease** ;
 - ▶ And predators attain a **maximal** population, and – because the lack of enough prey – it's population begins to **decrease**;
 - ▶ Meanwhile, prey get to a **minimum** and begin to **recover**, as the number of predators has **decreased**;
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- Makes sense!
- **But, is it true?**

The real world

- Does the Lotka-Volterra equations describe real situations?
- Partially.
- There are some elements that are clearly not realistic:
 - ▶ The growth of prey in the absence of predator is exponential; it does not saturate.
 - ★ No big deal. Just put a logistic term there. We can still have oscillating solutions. Great!
 - ▶ On the other hand... the growth rate of the predator is given by $(cV - d)$.
 - ▶ The larger V , the higher the rate. This predator is voracious!
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 - ▶ It would be rather natural to suppose that the conversion rate also saturates. An effect of the predators becoming satiated or because there is handling time to consume prey.
 - ▶ We can modify the above equations to take this into account.
- Cycling can still be present.

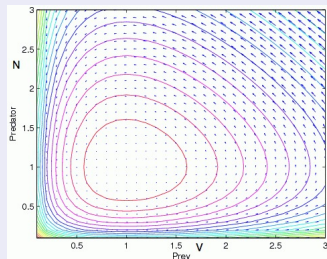


Glory and Misery of the Lotka-Volterra Equations

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- The lesson of the Lotka-Volterra equation is: although being an oversimplified equation for predator-prey system it captures an important feature: this kind of system exhibits oscillations – which are intrinsic to the dynamics.

Misery



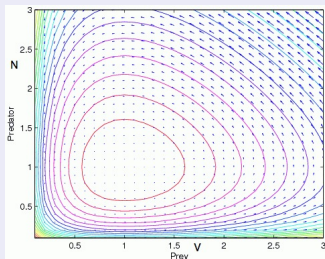
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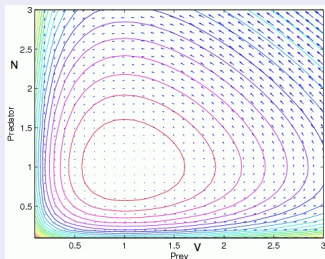
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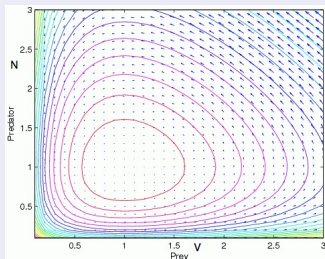
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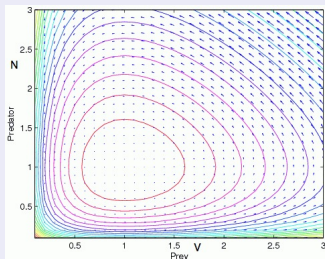
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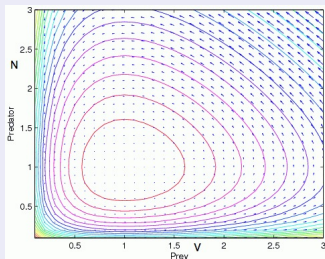
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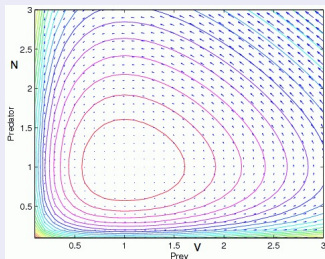
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- Real predator-prey oscillations would be better described by limit-cycles. **What's a limit cycle???**

Further beyond the Lotka-Volterra equations

- Obviously real interactions occur in interaction webs that can involve many species through predation, competition and mutualism.

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- Obviously real interactions occur in interaction webs that can involve many species through predation, competition and mutualism.
- Simple questions:
 - ▶ Whereupon does the prey feed?
 - ▶ This is not taken into account in the Lotka-Volterra equations.
 - ▶ If resource availability for prey is approximatively constant than a (generalized) Lotka-Volterra dynamics is maybe a good model.
 - ▶ But, on the other hand, the possibility exists that the prey and its resource are dynamically coupled... In this case we need to consider at least three species.
 - ▶ But beware!!! Do not try to put all species in a model.
- In summary, the Lotka-Volterra equations are rather a starting point than a final point for predator-prey models. .

A last comment

Host-parasitoid relations

- In close relation to the predator-prey dynamics there is the relation a parasitoid and its host ,
- The parasitoid plays a role analogous to the one of the predator and the host, that of the prey.
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- Although these may be seen as different biological interactions, the dynamics is similarly described.
- Note, however, that many insect species have non-overlapping generations.
- which takes us to the realm of discrete-time equations, or coupled mappings.



What I should remember

- Two-species interactions are the building blocks of larger networks of interactions:
- In a rough way, we can divide them as:
 - ▶ **predator-prey;**
 - ▶ **competition;**
 - ▶ **mutualism.**
- Predator-Prey tend to produce oscillations.
- Just don't forget that not every oscillation comes from a predator-prey dynamics.

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Online Resources

- <http://www.ictp-saifr.org/mathbio4>
- <http://ecologia.ib.usp.br/ssmb/>

Thank you for your attention