1. The spherical evolution model provides a simple relation between the linear and nonlinear densities. This relation is well approximated by

$$1 + \delta(t) = (1 - D(t)\,\delta_{init}/\delta_c)^{-\delta_c}.$$

a/ Use the fact that the nonlinear density is mass/(density × volume) to derive an approximation for the speed with which the edge of the object changes with time. Express your answer in terms of $f \equiv d\ln D/d\ln a$.

b/ Use this approximation for $\delta(\delta_{init})$ to estimate the linear theory overdensity $D(t)\delta_i$ associated with a 'void' (a region whose density is 0.2 times that of the background). This number is less than -1; why is this not problematic? (In contrast, δ cannot be less than -1.)

2. It is often stated that (in standard gravity) one may treat the evolution of a patch embedded in an over/under density as that within an effective cosmology that has a higher/lower density. However, to do this correctly, one must account both for the change in density, and for the fact that the Hubble constant of the effective cosmology is modified (so as to produce a universe of the same age). Explain how the excursion set approach incorporates this effect correctly. If one wishes to simulate the evolution of structure in such an environment, should one: a) change σ_8? b) run the simulation for a longer or shorter time? c) do something else? if so, what? (See Martino & Sheth 2009 for the solution.)

3. In standard gravity, the linear theory growth factor D is a function of time t but not of wave number k. Use this to argue that an average density patch of scale R is predicted to remain average density at later times. In modified gravity models, the linear growth factor is $D(k,t)$. Argue that, as a result, the density within a patch is predicted to evolve, even if it was average density initially. Since this is true for any R, should one worry about the definition of a 'homogeneous' universe in such models? If yes, should one require that physically reasonable models must have $D(k,t)$ independent of k for sufficiently large scales (small k)?

4. The characteristic mass scale m_* is set by requiring that the linear theory

$$\sigma^2(m_*) \equiv \int \frac{dk}{k}\,\frac{k^3 P_{\rm L}(k)}{2\pi^2}\,W^2(kR_*) = \delta_c^2.$$

a/ For a power law spectrum $P_{\rm L}(k) \propto k^n$, how does m_* evolve with time? (Express your answer in terms of the linear theory growth factor.)

b/ Since all objects have the same density ($\sim 200\times$ the background), how do the size and velocity dispersions of m_* evolve with time?

c/ At fixed mass, how do the size and velocity dispersions evolve with time?

d/ How do these scalings change if the objects have the same density relative to the critical (rather than the background) density?

5. See Section 4.3 of Cooray & Sheth (2002). How does the discussion change if the initial power spectrum is of the form $P_0(k) = A_0\, k^{-3/2}\, \exp(-k^2 R_*^2)$ rather than $A_0\, k^{-3/2}$, where R_* is the scale associated with what would have been m_* halos if there were no $\exp(-k^2 R_*^2)$ term. In particular, sketch and discuss how the mass function and the nonlinear power-spectrum of the dark matter are affected. Discuss why this might provide insight into structure formation in WDM models.

6. Suppose that the density profile of a halo is $\rho(r) \propto r^{-\epsilon}$.

a/ Show that the one-halo contribution to the two point correlation function is $\xi \propto r^{-\gamma}$ with $\gamma = 2\epsilon - 3$.

b/ The stable clustering approximation assumes that nonlinear, virialized objects no longer participate in the expansion of the background Universe: they maintain their shapes in physical coordinates, so they shrink in comoving coordinates. Combine this with the assumption that halos are 200× denser than the background at the time of virialization to show that $\gamma = 3(3+n)/(5+n)$, if the initial $P(k) \propto k^n$ with $n > -3$.

7. Suppose that nonlinear stucture formation occurs by rearranging matter on small scales. Argue that, as a result, halos must have compensated profiles such as those discussed in Section 4.4 of Cooray & Sheth (2002). Then show that, in such models, $P(k) \propto k^4$ on large scales ($k \ll 1$). This shows explicitly that local rearrangements of matter cannot generate $P(k) \propto k$: if we see such a scaling, then it must be primordial! What does this imply for the 'back-reaction' of small-scale rearrangements of matter on the large scale distribution?

8. Assume that the universe is flat with a cosmological constant ($\Omega_\Lambda = 1 - \Omega_m$), that $\Omega_m = \Omega_c + \Omega_b = 0.25$ with $\Omega_b = \Omega_m/6$, $\sigma_8 = 0.8$, and that the (linear theory) power spectrum of matter fluctuations is given by

$$P(k) = Ak^n\,[(\Omega_c/\Omega_m)\,T_c(k) + (\Omega_b/\Omega_m)\,T_b(k)]^2$$

with $n = 1$, $T_c(k) = \exp(-kR_H/2)$, $T_b(k) = 2\,\exp(-k^2 R_S^2)\sin(kR_B)/(kR_B)$, and $(R_H, R_S, R_B) = (64, 8, 100)h^{-1}$Mpc.

a/ What is the physical significance of this value of n, and of these three scales?

b/ It is common to approximate

$$P(k) \approx Ak^n\,(\Omega_c/\Omega_m)^2\,[T_c(k) + 2(\Omega_b/\Omega_c)\,T_c(k)T_b(k)].$$

Why is this reasonable?

c/ We usually express the amplitude A in terms of σ_8, which is the square root of the variance of the linear theory fluctuation field when smoothed with a tophat filter of radius $8h^{-1}$Mpc. What is A if $\sigma_8 = 0.8$?

d/ Make a plot showing $dn/d\ln m$, the comoving number density of halos, as a function of $\ln m$, at $z = 0$ and at $z = 1$. Assume that

$$\frac{dn}{d\ln m} = \frac{\bar{\rho}}{m} \frac{\nu \exp(-\nu^2/2)}{\sqrt{2\pi}} \left[1 - \frac{\mathrm{erfc}(\Gamma\nu/\sqrt{2})}{2} + \frac{e^{-\Gamma^2\nu^2/2}}{\sqrt{2\pi}\Gamma\nu}\right] \frac{d\ln\nu}{d\ln m},$$

where $\bar{\rho}$ is the comoving background density, $\Gamma^2 \approx 1/3$ and $\nu = \delta_c(z)/\sigma(m)$. Discuss why cosmological constraints on, e.g., Ω_m and σ_8, from halo abundances are usually derived from measurements at more than one epoch. (You may wish to make a similar plot, but for a different value of σ_8.)

e/ Assume that the density profile around the center of a halo scales as

$$\rho(r) \propto \frac{(r/r_s)^{-2}}{1 + (r/r_s)^2}.$$

To set the constant of proportionality, assume that the average density within the virial radius of a halo is $200\times$ the background density, and that the scale radius satisfies $c = r_{vir}/r_s = c_* \, (m_*/m)^{1/6}$, where m_* is that mass at which $\nu = 1$. Make a plot showing the correlation function of the dark matter at $z = 0$ and $z = 1$, if $P(k) \approx Ak^{-3/2}$ and $\sigma_8 = 0.8$ at $z = 0$. Compare this plot with one in which you assume that the virial radius of a halo is 200 times the critical density. Discuss the differences at low and high redshift.

f/ Assume that, at $z = 1$, the mean number of galaxies in a halo is given by $\langle N|m\rangle = 1 + (m/10^{12}h^{-1}M_\odot)$ and that there are no galaxies in halos with mass less than $10^{11}h^{-1}M_\odot$. What is the comoving number density of these galaxies?

g/ Make a plot showing the two-point correlation function of galaxies at $z = 1$, if, in halos which host a galaxy, the distribution of non-central galaxies is Poisson. Comment on why the transition scale between the 1- and 2-halo terms occurs where it does. What is the large scale bias factor of these galaxies?

h/ Suppose that, between $z = 1$ and $z = 0$, galaxies do not merge even though their host halos do. Show that the large scale bias factor evolves as $(b_1 - 1) = (b_0 - 1) \, D(z_0)/D(z_1)$ where D is the linear theory growth factor. How does the two-halo term for these galaxies evolve?

i/ Suppose that the number of subhalos more massive than m in a parent halo of mass M is $N(\geq m|M) = 0.01(M/m)$. What does this imply about the amount of mass that must be stripped from a galaxy's halo after it enters a cluster? (See discussion in Skibba et al. 2007.)