The Standard Model and Neutrino Mass Models

André de Gouvêa

Northwestern University

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Quick Preliminaries

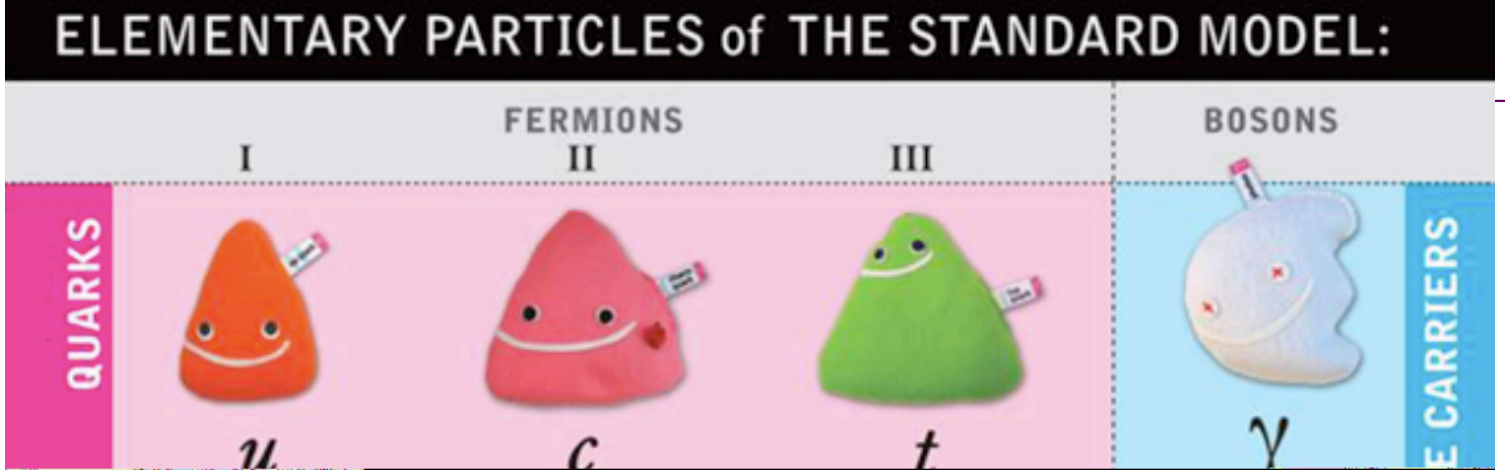
I will assume that everyone has been exposed to Quantum Field Theory and some Particle Physics in the past. These lectures are aimed at helping us understand the consequences of non-zero neutrino masses for particle physics and what might lie beyond the Standard Model.

Some good references include:

- "Introduction to Quantum Field Theory," Peskin and Schroeder;
- "Gauge Theories of the Strong, Weak, and Electromagnetic Interactions," Quigg;
- Dynamics of the Standard Model," Donoghue, Golowich, Holstein;
- S. Willenbrock, TASI 2004 Lectures, arXiv:hep-ph/0410370.

Tentative Outline for the Five Lectures

- SM Overview and QFT Basics;
- The Fundamental Fields, including Weyl, Dirac and Majorana fermions;
- Gauge Symmetries and SM Interactions;
- Spontaneous Symmetry Breaking and the Origin of Mass;
- Evidence for New Physics;
- Neutrino Masses.



21st Century Periodic Table

(Now with Higgs boson!)

http://www.particlezoo.net

- Result of over 60 years of particle physics theoretical and experimental research.
- Theoretical formalism based on the marriage of Quantum Mechanics and Special Relativity – Relativistic Quantum Field Theory.
- Very Powerful – once we specify the model ingredients: field content (matter particles) and the internal symmetries (interactions), the dynamics of the system is uniquely specified by a finite set of free parameters.

[courtesy of Pilar Hernández]

Quantum Field Theory in a (Tiny) Nutshell

The objects we care about are quantum fields, real or complex functions of space-time. They are operator-value objects. In the case of free fields, the differential equation that describes the different fields are solvable. These act on a Fock-space, and create and destroy free-particles with well-defined energy and momentum (including a dispersion relation: $E^2 - |\vec{p}|^2 = m^2$).

In the presence of interactions, the differential equations that describe the operators are non-linear and, in general, cannot be solved exactly. We make use of perturbation theory in order to compute useful observables. Mostly, we care about scattering: distinct asymptotic states in $t = \pm\infty$, and the matrix elements that connect them (e.g., $\mathcal{S} = {}_{+\infty}\langle \mu(p_\mu)\bar{\nu}(p_\nu)|\pi(p_\pi)\rangle_{-\infty}$). These asymptotic states are treated as if they were destroyed and created by the free fields.

[When perturbation theory is not effective, we need to get smarter. I won't talk about this at all!]

Elementary particles or interactions represented by (complex) causal quantum fields (operators in Fock space):

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(u_s^a(\mathbf{p}) \hat{a}^s_{\mathbf{p},a} e^{-ipx} + v_s^a(\mathbf{p}) \hat{b}^{s\dagger}_{\mathbf{p},a} e^{ipx} \right) \Big|_{p^0 = E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m^2}}$$

a : flavour index

s : spin index

One (anti)particle states:

$$\begin{cases} |\mathbf{p}, s, a\rangle_+ = \hat{a}^{s\dagger}_{\mathbf{p},a} |0\rangle \\ |\mathbf{p}, s, a\rangle_- = \hat{b}^{s\dagger}_{\mathbf{p},a} |0\rangle \end{cases}$$

$u_s^a(\mathbf{p}), v_s^a(\mathbf{p})$ wave functions in spin and internal space

[courtesy of Pilar Hernández]

Lorentz Invariance

We only care about QFTs which are Lorentz Invariant (unlike, say, folks in condensed matter physics). This is because, of course, Nature seems to like Lorentz invariance, at least at "small" distances and in the limit where we can ignore gravitational interactions. We will always ignore gravity, for two reasons. (i) Gravitational interactions are really weak compared to all other interactions for the types of processes we care about, and (ii) we don't know how to write down a mathematically well-defined QFT for the gravitational interactions.

Lorentz invariance dictates the objects (fields) we are allowed to use as ingredients. They are

- Massive fields with spin 0, 1/2, 1, 3/2, 2, etc
- Massless fields with spin 0, 1/2, 1, 3/2, 2, etc

Lorentz Invariance

If we want the theory to be renormalizable, we must stick to the following "types" of fundamental fields:

- Real Scalar. One degree of freedom.
- Complex Scalar. Two degrees of freedom.
- Spin 1/2 Fermion. Two degrees of freedom (Weyl and Majorana fermions) or four degrees of freedom (Dirac fermion). More on this later.
- Vector field. Two degrees of freedom when massless, three when massive. The latter are problematic. More on this later.

This is it! [Composite objects can be more complicated. As you know, the $\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}$ baryons have spin-3/2, the f_2 mesons have spin 2, etc. Check out the Particle Data Book(let) if you want to be impressed!]

Weyl, Majorana, and Dirac Fermions

The generators of the Lorentz group are the rotations, J_i, and the boosts, K_i. They satisfy the algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$
$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$
$$[J_i, K_j] = i\epsilon_{ijk} K_k \ .$$

The J_i are Hermitian, and the K_i are anti-Hermitian. The J_i satisfy the algebra of the rotation group, $SU(2)$. The last commutation relation expresses the fact that a boost transforms as a three-vector under rotations.

To disentangle the algebra, define the Hermitian generators

$$\begin{aligned} A_i &= \frac{1}{2}(J_i + iK_i) \\ B_i &= \frac{1}{2}(J_i - iK_i) \ . \end{aligned}$$

It is easy to show (try it!) that the A_i and B_i satisfy the algebra

$$\begin{aligned} [A_i, A_j] &= i\epsilon_{ijk}A_k \\ [B_i, B_j] &= i\epsilon_{ijk}B_k \\ [A_i, B_j] &= 0 \ . \end{aligned}$$

A_i and the B_i: algebra of two independent $SU(2)$.

The Lorentz group, $SO(3,1)$, is locally isomorphic to $SU(2) \times SU(2)$.

Representations of $SU(2)$ are familiar from the representations of the rotation group: each representation is label by "spin," which can have integer or half-integer values.

Representations of the Lorentz group are labeled (a, b), where $a, b = 1/2, 1, 3/2, \ldots$. The simplest representation is $(0, 0)$, which corresponds to a scalar field. The simplest nontrivial representation is $(1/2, 0)$, which corresponds to a Weyl spinor, χ.

A Weyl spinor is a 2-component object that transforms under rotations and boosts as

$$\begin{aligned} \chi &\rightarrow e^{-\frac{i}{2}\sigma\cdot\theta}\chi \\ \chi &\rightarrow e^{-\frac{1}{2}\sigma\cdot\eta}\chi \end{aligned}$$

where η is the rapidity, which is related to the velocity by $\beta = \tanh\eta$. The transformation under rotations shows that a Weyl spinor carries spin 1/2.

Massive Weyl Spinor

$$\mathcal{L} \supset \frac{1}{2} m (\chi^T \epsilon \chi + h.c.)$$

where $\epsilon \equiv i\sigma_2$ is the 2×2 antisymmetric matrix. This is Lorentz invariant.

(Proof: Denote a Lorentz transformation acting on χ by the matrix M, where

$$M = e^{-\frac{i}{2}\sigma\cdot\theta} \text{ or } e^{-\frac{1}{2}\sigma\cdot\eta} .$$

Thus under a Lorentz transformation,

$$\chi^T \epsilon \chi \to \chi^T M^T \epsilon M \chi .$$

Displaying indices,

$$(M^T)_{\alpha\beta}\epsilon_{\beta\gamma}M_{\gamma\delta} = \epsilon_{\beta\gamma}M_{\beta\alpha}M_{\gamma\delta} = \epsilon_{\alpha\delta}\det M = \epsilon_{\alpha\delta}$$

where the last step uses the fact that $\det M = 1$. Thus $M^T \epsilon M = \epsilon$, which completes the proof in This also shows that the Lorentz group is locally isomorphic to $SL(2, \mathcal{C})$, the group of 2×2 complex matrices of unit determinant.)

Although a Majorana mass is less familiar than a Dirac mass, it is actually a more basic quantity, constructed from a single Weyl spinor. In this sense a Majorana mass is the simplest fermion mass term. However, if χ carries an unbroken global or local $U(1)$ charge, a Majorana mass is forbidden, since it would violate this symmetry.

None of the fermions of the standard model (except neutrinos – much more on this later!) can have a Majorana mass, since they carry electric charge. This is why we often don't hear about them.

[More generally, if χ transforms under a complex or pseudoreal representation of an unbroken global or local internal symmetry, a Majorana mass is forbidden. Let

$$\chi \to U\chi$$

where U is a unitary transformation acting on a set of Weyl spinors. The Majorana mass term transforms as

$$\chi^T \epsilon \chi \to \chi^T U^T \epsilon U \chi = \chi^T \epsilon U^T U \chi$$

This is invariant only if $U^T U = 1$, which is true only if the unitary transformation U is real ($U^* = U$).]

Physically, a fermion with a Majorana mass is its own antiparticle. It is referred to as a Majorana fermion. It cannot carry an unbroken global or local $U(1)$ charge (or, more generally, transform under a complex or pseudoreal representation) because a particle and an antiparticle must carry opposite charge.

Exercise - Gluinos are hypothetical Majorana fermions that are the superpartners of gluons. Why can they carry color charge?

Exercise - Consider a Weyl fermion that transforms under the defining representation of an unbroken $SU(2)$ group. Show that the Majorana mass term

$$\mathcal{L} = \frac{1}{2} m (\epsilon_{ab} \chi^{aT} \epsilon \chi^{b} + h.c.)$$

is invariant under the $SU(2)$ symmetry. However, show that this term vanishes.

If a Weyl fermion transforms under a complex or pseudoreal representation of an unbroken global or local symmetry, then we need to introduce a second Weyl fermion that transforms under the complex-conjugate representation in order to construct a mass term. This is a Dirac mass. Let χ, ξ transform under the $(1/2, 0)$ representation of the Lorentz group, and transform under some unbroken global or local symmetry as

$$\begin{aligned} \chi &\to U\chi \\ \xi &\to U^*\xi \,. \end{aligned}$$

Then a Lorentz-invariant mass term may be formed which respects the unbroken symmetry,

$$\mathcal{L} = m(\xi^T \epsilon \chi + h.c.)$$

since

$$\xi^T \epsilon \chi \to \xi^T U^\dagger \epsilon U \chi = \xi^T \epsilon \chi \,.$$

Thus it takes two Weyl spinors to construct a Dirac mass. A fermion with a Dirac mass is called a Dirac fermion.

The Dirac Spinor (The One With Which Everyone is Familiar)

Dirac spinor, which is a four-component object constructed from a pair of $(1/2, 0)$ Weyl spinors χ, ξ via

$$\psi = \begin{pmatrix} \chi \\ \epsilon\xi^* \end{pmatrix} .$$

In terms of a Dirac spinor, a Dirac mass is written in the familiar form

$$\begin{aligned} \mathcal{L} = -m\bar{\psi}\psi &= -m\left(\chi^\dagger, -\xi^T\epsilon\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \chi \\ \epsilon\xi^* \end{pmatrix} \\ &= m(\xi^T\epsilon\chi - \chi^\dagger\epsilon\xi^*). \end{aligned}$$

We are using the so-called Weyl or chiral basis for the gamma matrices,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad ,$$

where each entry in the above matrices is itself a 2×2 matrix. In this basis, the chiral projection operator $(1 \pm \gamma_5)/2$ projects out the Weyl spinors,

$$\psi = \frac{1-\gamma_5}{2}\psi + \frac{1+\gamma_5}{2}\psi = \psi_L + \psi_R$$

where

$$\psi_L = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$$

$$\psi_R = \begin{pmatrix} 0 \\ \epsilon\xi^* \end{pmatrix} .$$

ψ_L is the four-component Dirac-spinor version of the Weyl spinor χ, and similarly for ψ_R and $\epsilon\xi^*$.

Exercise - If ξ transforms under the $(1/2, 0)$ representation of the Lorentz group, show that $\epsilon\xi^*$ transforms under the $(0, 1/2)$ representation. [Hint: recall $M^T \epsilon M = \epsilon$].

A Dirac spinor transforms under the $(1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group, corresponding to $\psi = \psi_L + \psi_R$.

The Majorana Spinor

While a Dirac spinor is composed of two Weyl spinors, a Majorana spinor is a four-component object composed of a single Weyl spinor,

$$\psi_M = \begin{pmatrix} \chi \\ \epsilon\chi^* \end{pmatrix} .$$

Thus it is simply a four-component version of a Weyl spinor.

Exercise - Show that

$$\mathcal{L} = -\frac{1}{2} m \bar{\psi}_M \psi_M$$

is a Majorana mass term.

The charge-conjugation matrix C in the Weyl representation is

$$C = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} .$$

Given a Dirac spinor ψ, we can form the conjugate spinor via

$$\begin{aligned} \psi^c &\equiv C\gamma^0\psi^* \\ &= \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi^* \\ \epsilon\xi \end{pmatrix} \\ &= \begin{pmatrix} \xi \\ \epsilon\chi^* \end{pmatrix} . \end{aligned}$$

Thus

$$\psi_L = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$$

$$(\psi^c)_L = \begin{pmatrix} \xi \\ 0 \end{pmatrix} = (\psi_R)^c .$$

Exercise - Show that

$$\mathcal{L} = -m((\psi^c)_L^T C\psi_L + h.c.)$$

is a Dirac mass.

Exercise - Show that

$$\mathcal{L} = -\frac{1}{2}m(\psi_L^T C\psi_L + h.c.)$$

is a Majorana mass. Thus one can write a Majorana mass in terms of a Dirac spinor.

Exercise - Show that

$$\psi_M^c = \psi_M \ .$$

This is called the Majorana condition.

One last point: We can write a Dirac mass in terms of Majorana spinors

Consider a Dirac mass written in terms of a Dirac spinor,

$$\mathcal{L} = -m\bar{\psi}\psi = -\frac{1}{2}m(\bar{\psi}\psi + \bar{\psi}^c\psi^c)$$

where I'll let you verify the last equality. Now define the Majorana spinors

$$\begin{aligned} \psi_M^1 &\equiv \frac{1}{\sqrt{2}}(\psi + \psi^c) \\ \psi_M^2 &\equiv \frac{1}{\sqrt{2}}(\psi - \psi^c)\ . \end{aligned}$$

and

$$\mathcal{L} = -\frac{1}{2}m(\bar{\psi}_M^1\psi_M^1 + \bar{\psi}_M^2\psi_M^2)\ .$$

Thus a Dirac fermion is equivalent to two degenerate Majorana fermions. However

$$\begin{aligned} (\psi_M^1)^c &= \frac{1}{\sqrt{2}}(\psi^c + \psi) = \psi_M^1 \\ (\psi_M^2)^c &= \frac{1}{\sqrt{2}}(\psi^c - \psi) = -\psi_M^2\ . \end{aligned}$$

The two Majorana spinors have the opposite sign under charge conjugation.

spinor	Majorana mass	Dirac mass
Weyl	$\frac{1}{2}m(\chi^T\epsilon\chi + h.c.)$	$m(\xi^T\epsilon\chi + h.c.)$
Majorana	$-\frac{1}{2}m\bar{\psi}_M\psi_M$	$-\frac{1}{2}m(\bar{\psi}^1_M\psi^1_M + \bar{\psi}^2_M\psi^2_M)$
Dirac	$-\frac{1}{2}m(\psi^T_L C\psi_L + h.c.)$	$-m((\psi^c)^T_L C\psi_L + h.c.)$
Dirac	$-\frac{1}{2}m(\overline{(\psi^c)}_R\psi_L + h.c.)$	$-m\bar{\psi}\psi$

Table 1: A Majorana mass and a Dirac mass may be constructed from Weyl, Majorana, or Dirac spinors.

[from S. Willenbrock, arXiv:hep-ph/0410370.]

Very quickly: Kinetic Energy terms.

$$\chi^T \left(\bar{\sigma}^\mu\right) \chi, \quad \bar{\sigma}^\mu = \left(1, -\vec{\sigma}\right),$$

transforms like a four-vector, similar to $\bar{\psi}\gamma^\mu\psi$. In fact, (check this!)

$$\bar{\psi}\gamma^\mu\partial_\mu\psi = \chi^T \left(\bar{\sigma}^\mu\partial_\mu\right) \chi + \xi^T \left(\bar{\sigma}^\mu\partial_\mu\right) \xi$$

These don't "mix" χ and ξ. As far as they are concerned, these two objects are completely unrelated. The same is true of the four-component ψ_R and ψ_L. As far as the kinetic energy terms are concerned, they are different, unrelated objects.

Gauge Invariance and the Gauge Interactions

All fundamental interactions – strong, weak, electromagnetic (remember we pretend gravity does not exist) – except for the so-called Yukawa interactions, are mediated by vector fields.

We need to be careful when it comes to considering the vector fields. It turns out that we can only write renormalizable QFTs with vector fields if we also introduce gauge invariance.

Gauge invariance is not a choice.

Gauge Symmetry: U(1)

Maxwell eqs. in terms of gauge potentials are invariant under

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$$

Eqs. of motion for any charged field will also remain invariant if

$$\Psi(x) \to e^{iq\alpha(x)}\Psi(x)$$

e.m. gauge invariance <-> U(1) local gauge transformation

$$D_\mu \Psi \equiv (\partial_\mu - iqA_\mu)\Psi \to e^{iq\alpha} D_\mu \Psi$$

U(1) invariants:

$$\bar{\Psi}\gamma_\mu D_\mu \Psi \qquad \bar{\Psi}\Psi \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

[courtesy of Pilar Hernández]

Gauge Symmetry: U(1)

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\Psi}(i\not{D} - m)\Psi$$

$$D_\mu = \partial_\mu - iqA_\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

A mass term $-\frac{m_A^2}{2}A_\mu^2$ breaks symmetry: photon massless

[courtesy of Pilar Hernández]

Gauge Symmetry: SU(N)

Gauge transformation belongs to the special unitary group

$\Psi^{i=1,\ldots,N}$ fundamental rep.

$$\Psi(x) \to \Omega(x)\Psi(x) \qquad \Omega(x) \to \mathrm{N} \times \mathrm{N} \text{ unitary matrix}$$

$$\Omega(x) = \exp\left(i\alpha_a T^a\right)$$

$$T^{a\dagger} = T^a \leftrightarrow \text{generators}, a = 1, \ldots, N^2 - 1$$

(For N=2 $T^a = \frac{\sigma_a}{2}$)

$A_\mu^{a=1,..,N^2-1}$ Adjoint rep.

$$gA_\mu^a T^a \to \Omega(x) g A_\mu^a T^a \Omega^\dagger(x) + i\Omega(x)\partial_\mu \Omega^\dagger(x)$$

[courtesy of Pilar Hernández]

Gauge Symmetry SU(N)

$$D_\mu = \partial_\mu - igA^a_\mu T^a \qquad F^a_{\mu\nu}T^a = \frac{i}{g}[D_\mu, D_\nu]$$

$$D_\mu \Psi \to \Omega\, D_\mu \Psi \qquad F^a_{\mu\nu}T^a \to \Omega F^a_{\mu\nu}T^a \Omega^\dagger$$

$$\mathcal{L}_{SU(N)} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\Psi}\,(i\not{D} - m)\,\Psi$$

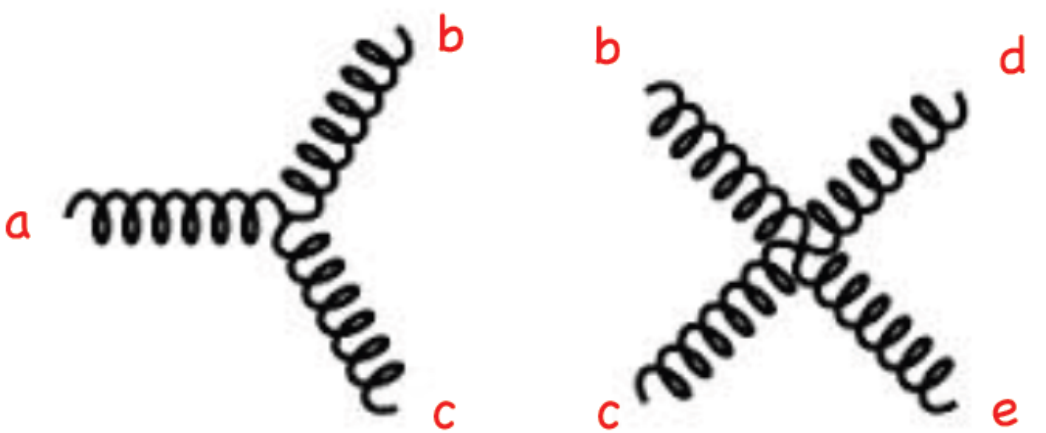
A mass term $-\frac{1}{2}m_A^2 A^a_\mu$ would also break the gauge symmetry

[courtesy of Pilar Hernández]

Gauge Symmetry SU(N)

$$\mathcal{L}_{SU(N)} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\Psi}\left(i\not{D} - m\right)\Psi$$

Mediators self-interactions:



$\propto \mathrm{Tr}[T^a[T^b, T^c]]$ $\propto \mathrm{Tr}[T^a[T^b, T^c]]\mathrm{Tr}[T^a[T^d, T^e]]$

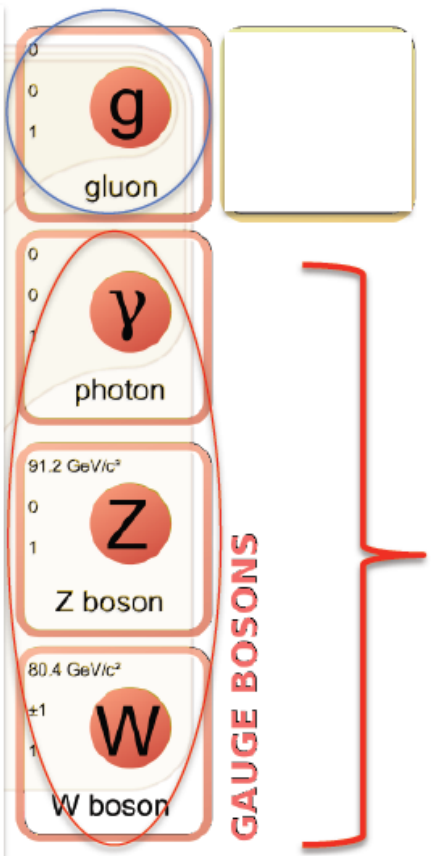
[courtesy of Pilar Hernández]

SM is a gauge theory

$$SU(3) \times SU(2) \times U(1)_Y$$

$SU(3)$

$SU(2) \times U(1)_Y$



$G_\mu^{a=1,..,8}$

$W_{\mu\nu}^{a=1,2,3}, B_\mu$

[courtesy of Pilar Hernández]

				$SU(3)$	$SU(2)$	$U(1)_Y$
$Q^i =$	$\begin{pmatrix} u \\ d \end{pmatrix}$	$\begin{pmatrix} c \\ s \end{pmatrix}$	$\begin{pmatrix} t \\ b \end{pmatrix}$	3	2	$\frac{1}{6}$
$(u^c)^i =$	u^c	c^c	t^c	$\bar{3}$	1	$-\frac{2}{3}$
$(d^c)^i =$	d^c	s^c	b^c	$\bar{3}$	1	$\frac{1}{3}$
$L^i =$	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$	1	2	$-\frac{1}{2}$
$(e^c)^i =$	e^c	μ^c	τ^c	1	1	1

Table 2: All fields can be interpreted as two-component Weyl fermions of the left-handed type.

$$\mathcal{L} \supset i\bar{Q}^i_L \not{D} Q^i_L + i\overline{(u^c)}^i_L \not{D} (u^c)^i_L + i\overline{(d^c)}^i_L \not{D} (d^c)^i_L + i\bar{L}^i_L \not{D} L^i_L + i\overline{(e^c)}^i_L \not{D} (e^c)^i_L \ .$$

or

$$\mathcal{L} \supset i\bar{Q}^i_L \not{D} Q^i_L + i\bar{u}^i_R \not{D} u^i_R + i\bar{d}^i_R \not{D} d^i_R + i\bar{L}^i_L \not{D} L^i_L + i\bar{e}^i_R \not{D} e^i_R \ .$$

All SM gauge interactions are included in the $\not{D}$ terms. At this point, this is a theory of chiral, charged massless fermions which interact with massless spin-1 particles.

All fermion fields are completely independent. They are interpreted as follows:

- Q destroys left-handed quark doublets and creates right-handed anti-quark-doublets.
- u^c destroys left-handed anti-up-quarks and creates right-handed up-quarks.
- L destroys left-handed lepton doublets and creates right-handed anti-lepton-doublets.
- etc

What is wrong with this picture

- Aren't the W-boson and the Z-boson massive? Where are the longitudinal gauge bosons (extra degrees of freedom of the spin-1 massive fields)?
- Where is the photon? Where are the electric charges of all the different fermion fields?
- Aren't the fermions massive?

The answer is the introduction of new field(s) and spontaneous gauge symmetry breaking, as we will discuss momentarily.

A couple of asides

- Global Symmetries: In the absence of fermion masses, $\mathcal{L}$ has a good deal of (accidental) global symmetry, $U(3)^5$

$$\begin{aligned} Q^i &\rightarrow U_Q^{ij} Q^j, \\ (u^c)^i &\rightarrow U_u^{ij} (u^c)^j, \\ (d^c)^i &\rightarrow U_d^{ij} (d^c)^j, \\ L^i &\rightarrow U_L^{ij} L^j, \\ (e^c)^i &\rightarrow U_e^{ij} (e^c)^j \ . \end{aligned}$$

- We have what is called a chiral gauge theory. This means that, unlike electromagnetism, there are no Weyl fermions with opposite charges. These theories are susceptible to anomalies!

Anomalies in a Tiny Nutshell

There are circumstances where Lagrangian possesses a classical symmetry that is broken at the quantum mechanical level.

This is not a problem for global symmetries. Indeed, anomalies help explain certain low-energy phenomena ($\pi^0 \to \gamma\gamma$).

This is a disaster for gauge symmetries. If these are anomalous, gauge invariance is not around to ensure the theory is mathematically consistent.

The "poster child" for potentially anomalous theories are chiral gauge theories. A.k.a. the SM. It requires anomaly cancellations. This means one adds up the contributions to the anomaly of all different fields and hopes that they add up to zero. While this is mysterious, it is stable.

Constraints include

$$\sum_{i=Q,u,d,L,e} Y_i = 0, \qquad \sum_{i=Q,u,d,L,e} Y_i^3 = 0, \qquad \sum_{i=Q,u,d} Y_i^3 = 0, \sum_{i=Q,L} Y_i^3 = 0.$$

If you have never seen this before, check it!

SSB of Gauge symmetry

The effective field theory of superconductivity: a complex scalar field with charge q coupled to the U(1) gauge field

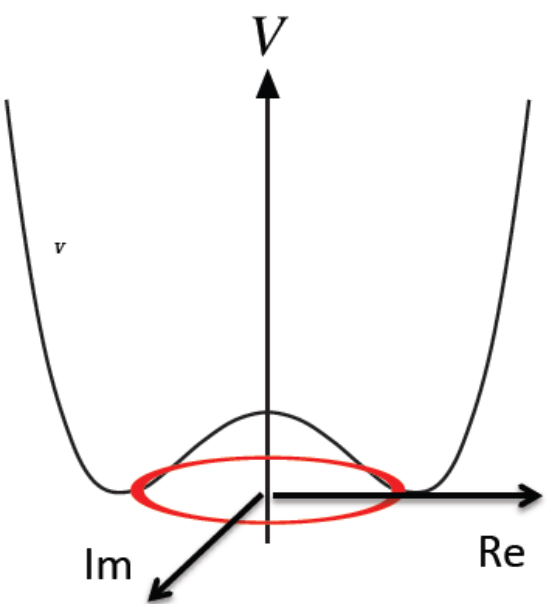
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger D_\mu\phi - V(\phi) \qquad D_\mu = \partial_\mu - iqA_\mu$$

$$V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$$

$$\lambda > 0$$

Lowest energy configuration (vacuum):

$$\phi^\dagger\phi = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} \qquad \phi = \frac{v}{\sqrt{2}}e^{i\theta}$$



[courtesy of Pilar Hernández]

[courtesy of Pilar Hernandez]

SM BEH mechanism

Of the full symmetry group: $\phi \to e^{iT^a\alpha^a}\phi \quad T^a = \left(\frac{\sigma^0}{2}, \frac{\vec{\sigma}}{2}\right)$

A U(1) subgroup remains unbroken

$$(T^0 + T^3)\langle\phi\rangle = 0$$

$$SU(2) \times U(1) \to U(1)_{em}$$

v

three broken generators: three massive gauge fields (W^{+-}, Z^0) and one massless photon

[courtesy of Pilar Hernández]

Gauge boson masses

$$\phi = e^{i\alpha^a T^a} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

In unitary gauge:

$$D_\mu \phi^\dagger D_\mu \phi = \frac{1}{2}\partial_\mu h \, \partial_\mu h + \frac{1}{2}(0 \quad v+h) \left(g W_\mu^a \frac{\sigma^a}{2} + \frac{1}{2} g' B_\mu \right)^2 \begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

$$= \frac{1}{2}\frac{v^2}{4} \left\{ g^2 \left((W_\mu^1)^2 + (W_\mu^2)^2 \right) + (g' B_\mu - g W_\mu^3)^2 \right\} = m_W^2 W_\mu^+ W_\mu^- + \frac{1}{2} m_Z^2 Z_\mu Z_\mu + O(h)$$

Charged weak: $W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \pm i W_\mu^2 \right)$ $\qquad m_W \equiv g \frac{v}{2}$

Neutral weak $Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g W_\mu^3 - g' B_\mu \right)$ $\qquad m_Z \equiv \sqrt{g^2 + g'^2} \frac{v}{2}$

Electromagnetic $A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g' W_\mu^3 + g B_\mu \right)$

[courtesy of Pilar Hernández]

Gauge Boson masses

Weak mixing angle

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g W_\mu^3 - g' B_\mu \right)$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g' W_\mu^3 + g B_\mu \right)$$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$$

$$\cos\theta_W \equiv \frac{g}{\sqrt{g^2 + g'^2}} \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}$$

$$m_W = m_Z \cos\theta_W$$

[courtesy of Pilar Hernández]

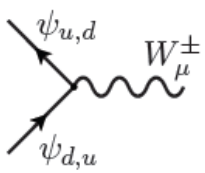
Neutral currents

All fermions are doublets $D_\mu \Psi = (\partial_\mu - i g W_\mu^a \frac{\sigma^a}{2} - i Y_\Psi g' B_\mu)\Psi$

$$D_\mu \Psi = \left(\partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{g}{\cos\theta_W}(T^3 - \sin^2\theta_W Q_\Psi) Z_\mu - ieQ_\Psi A_\mu \right) \Psi$$

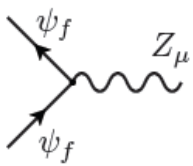
$$Q_\Psi \equiv T^3 + Y_\Psi \quad e \equiv \frac{g g'}{\sqrt{g^2 + g'^2}} = g \sin\theta_W$$

Off-diagonal in isospin!

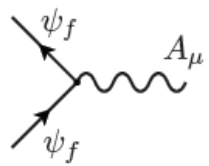


$$i\frac{g}{\sqrt{2}}\gamma_\mu \frac{1-\gamma_5}{2}$$

Diagonal in isospin!



$$i\frac{g}{\cos\theta_W}\gamma_\mu \left(g_V^f - g_A^f \gamma_5\right)$$



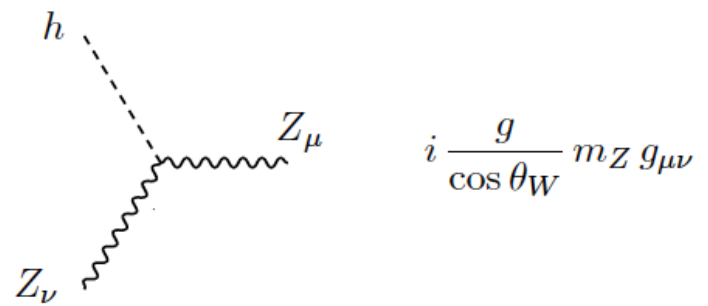
$$-ieQ_f\gamma_\mu$$

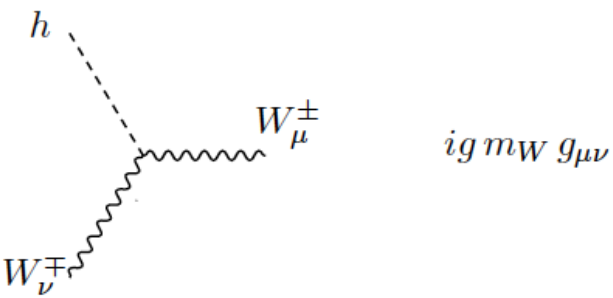
$$g_V^f = \frac{1}{2}T_f^3 - Q_f \sin^2\theta_W, \quad g_A^f = \frac{1}{2}T_f^3 \,.$$

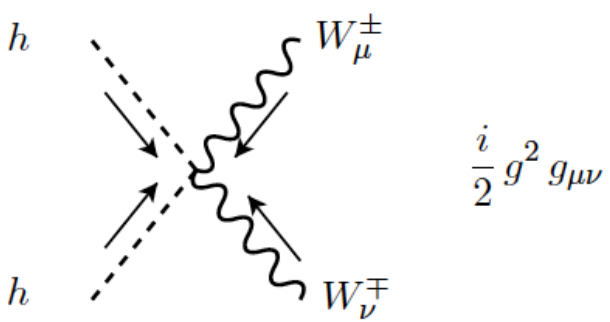
[courtesy of Pilar Hernández]

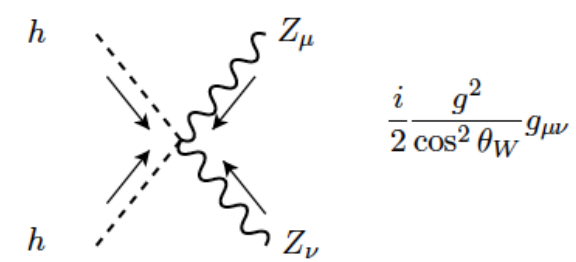
Higgs-Gauge couplings

$$(1 + h/v)^2 (m_W^2 W_\mu^+ W_\mu^- + \frac{1}{2} m_Z^2 Z_\mu Z_\mu)$$









[courtesy of Pilar Hernández]

Fermion Masses

Given the $SU(2)$ and $U(1)$ charges of the Higgs field and the fermions, Yukawa interactions are allowed:

$$-\mathcal{L} \supset Y_u Q u^c \phi + Y_d Q d^c \tilde{\phi} + Y_e L e^c \tilde{\phi} + h.c.,$$

where $\tilde{\phi} \equiv i\sigma_2 \phi^*$. Using

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle \tilde{\phi} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix},$$

$$-\mathcal{L} \supset (m_u)_{ij} u^i (u^c)^j + (m_d)_{ij} d^i (d^c)^j + (m_e)_{ij} e^i (e^c)^j + h.c.,$$

where $m_f \equiv Y_f v/\sqrt{2}$.

$u + u^c$, $d + d^c$, $e + e^c$ merge into Dirac fermions [note that all these fields have equal-but-opposite electric charges]. Neutrinos are left unpaired! $\rightarrow$ neutrinos are massless. Robust prediction, stable under quantum corrections.

Global Symmetries: $U(3)^5$ explicitly broken down to $U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau$. I will come back to this.

Quark mixing: Cabbibo-Kobayashi-Maskawa

The Yukawa couplings are generic matrices in flavour space:

basis where CC and NC diagonal ≠ mass eigenbasis

$$\mathcal{L}_{SM} \supset -(\bar{d}_L, \bar{s}_L, \bar{b}_L) \underbrace{m_d}_{3\times3} \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} - (\bar{u}_L, \bar{c}_L, \bar{t}_L) \underbrace{m_u}_{3\times3} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} - (\bar{e}_L, \bar{\mu}_L, \bar{\tau}_L) \underbrace{m_u}_{3\times3} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}$$

$$m_i = U_{Li}^{\dagger} \mathrm{Diag}(m_i) V_{Ri}$$

$$u'_L = U_{Lu} u_L, \;\; d'_L = U_{Ld} d_L, \;\; l'_L = U_{Ll} l_L, \;\; u'_R = V_{Ru} u_R, \;\; d'_R = V_{Rd} d_R, \;\; l'_R = V_{Rl} l_R$$

$$\mathcal{L}_{SM}^{CC} \supset -\frac{g}{\sqrt{2}} \bar{u}'_{Li} \underbrace{(U_{Lu} U_{Ld}^{\dagger})_{ij}}_{CKM} \gamma_\mu W_\mu^+ d'_{Lj} + h.c.$$

[courtesy of Pilar Hernández]

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The Yukawa couplings are generic matrices in flavour space:

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$$\mathcal{L}_{SM} \supset -(\bar{d}_L, \bar{s}_L, \bar{b}_L) \underbrace{m_d}_{3\times3} \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} - (\bar{u}_L, \bar{c}_L, \bar{t}_L) \underbrace{m_u}_{3\times3} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} - (\bar{e}_L, \bar{\mu}_L, \bar{\tau}_L) \underbrace{m_u}_{3\times3} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}$$

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$$u'_L = U_{Lu} u_L, \;\; d'_L = U_{Ld} d_L, \;\; l'_L = U_{Ll} l_L, \;\; u'_R = V_{Ru} u_R, \;\; d'_R = V_{Rd} d_R, \;\; l'_R = V_{Rl} l_R$$

$$\mathcal{L}_{SM}^{NC} \supset -\frac{g}{\cos\theta_W} \bar{d}'_{Li} \left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W \right) \underbrace{(U_{Ld} U_{Ld}^{\dagger})_{ij}}_{\delta_{ij}} \gamma_\mu Z_\mu d'_{Lj}$$

$$-\frac{g}{\cos\theta_W} \bar{u}'_{Li} \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_W \right) \underbrace{(U_{Lu} U_{Lu}^{\dagger})_{ij}}_{\delta_{ij}} \gamma_\mu Z_\mu u'_{Lj}$$

[courtesy of Pilar Hernández]

Quark mixing: Cabbibo-Kobayashi-Maskawa

Neutral currents diagonal also in the mass eigenbasis: only quarks in the same family can exchange a Z boson

It was quite of a challenge to come up with this when only 1.5 quark family was known: u, d, s -> prediction of the charm !

Charged currents not diagonal: CKM 3x3 unitary matrix

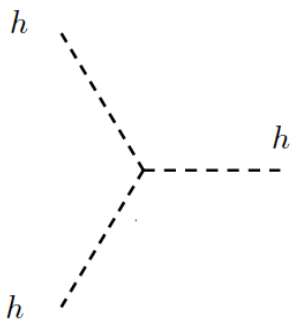
$$|V|_{\rm CKM} = \begin{pmatrix} 0.97427 \pm 0.00015 & 0.22534 \pm 0.0065 & (3.51 \pm 0.15) \times 10^{-3} \\ 0.2252 \pm 0.00065 & 0.97344 \pm 0.00016 & (41.2^{+1.1}_{-5}) \times 10^{-3} \\ (8.67^{+0.29}_{-0.31}) \times 10^{-3} & (40.4^{+1.1}_{-0.5}) \times 10^{-3} & 0.999146^{+0.000021}_{-0.000046} \end{pmatrix}$$

PDG

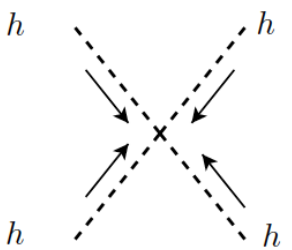
[courtesy of Pilar Hernández]

Higgs self-couplings

From the Higgs potential:



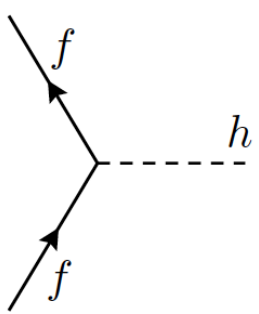
$$-\frac{3}{2}\,i\,g\frac{m_h^2}{m_W^2}$$



$$-\frac{3}{4}\,i\,g^2\frac{m_h^2}{m_W^2}$$

Higgs-fermion couplings

$$(1+h/v)\left(-m_d\bar{d}_L d_R - m_u\bar{u}_L u_R - m_l\bar{l}_L l_R\right) + h.c.$$



$$-i\,\frac{g}{2}\,\frac{m_f}{m_W}$$

[courtesy of Pilar Hernández]

How many physically relevant parameters remain after the field redefinitions are performed?

The number of parameters contained in the complex matrices Y_u, Y_d is $2 \times 3 \times 3 \times 2 = 36$.

The unitary symmetries U_Q, U_u, U_d are a subset of the quark field redefinitions. There are $3 \times 3 \times 3$ degrees of freedom in these symmetries (a unitary $N \times N$ matrix has N^2 free parameters), so the total number of parameters that remain in the full Lagrangian after field redefinitions is

$$2 \times 3 \times 3 \times 2 - (3 \times 3 \times 3 - 1) = 10,$$

subtracting baryon number from the subset of field redefinitions that are symmetries of the matter Lagrangian. $U(1)_B$ is a symmetry of the Yukawa Lagrangian, and hence cannot be used to diagonalize the mass matrices. The ten remaining parameters correspond to the six quark masses and the four parameters of the Cabibbo-Kobayashi-Maskawa (CKM) matrix (three mixing angles and one CP-violating phase).