Exploring Universal behavior in few-body systems

A. Kievsky

INFN, Sezione di Pisa (Italy)

ICTP-Saifr, Minischool on few-body physics,
Sao Paolo, October 2014

Collaborators

- M. Gattobigio - *INLN & Nice University, Nice (France)*
- E. Garrido - *CSIC, Madrid (Spain)*
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Glossary

Universality in weakly bound N-boson systems

- The particles are most of the time outside the interaction range
- The dynamics is insensible to the short-range part
- There is a control parameter: the scattering length a

Scaling

- Universal dynamics $\rightarrow$ non-dimensional quantities
- non-dimensional quantities $\rightarrow$ scale parameter $\rightarrow$ real world
- real world $\rightarrow$ finite-size scaling parameter $\rightarrow$ real system

finite-size scaling

- $x = \kappa_* a \longrightarrow \kappa_*$ define the scale
- zero-range potential $g\delta(|\mathbf{r}_1 - \mathbf{r}_2|)$ vs finite-range potentials $V(|\mathbf{r}_1 - \mathbf{r}_2|) \longrightarrow$ small parameter r_0/a
- $\Gamma \propto \kappa_* r_0 \longrightarrow$ finite-size scale parameter

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Universality in the two-body system

Shallow states (short range interactions)

- Low energy scattering $E = \hbar^2 k^2/m << \hbar^2/m r_0^2$ or $k r_0 << 1$

$$S - \text{wave scattering} \longrightarrow k \cot\delta \approx -\frac{1}{a} + \frac{1}{2} r_0 k^2$$

- Shallow bound states: $a >> r_0 \rightarrow$ shallow dimer $E_D \approx \hbar^2/ma^2$

- $r_0 \rightarrow 0$, $k \cot\delta = -1/a$ and $E_D = \hbar^2/ma^2$

Continuous Scale Invariance ($r_0/a \rightarrow 0$)

- $\Psi_D \rightarrow \frac{e^{-r/a}}{r}$ $< r^2 > \rightarrow a^2/2$ $\sigma \rightarrow 2\pi \frac{4a^2}{1+a^2k^2}$
- $a \rightarrow \lambda a$

  $E \rightarrow \lambda^{-2} E$

  $\sigma \rightarrow \lambda^2 \sigma$

  $< r^2 > \rightarrow \lambda^2 < r^2 >$

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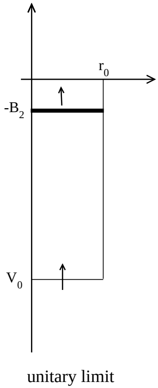
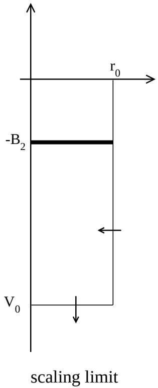
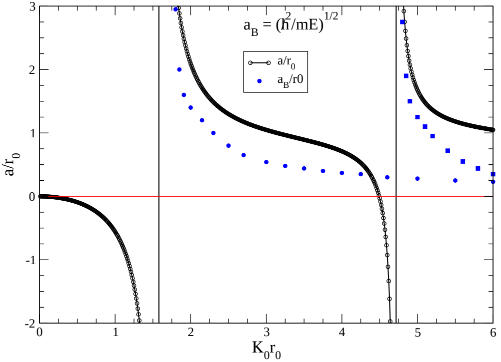
$N = 3$: The Efimov (1970) and Thomas (1935) effects





$$\frac{r_0}{a} << 1 \qquad B_2 \approx \hbar^2/ma^2 \qquad B_2 = \hbar^2/ma^2$$

Natural and unnatural states (shallow states)



Most of the time $E \approx -\hbar^2/mr_0^2 \rightarrow$ natural state

In same cases $a >> r_0$ and $E \approx -\hbar^2/ma^2 \rightarrow$ fine tunning

$N = 3$

Thomas collapse (scaling limit)

- $r_0 \to 0$, $V_0 \to -\infty$
- $B_2 = constant \to \hbar^2/ma^2$
- The three-body bound state $E_3 \to -\hbar^2/mr_0^2 \to -\infty$
- The three-body bound state is unbound from below
- curiosity: even if $B_2 \approx 0$, $E_3 \to -\infty$

Efimov effect (unitary limit)

- $E_2 \to 0$, $a \to \infty$ (CSI)
- $r_0 = constant$
- a series of states appears between $-\hbar^2/mr_0^2 \le E_3^n \le -\hbar^2/ma^2$
- two consecutive states have ratio: $E_3^{(n+1)}/E_3^{(n)} \to e^{-2\pi/s_0}$
- The series of states is infinite at the unitary limit
- s_0 is an universal number (DSI)

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The Efimov Effect

- The three body Hamiltonian:

$H = T + V(1,2) + V(2,3) + V(3,1)$

- Hyperspherical coordinates:

$\rho^2 = x^2 + y^2 \qquad \boldsymbol{x} = \boldsymbol{r}_1 - \boldsymbol{r}_2\,, \qquad \boldsymbol{y} = \boldsymbol{r}_3 - (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2\,,$

$[\boldsymbol{x},\boldsymbol{y}] \equiv [\rho,\Omega] = [\rho,\hat{x},\hat{y},\arctan(x/y)] = [\rho,\hat{x},\hat{y},\alpha]$

- The kinetic energy:

$$T = T_\rho - \frac{L^2(\Omega)}{\rho^2} = \frac{\partial^2}{\partial\rho^2} + \frac{5}{\rho}\frac{\partial}{\partial\rho} - \frac{L^2(\Omega)}{\rho^2}$$

- Adiabatic Hyperspherical expansion:

$$\Psi(\boldsymbol{x},\boldsymbol{y}) = \Psi(\rho,\Omega) = \rho^{-5/2}\sum_n w_n(\rho)\phi_n(\rho,\Omega)$$

- the adiabatic functions $\phi_n(\rho,\Omega)$ are solutions of:

$$\left(\frac{\hbar^2}{m}\frac{L^2(\Omega)}{\rho^2} + V(1,2) + V(2,3) + V(3,1)\right)\phi_n(\rho,\Omega) = U_n(\rho)\phi_n(\rho,\Omega)$$

The Efimov Effect

- The hyperadial functions $w_n(\rho)$ are solutions of:

$$\left[\frac{\hbar^2}{m}\left(-\frac{\partial^2}{\partial\rho^2}+\frac{15}{4\rho^2}\right)+U_n(\rho)\right]w_n(\rho)+\sum_m C_{nm}w_m(\rho)=Ew_n(\rho)$$

- The equation for the lowest hyperadial functions $w_0(\rho)$ as $\rho\to\infty$ is:

$$\frac{\hbar^2}{m}\left(-\frac{\partial^2}{\partial\rho^2}-\frac{s_0^2+1/4-\rho^2/a^2}{\rho^2}\right)w_0(\rho)=Ew_0(\rho)$$

- in the unitary limit $a\to\infty$ and the solutions are

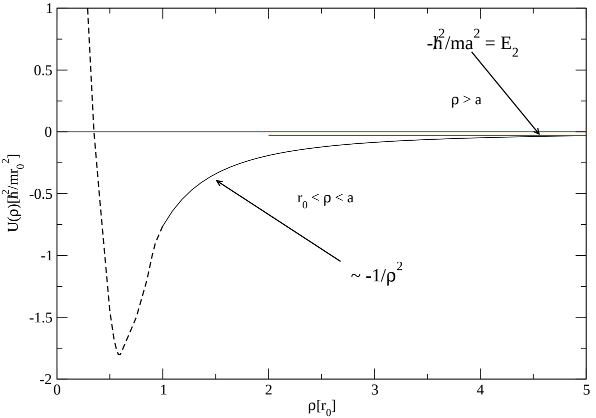
$w_0(\rho)=\rho^{1/2}K_{is_0}(\sqrt{2}\kappa\rho)$ with $\kappa^2=E/(\hbar^2/m)$

- The discrete spectrum arises from the boundary condition at $\rho\approx 0$

$$E^{(i+1)}=\mathrm{e}^{-2\pi/s_0}E^{(i)}$$

- a number of discrete states appears between $\hbar^2/mr_0^2 < E_i < \hbar^2/ma^2$

$$N \rightarrow \frac{s_0}{\pi} \ln \frac{|a|}{r_0}$$



Three-boson spectrum for $r_0/a << 1$

Matching conditions

- working equation: $\frac{\hbar^2}{m}\left(-\frac{\partial^2}{\partial\rho^2}-\frac{s_0^2+1/4}{\rho^2}\right)w_0(\rho)=Ew_0(\rho)$
- solution ($\rho << |a|$): $w_0(\rho)\approx(H\rho)^{1/2}\sin[s_0\ln(cH\rho)+\theta_*)$
- boundary conditions at short distances:
  $s_0\cot[s_0\ln(\Lambda_0\rho_0)]=\rho_0\frac{w_0'(\rho_0)}{w_0(\rho_0)}-\frac{1}{2}$
- $\theta_*=-s_0\ln(cH/\Lambda_0)$
- defining: $1/a=H\cos\xi$ and $\kappa=H\sin\xi$
  $2\theta_*=-\Delta(\xi)\qquad \mod 2\pi$

$$E_3^n+\frac{\hbar^2}{ma^2}=\left[e^{-2\pi(n-n_*)/s_0}\right]\left[e^{\Delta(\xi)/s_0}\right]\frac{\hbar^2\kappa_*^2}{m}$$

$$E_3^{n_*}=\frac{\hbar^2\kappa_*^2}{m}$$

Three-boson spectrum for $r_0/a << 1$

Zero-Range Theory

$$E_3^n + \frac{\hbar^2}{ma^2} = \left[e^{-2\pi(n-n_*)/s_0}\right]\left[e^{\Delta(\xi)/s_0}\right]\frac{\hbar^2\kappa_*^2}{m}$$

- the ratios at $\xi = const$. are $E_3^{(n+1)}/E_3^{(n)} = e^{-2\pi/s_0} \approx 1/22.7^2$
- κ_* is the three-body parameter
- $\Delta(\xi)$ is an universal function (known)
- parametric form:

$$E_3^n/(\hbar^2/ma^2) = \tan^2\xi$$

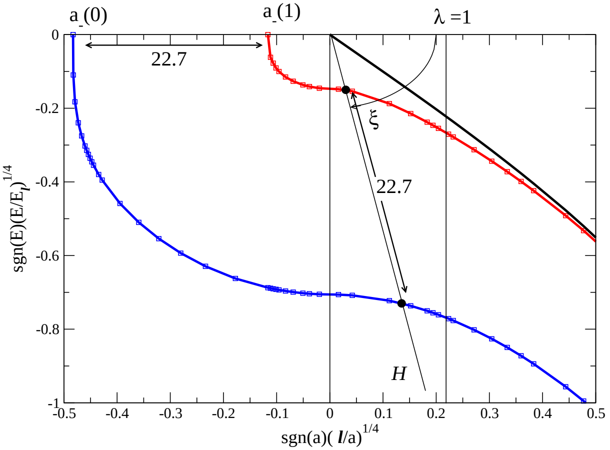
$$\kappa_* a = [e^{\pi(n-n_*)/s_0}][e^{-\Delta(\xi)/2s_0}]/\cos\xi$$

$$x = e^{\pi(n-n_*)/s_0} y(\xi)$$

Discrete Scale Invariance

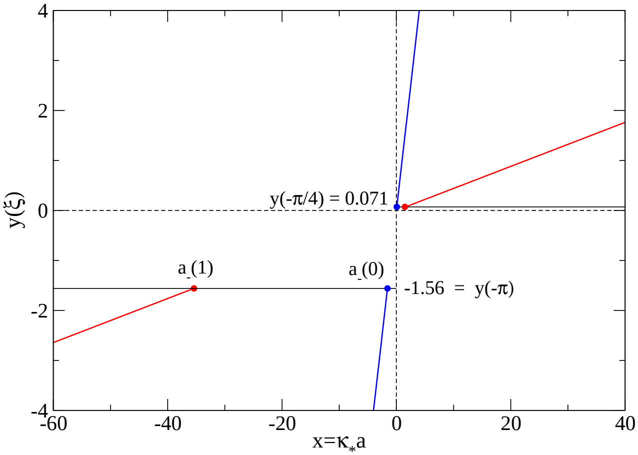
Polar Coordinates: $1/a = H\cos\xi\,,\quad K = H\sin\xi$

$$E_3^n + \frac{\hbar^2}{ma^2} = e^{-2n\pi/s_0} e^{\Delta(\xi)/s_0} \frac{\hbar^2\kappa_*^2}{m} \rightarrow H = \kappa_*\, e^{-n\pi/s_0}\, e^{\Delta(\xi)/2s_0}$$



Discrete Scale Invariance in the $(x - y)$-plane

$$x = \kappa_* a = e^{-n\pi/s_0} \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi} = e^{-n\pi/s_0} y(\xi)$$



Studying Universality

Efimov physics with potential models

- Tunable strength
- Finite-range versus zero-range theory
- Finite-range effects
- Equivalent to results from EFT
- Comparisons to experimental results

Discrete Scale Invariance

- Evolution with N
- Constrains imposed by DSI
- N-boson spectrum with a (regularized) contact interaction
- Quantum Mechanics of shallow states
- Experimental studies in nuclear physics

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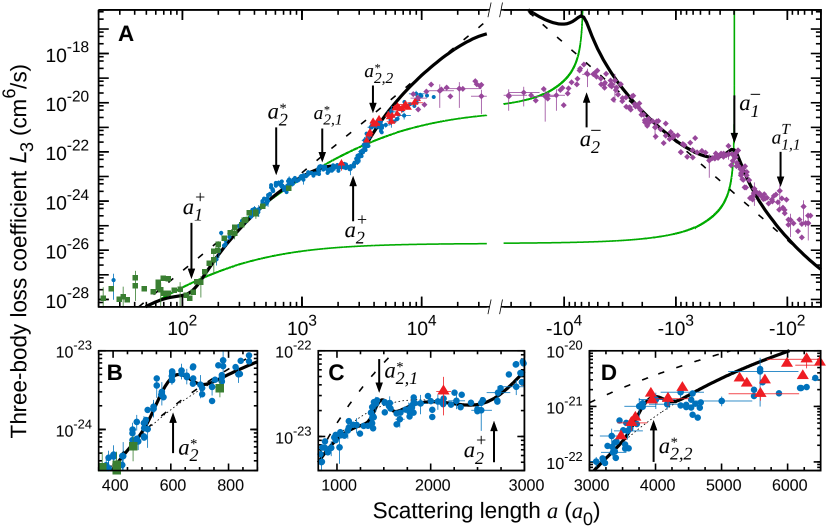
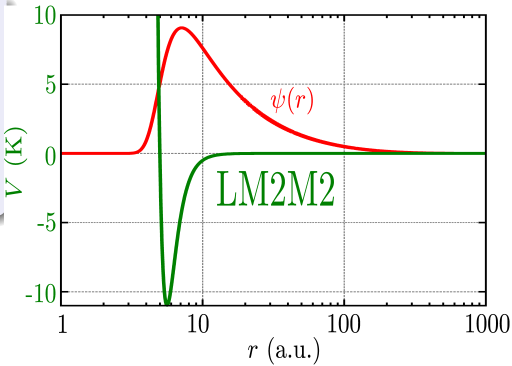
S.E. Pollack et al., Science 326 (2009) 1683



Efimov physics with potential models

The He-He system as example:

$E\,(\mathrm{He}-\mathrm{He}) \approx -1.3$ mK
$a = 190$ a.u.
$r_0 = 13$ a.u.
$a >> r_0$
$E\,(\mathrm{He}-\mathrm{He}) \approx -\frac{\hbar^2}{m}\frac{1}{a^2}$
≈ -1.2 mK



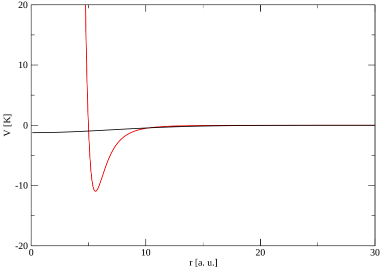
Soft Two-Body Gaussian Potential

Effective low-energy soft potential

- $V(r) = V_0\, \mathrm{e}^{-r^2/R^2}$
  - Regularized contact interaction
  - Fix V_0 to reproduce one low-energy LM2M2 datum
  - Use the cut-off R to reproduce a second datum

$V_0 = -1.2344$ K, $R = 10.0$ a.u.

	Gaussian	LM2M2
a_0 (a.u.)	189.41	189.42
r_0 (a.u.)	13.81	13.84
E_2 (mK)	-1.303	-1.303



Soft Hyper-Central Three-Body Potential

Problem in the three-body sector

	Soft-Gaussian	LM2M2
$E_3^{(0)}$ (mK)	-150.4	-126.4
$E_3^{(1)}$ (mK)	-2.467	-2.271

Effective low-energy three-body-soft potential

$$W(\rho_{ijk}) = W_0 \ e^{-2\rho_{ijk}^2/\rho_0^2} \quad (\rho_{ijk}^2 \propto r_{ij}^2 + r_{jk}^2 + r_{ki}^2)$$

potential	$E_{3b}^{(0)}$ (mK)	$E_{3b}^{(1)}$ (mK)
LM2M2	-126.4	-2.265
gaussian	-150.4	-2.467
(W_0 [K], ρ_0 [a.u.])		
$(0.422, 14)$	-126.4	-2.299

- LO Effective Field Theory

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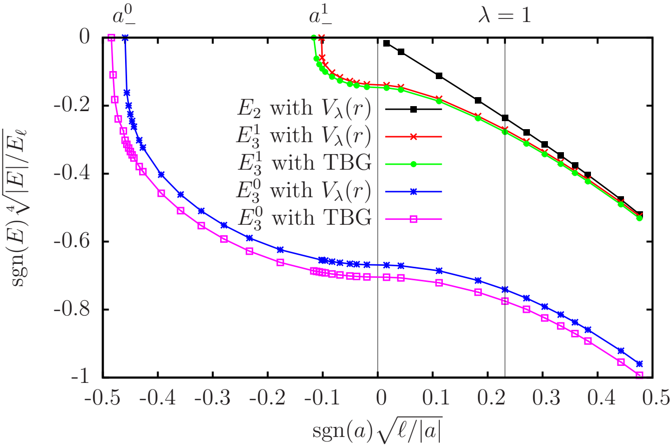

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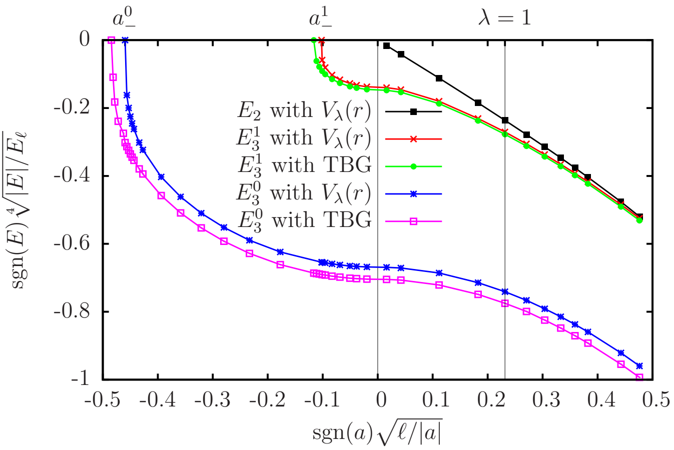

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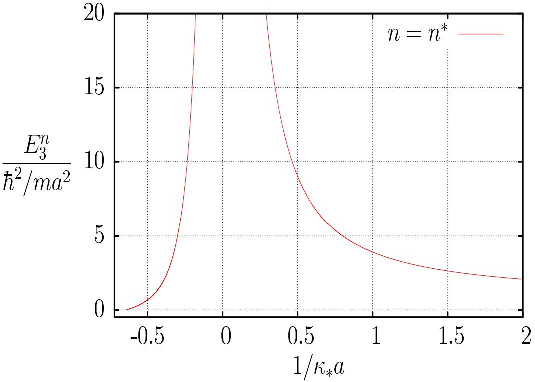

$$E_3^n/(\hbar^2/ma^2) = \tan^2 \xi$$

$$\kappa_* a = e^{(n-n^*)\pi/s_0} \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}$$

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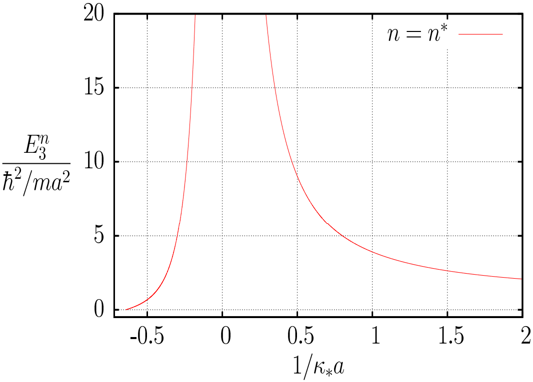

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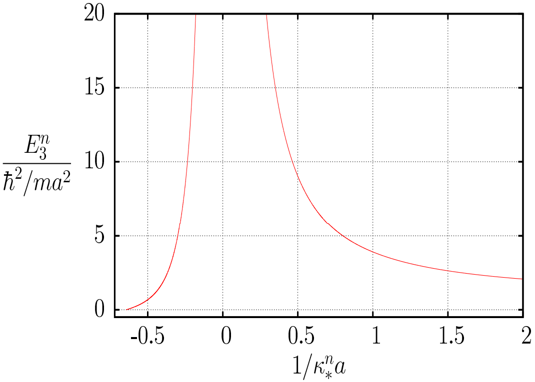

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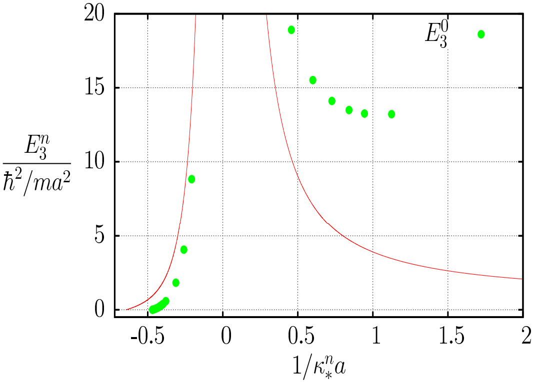

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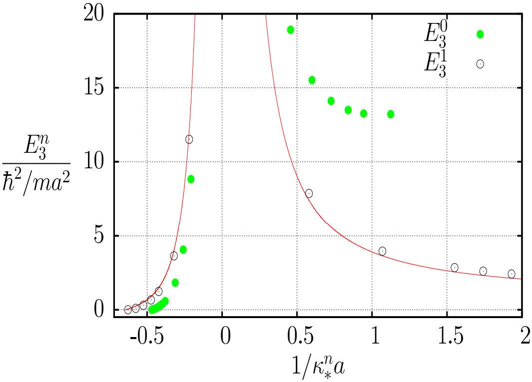

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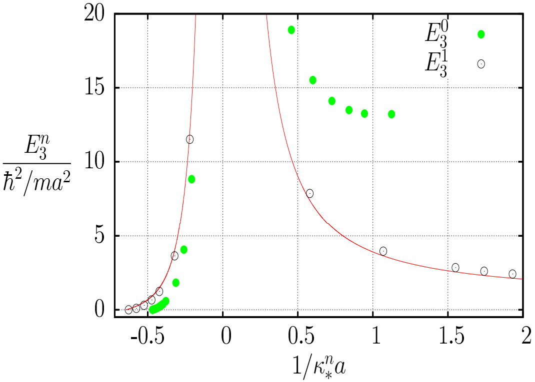
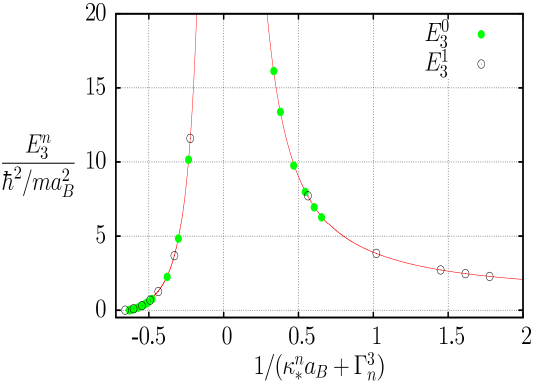


$$E_3^n/(\hbar^2/ma_B^2) = \tan^2\xi \qquad \hbar^2/ma_B^2 = E_2 \quad a > 0$$

$$\kappa_*^n a_B + \Gamma_3^n = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi} \qquad \hbar^2/ma_B^2 = E_v \quad a < 0$$

Three-Body Bound States



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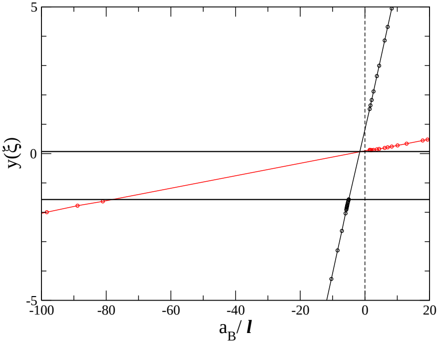

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$$\kappa_*^n a_B + \Gamma_3^n = y(\xi) \qquad \hbar^2/ma_B^2 = E_v \quad a < 0$$

Origin of the finite-range scaling parameter Γ_3

- From the matching condition we have: $\theta_* = -s_0 \ln(cH/\Lambda_0)$
- Identifying $\Lambda_0 \to \kappa_*$
- remembering that $1/a = H \cos\xi$
- the Efimov equations are obtained
- However ...

finite-range potentials with large scattering length

The relation between Λ_0 and κ_* is: $\Lambda_0 = \kappa_* \left(1 + \mathcal{A}\frac{r_0}{a_B} + \dots\right)$
and defining $\Gamma_3 = \mathcal{A}\kappa_* r_0$, the shifted Efimov equation is obtained

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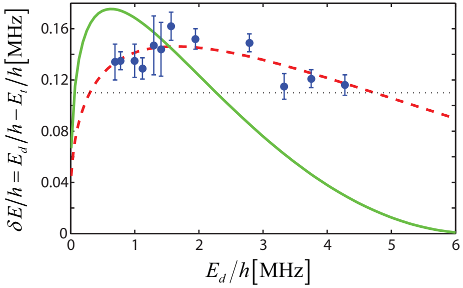
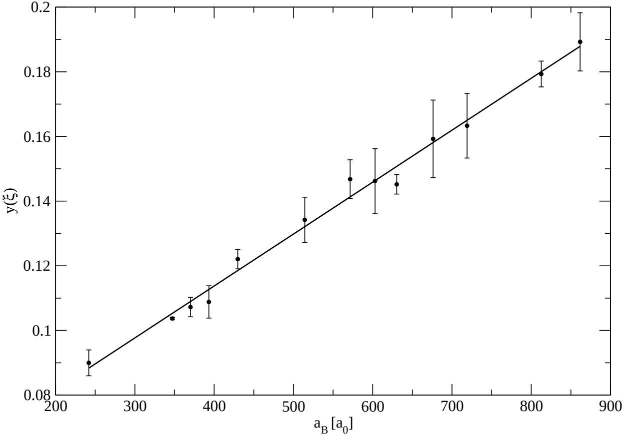
$$\kappa_*^n a_B + \Gamma_3^n = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

Analyzing data: O. Matchey et al.,Phys. Rev. Lett., 108, 210406 (2012)



- Trimer-dimer energy differences in ^{7}Li
- universal theory (zero-range)
- fit to the data

Analyzing data in the $(x - y)$ plane



$\kappa_* = 1.61 \times 10^{-4}\ a_0^{-1}$ $\Gamma = 4.95 \times 10^{-2}$

Bound States: Increasing N

- We know that there is a tree of two attached states

Bound States: Increasing *N*

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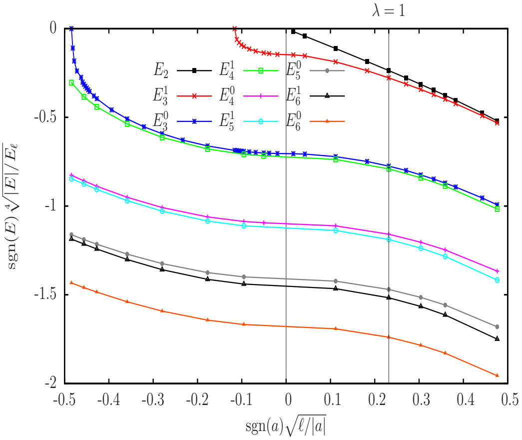


Bound States: Increasing N

- We know that there is a tree of two attached states

- We know that DSI is verified for $N = 3$:

$$E_3^n / E_2 = \tan^2 \xi$$

$$\kappa_3^n a_B + \Gamma_3^n = \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}$$

- Going to $N > 3$ we propose

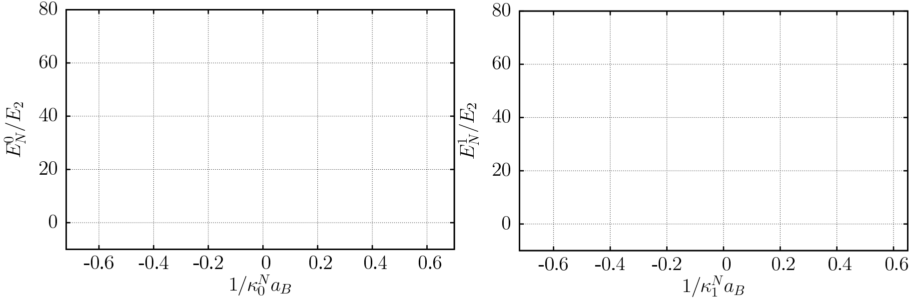
$$E_N^{n,m} / E_2 = \tan^2 \xi$$

$$\kappa_N^{n,m} a_B + \Gamma_N^{n,m} = \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}$$

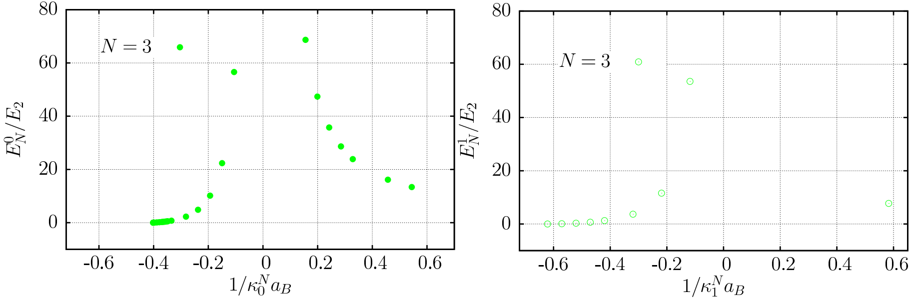
with $m = 0, 1$. We restric the analysis to $n = 0$. The states with $n > 0$ appear as resonances.

- We search rules for $\kappa_N^{0,m}$ and for $\Gamma_N^{0,m}$

Increasig N


80
60
40
20
0
E_N^0/E_2
-0.6 -0.4 -0.2 0 0.2 0.4 0.6
$1/\kappa_0^N a_B$
80
60
40
20
0
E_N^1/E_2
-0.6 -0.4 -0.2 0 0.2 0.4 0.6
$1/\kappa_1^N a_B$

Increasig N



$N = 3$
E_N^0/E_2
$1/\kappa_0^N a_B$
80
60
40
20
0
-0.6
-0.4
-0.2
0
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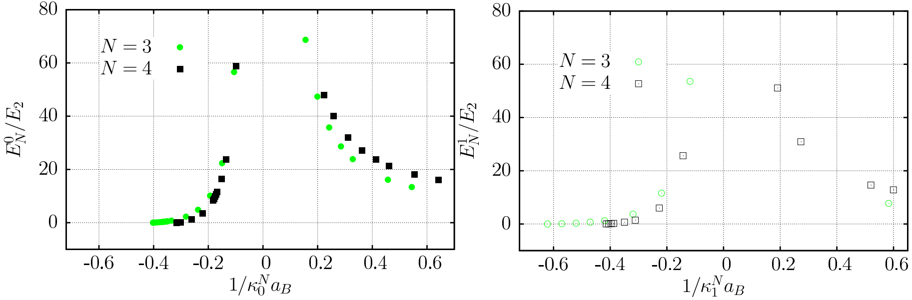
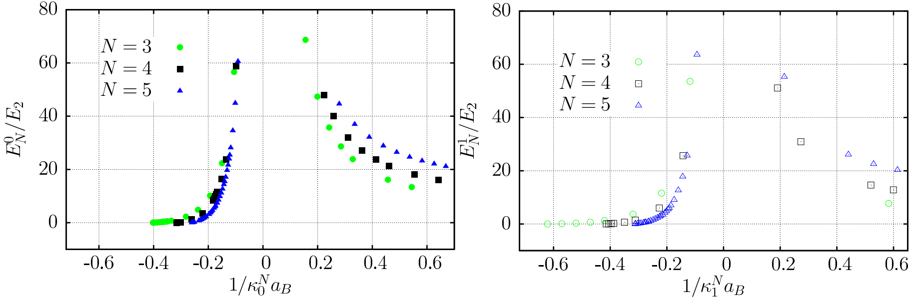
0.6

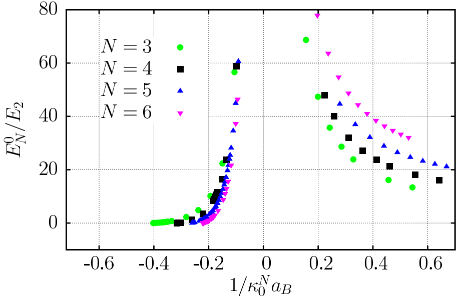
Increasig N



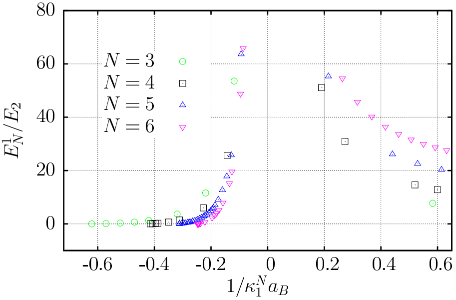
Increasig N



Increasig N



E_N^0/E_2
80
60
40
20
0
$N=3$
$N=4$
$N=5$
$N=6$
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
$1/\kappa_0^N a_B$



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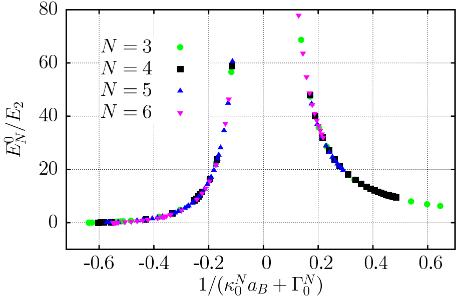
Increasig N



$N = 3$
$N = 4$
$N = 5$
$N = 6$
E_N^0/E_2
$1/(\kappa_0^N a_B + \Gamma_0^N)$
80
60
40
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-0.2
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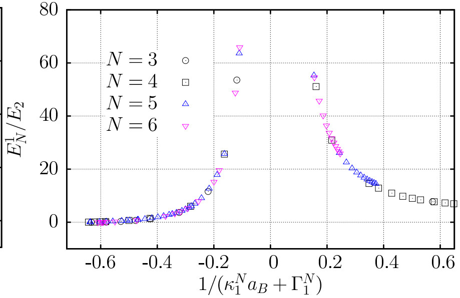
0.6



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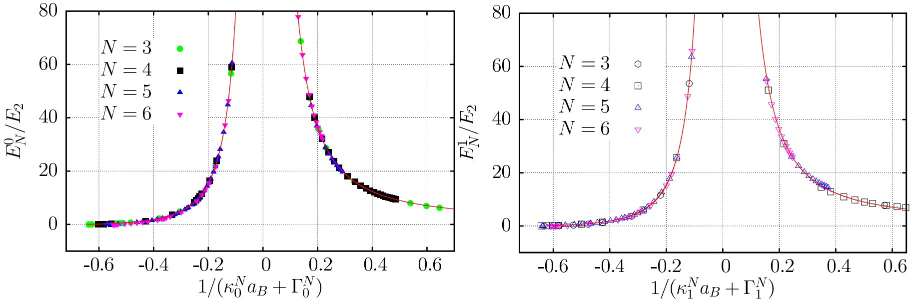
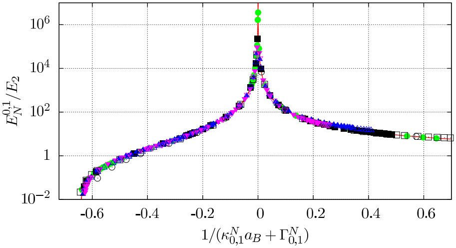
Increasig N



$$E_N^n/E_2 = \tan^2\xi$$

$$\kappa_N^n a_B + \Gamma_N^n = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

Increasig N



Universal Formula

$$E_N^n/E_2 = \tan^2\xi$$

$$\kappa_N^n a_B + \Gamma_N^n = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

Increasing N

- We know that there is a tree of two attached states

- We know that DSI is verified:

$$E_N^n/E_2 = \tan^2\xi$$

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- Going to $N > 6$ what we can expect?

- It is clear that increasing the number of particles the potential energy can grow faster than the kinetic energy destroying the Efimov picture (more excited states to appear)

- The problem of N interacting particles with contact interactions has a known solution?

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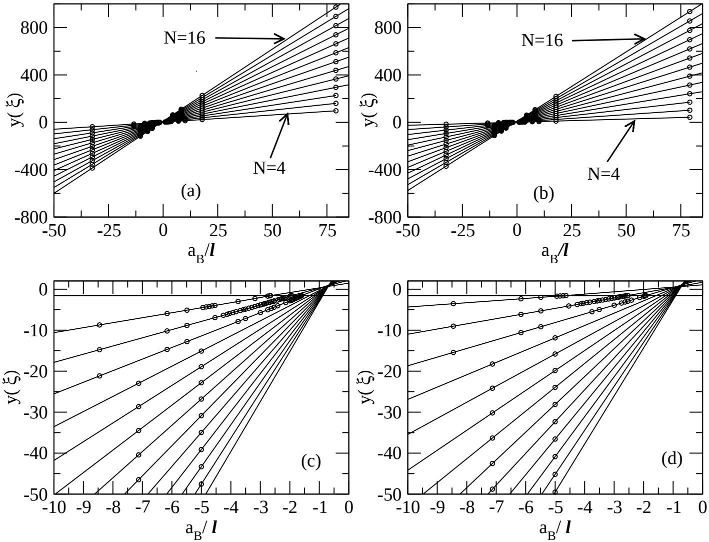
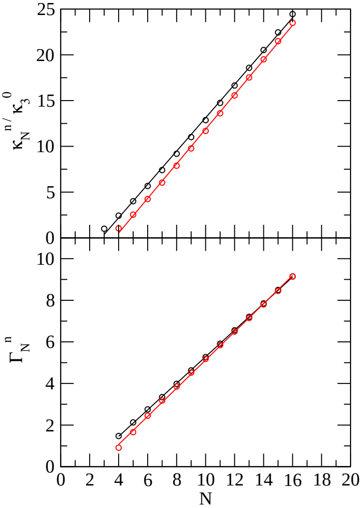
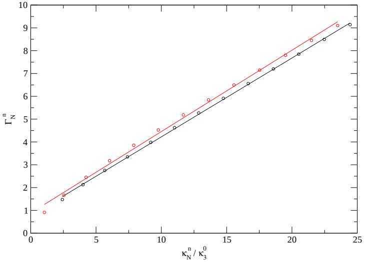
Efimov plot up to $N = 16$ using soft potentials



Studying κ_N^n and Γ_N^n with N





Analysis of the results

zero-range vs. finite-range results

zero-range	finite-range results
$E_N^n/(\hbar^2/ma^2) = \tan^2\xi$	$E_N^n/(\hbar^2/ma_B^2) = \tan^2\xi$
$\kappa_N^n a = y(\xi)$	$\kappa_N^n a_B + \Gamma_N^n = y(\xi)$

- general equations for the lowest states ($n = 0, 1$) of the N-boson system
- The zero-range theory is a one-parameter theory
- The finite-range theory is a two-parameter theory
- κ_N^n is a scale parameter
- Γ_N^n is a finite-size scale parameter
- The κ_N^n parameters are not independent (are linear in N):

$$\frac{\kappa_N^0}{\kappa_3^0} = 1 + (N-3)\left(\frac{\kappa_4^0}{\kappa_3^0} - 1\right)$$

- $\kappa_4^0/\kappa_3^0 = 2.147$ is an universal quantity (A. Deltuva, FBS 54, 569 (2013)).

The linear dependence on y at constant a in He clusters

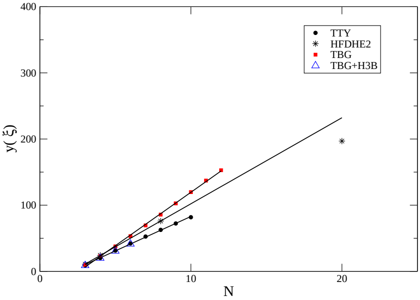
$$E_N/E_2 = \tan^2\xi$$

$$\kappa_N a_B + \Gamma_N = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi} = y(\xi)$$

M. Lewerenz, J. Chem. Phys. 106, 4596 (1997) for the TTY potential

V.R. Pandharipande et al., Phys. Rev. Lett. 50, 1676 (1983) for the HFDHE2 potential

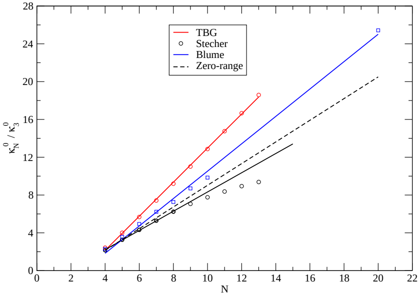
	TTY	HFD
a_0 (a.u.)	189.0	170.5
r_0 (a.u.)	13.94	13.90
E_2 (mK)	-1.310	-1.685
E_3 (mK)	-126	-133
E_4 (mK)	-558	-604
E_5 (mK)	-1302	.
.	.	.
.	.	.



The linear dependence on κ_N^0 with N

$$\frac{\kappa_N^0}{\kappa_3^0} = 1 + (N-3)(\frac{\kappa_4^0}{\kappa_3^0} - 1)$$

- For the zero-range theory $\kappa_4^0/\kappa_3^0 = 2.147$



J. von Stecher, J. Phys. B: At. Mol. Opt. Phys. 43, 101002 (2010)

G.J. Hanna and D. Blume, Phys. Rev. A **74**, 063604 (2006)

Conclusions

- The Efimov spectrum for N bosons has been analyzed
- The universal formula has been extended to general N
- The zero-range theory is a one-parameter theory
  - It can be represented as stright lines going through the origin
- The finite-range theory is a two-parameter theory
  - It can be represented as stright lines not going through the origin
  - The distance to the origin is the finite-size scale parameter Γ_N^n
- A linear dependence has been found for the scale parameter κ_N^n and for the finite-size scale parameter Γ_N^n
- Experimental data from Khaykovich has been analyzed within this theory
- Numerical results from Lewerenz, Pandharipande, Blume, and Stecher has been analyzed as well
- Possible four-body force to describe saturation
- We hope that this analysis will estimulate more experimental activity