ICTP LASS 2015 FINAL EXAM

CINVESTAV, Mexico City, October 26-th - November 6th, 2015

November 6th, 2015

1. a) Construct gamma matrices in $d = 4$ Minkowski space satisfying $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$ for $m = 0, ..., 3$.

   What is the minimum size of these matrices?

   b) Construct gamma matrices in $d = 5$ Minkowski space satisfying $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$ for $m = 0, ..., 4$.

   What is the minimum size of these matrices?

2. a) On a Riemmanian manifold show that the Laplacian with metric tg is related to that with metric g by

$$\Delta_{tg} = t^{-1}\Delta_g.$$

   b) Show that on a Riemmanian manifold manifold of real dimension n that the volume form satisfies

$$dV_{tg} = t^{n/2} dV_g.$$

   c) Let $X = M \times S^3$, A a 1-form, B a 2-form and C a 3-form on X and suppose that

$$\Delta_X \Phi = 0,$$

   with $\Phi = A$, B and C. Put a product metric on X.

   Put the product metric $g = g_M \oplus t g_{S^3}$ on X. How many massless forms do we see on M in the limit that $t \to 0$? Hint:

   $H^0(S^3, R) = R, H^1(S^3, R) = 0$, $H^2(S^3, R) = 0$ and, $H^3(S^3, R) = R$.

3. Construct the massless spectrum of the Type IIA and Type IIB superstring.

4. • **Closed Strings**

Remember the mass formula we derived for closed strings compactified on a circle of radius R (in the x^9 direction), for the left– and right–moving sectors:

$$M^2 = \frac{2}{\alpha'}(\alpha_0^9)^2 + \frac{4}{\alpha'}\left(N - \frac{1}{2}\right) , \qquad M^2 = \frac{2}{\alpha'}(\tilde{\alpha}_0^9)^2 + \frac{4}{\alpha'}\left(\tilde{N} - \frac{1}{2}\right) ,$$

where $N, \tilde{N}$ are the oscillator number operators in the NS sector, n and w are integers counting momentum and winding in the x^9 direction, and

$$\alpha_0^9 = \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} , \qquad \tilde{\alpha}_0^9 = \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} ,$$

The massless ($M^2 = 0$) state where $n = w = 0$ and $N = 1/2$, coming from acting with the oscillator mode $\psi^\mu_{-1/2}$ corresponds (with oscillator $\tilde{\psi}^9_{-1/2}$ acting on the right) to the $U(1)$ Kaluza–Klein vector $A^\mu(\mathbf{x})$.

  – With $\psi^\mu_{-1/2}$ turned on again, show that at the special radius $R = \sqrt{\alpha'}$, there are two extra massless vectors appearing. These three vectors at the special radius turn out to make an $SU(2)$ gauge symmetry.

• **Open Strings**

Let's consider a sector of open string theory with $U(2)$ Chan–Paton factors, with a Wilson line chosen such that the gauge group is broken to $U(1) \times U(1)$ when the theory is compactified on a circle of radius R in the x^9 direction. We denote a string state as $|ij>$, where i labels one end and j labels the other, and the indices take the value either 1 or 2, corresponding to either the first $U(1)$ or the second. As discussed, the state $|ij>$ has shifted momentum $p^9 = \frac{n}{R} + \frac{\theta_i - \theta_j}{2\pi R}$, and so the mass formula for the open string spectrum (in the NS sector) is:

$$M^2 = \left(\frac{n}{R} + \frac{\theta_i - \theta_j}{2\pi R}\right)^2 + \frac{1}{\alpha'}\left(N - \frac{1}{2}\right),$$

where N is the oscillator number operator and n labels the discrete Kaluza–Klein momentum in the x^9 direction.

- Rewrite the mass formula in terms of the dual radius $R' = \alpha'/R$ and explain, by reference to the fact that the string has tension $T = 1/2\pi\alpha'$, why n now has an interpretation as a winding number, and why $\theta_i R$ labels a position of the end of the string in state i.
- Which states give massless vectors of the $U(1) \times U(1)$?
- Show that when $\theta_1 = \theta_2$, there are two more massless vectors. This is in fact just enough to restore the $U(2)$ gauge symmetry.

5. The 11-dimensional Supergravity has the following metric

$$ds^2 = e^{-\frac{2}{3}\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4}{3}\phi}(dx^{10} + C_\mu^{(0)} dx^\mu)^2 \tag{1}$$

where $\mu,\ \nu = 0, 1, ..., 9$ are the indices in the non-compact 10-dim. space-time and x^{10} is compactified on a circle with periodicity $x^{10} = x^{10} + 2\pi$. The fields $g_{\mu\nu}$, ϕ and $C_\mu^{(0)}$ are the type IIA metric, dilaton and the RR 1-form potentials respectively. The 11-dimensional theory possesses $M2$ and $M5$ branes. Determine the dependence of the tensions of these branes on IIA dilaton in the following cases:
i) M2 and M5 branes are wrapped on the x^{10} circle.
ii) the world volumes of M2 and M5 branes are entirely living in the non-compact 10-dim. space time (i.e. they do not wrap the x^{10} circle). Can you identify the resulting objects in the IIA theory?

6. Consider the following metric on $AdS_5 \times S^5$:

$$ds^2 = \frac{L^2}{z^2}\left(dr^2 + r^2 d\phi^2 + dz^2 + dx_1^2 + dx_2^2\right) + L^2 d\Omega_5^2, \tag{2}$$

where $d\Omega_5^2$ denotes the standard metric on the round S^5.

(a) Assuming that the world sheet coordinates are identified with (r, ϕ), show that the action of the Nambu-Goto string describing an embedding determined by $z = z(r)$ can be written as:

$$S_{NG} = \frac{L^2}{2\pi\alpha'} \int dr d\phi \frac{r}{z^2} \sqrt{1 + (z')^2}, \tag{3}$$

where $z' = \frac{dz(r)}{dr}$.

(b) A solution satisfying the boundary conditions corresponding to a circular Wilson loop or radius a at the boundary ($z = 0$) is given by

$$z^2 = \sqrt{a^2 - r^2}. \tag{4}$$

Evaluate the action for this solution using a cutoff $z \geq \epsilon$ and show that the answer is

$$S = -\sqrt{\lambda} + \sqrt{\lambda}\frac{a}{\epsilon}, \tag{5}$$

where $\sqrt{\lambda} = L^2/\alpha'$.

7. See adjoint problems.

Problem 14.5 Counting states in heterotic $SO(32)$ string theory.

In heterotic (closed) string theory the right-moving part of the theory is that of an open superstring. It has an NS sector whose states are built with oscillators α_{-n}^I and b_{-r}^I acting on the NS vacuum. It also has an R sector whose states are built with oscillators α_{-n}^I and d_{-n}^I acting on the R ground states. The index I runs over 8 values. The standard GSO projection down to NS+ and R− applies.

The left-moving part of the theory is that of a peculiar bosonic open string. The 24 transverse coordinates split into eight bosonic coordinates X^I with oscillators $\bar{\alpha}_{-n}^I$ and 16 peculiar bosonic coordinates. A surprising fact of two-dimensional physics allows us to replace these 16 coordinates by 32 two-dimensional left-moving *fermion* fields λ^A, with $A = 1, 2, \ldots, 32$. The (anticommuting) fermion fields λ^A imply that the left-moving part of the theory also has NS′ and R′ sectors, denoted with primes to differentiate them from the standard NS and R sectors of the open superstring.

The left NS′ sector is built with oscillators $\bar{\alpha}_{-n}^I$ and λ_{-r}^A acting on the vacuum $|\text{NS}'\rangle_L$, declared to have $(-1)^{F_L} = +1$:

$$(-1)^{F_L}|\text{NS}'\rangle_L = +|\text{NS}'\rangle_L.$$

The *naive* mass formula in this sector is

$$\alpha' M_L^2 = \frac{1}{2}\sum_{n\neq 0} \bar{\alpha}_{-n}^I \bar{\alpha}_n^I + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} r\,\lambda_{-r}^A \lambda_r^A.$$

The left R′ sector is built with oscillators $\bar{\alpha}_{-n}^I$ and λ_{-n}^A acting on a set of R′ ground states. The naive mass formula in this sector is

$$\alpha' M_L^2 = \frac{1}{2}\sum_{n\neq 0}\Big(\bar{\alpha}_{-n}^I \bar{\alpha}_n^I + n\,\lambda_{-n}^A \lambda_n^A\Big).$$

Momentum labels are not needed in this problem so they are omitted throughout.

(a) Consider the left NS′ sector. Write the precise mass-squared formula with normal-ordered oscillators and the appropriate normal-ordering constant. The GSO projection here keeps the states with $(-1)^{F_L} = +1$; this defines the left NS′+ sector. Write explicitly and count the states we keep for the three lowest mass levels, indicating the corresponding values of $\alpha' M_L^2$. [This is a long list.]

(b) Consider the left R$'$ sector. Write the precise mass-squared formula with normal-ordered oscillators and the appropriate normal-ordering constant. We have 32 zero modes λ_0^A and 16 linear combinations behave as creation operators. As usual, half of the ground states have $(-1)^{F_L} = +1$ and the other half have $(-1)^{F_L} = -1$. Let $|R_\alpha\rangle_L$ denote ground states with $(-1)^{F_L} = +1$. How many ground states $|R_\alpha\rangle_L$ are there? Keep only states with $(-1)^{F_L} = +1$; this defines the left R$'$+ sector. Write explicitly and count the states we keep for the two lowest mass levels, indicating the corresponding values of $\alpha' M_L^2$. [This is a shorter list.]