CFTC Centro de Física Teórica e Computacional

Ciências ULisboa

Percolation theory in complex networks

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School on Complex Networks and Applications to Neuroscience
01/October/2015

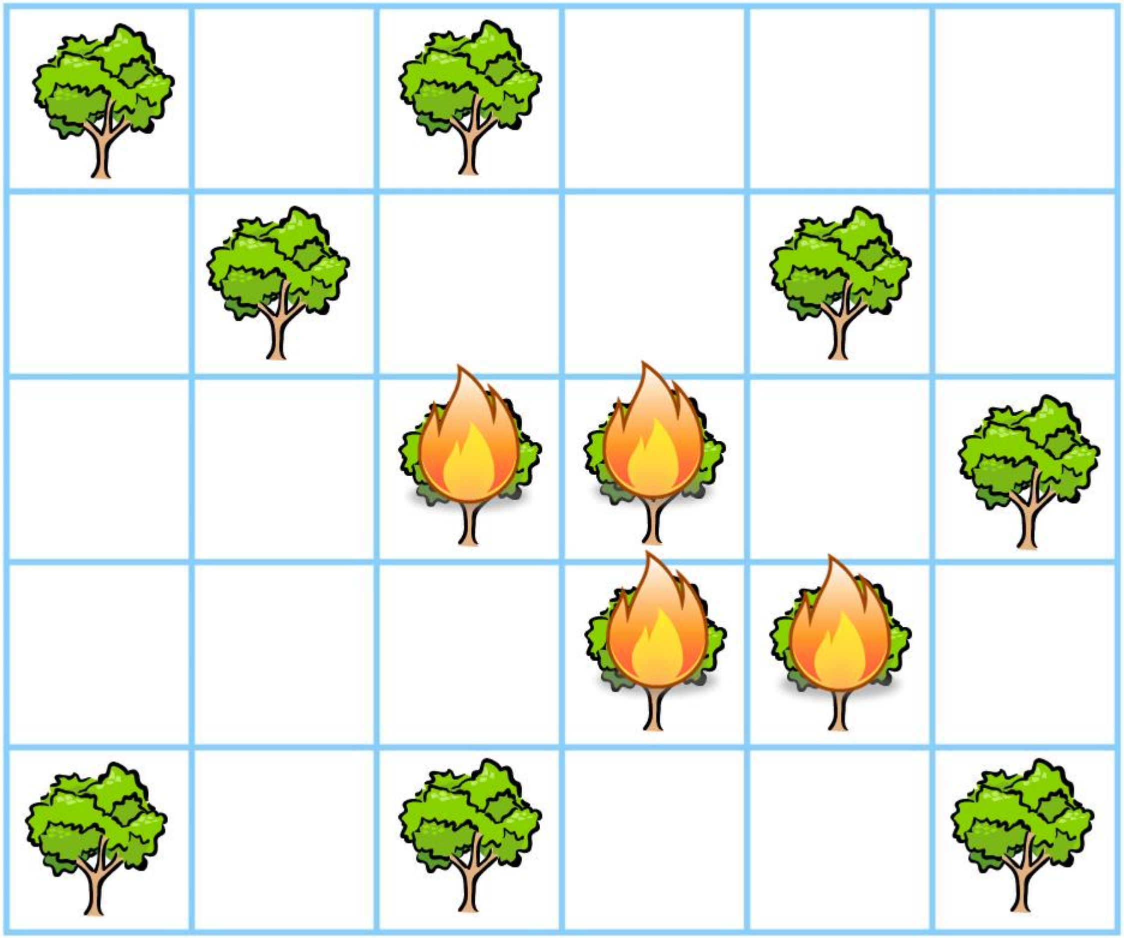
http://www.namaraujo.net

POPH

IF INVESTIGADOR FCT

Forest fire

Forest fire



Spreading of epidemics

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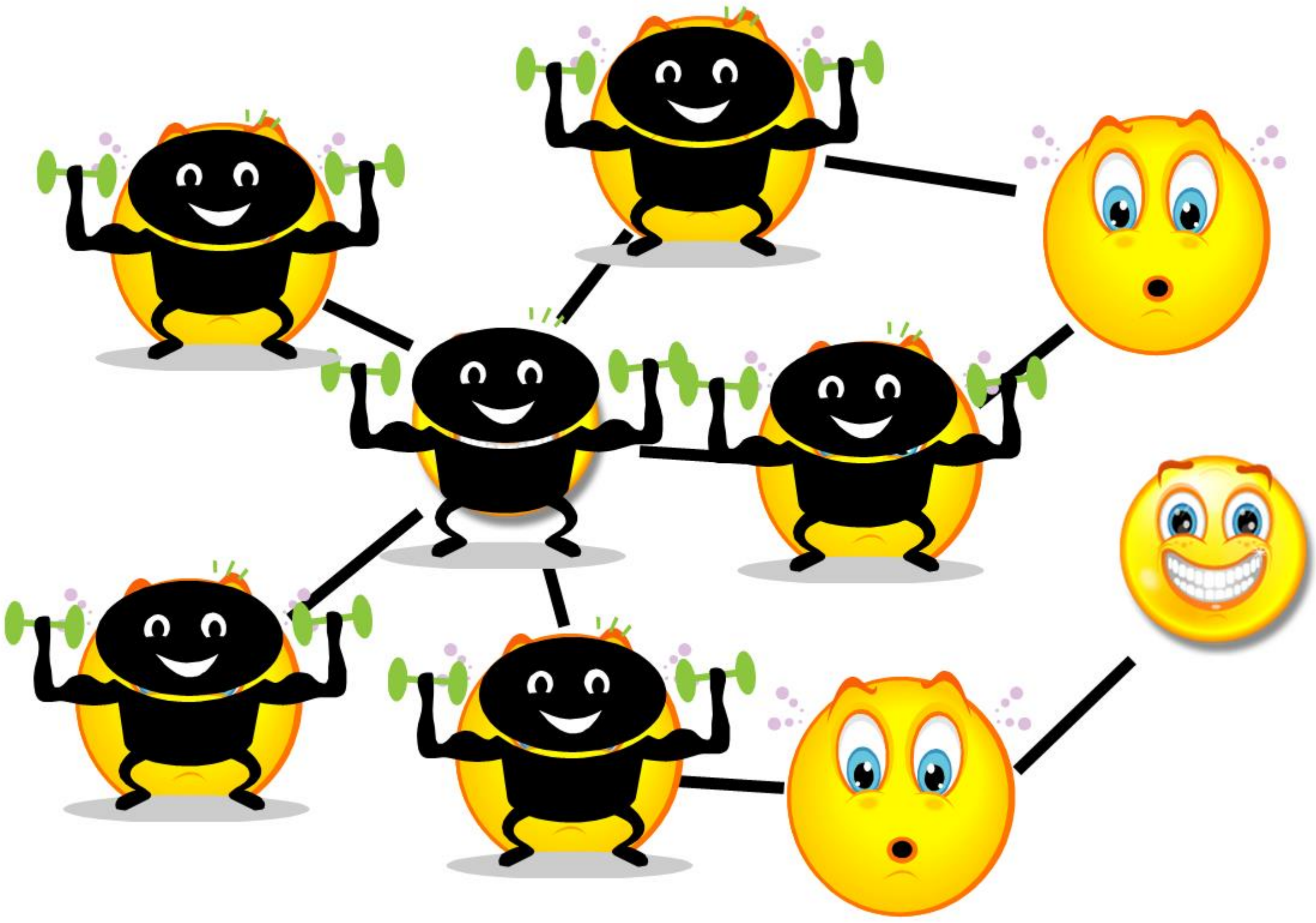
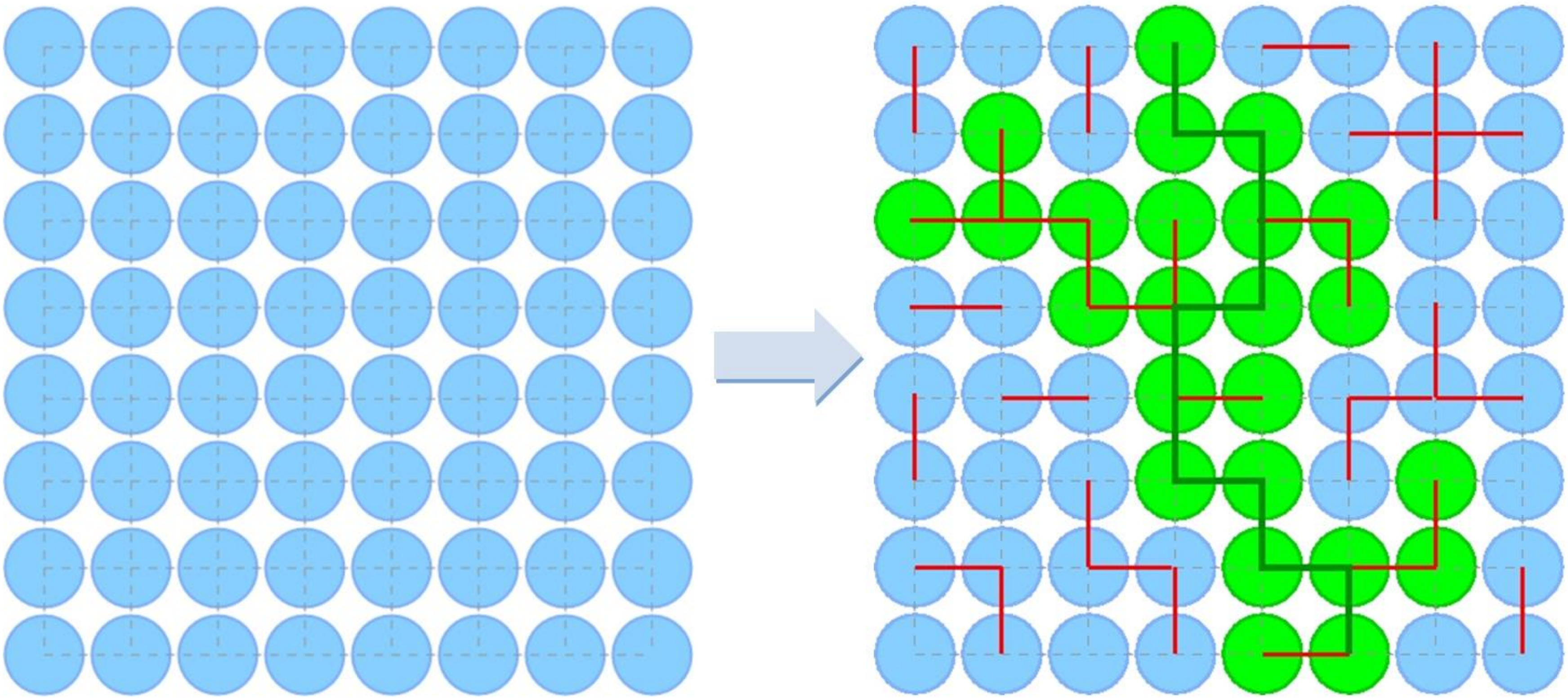


Oil fields

at *Barrancabermeja (Colombia)*, photo by *Melissa Jiménez*.

Percolation model

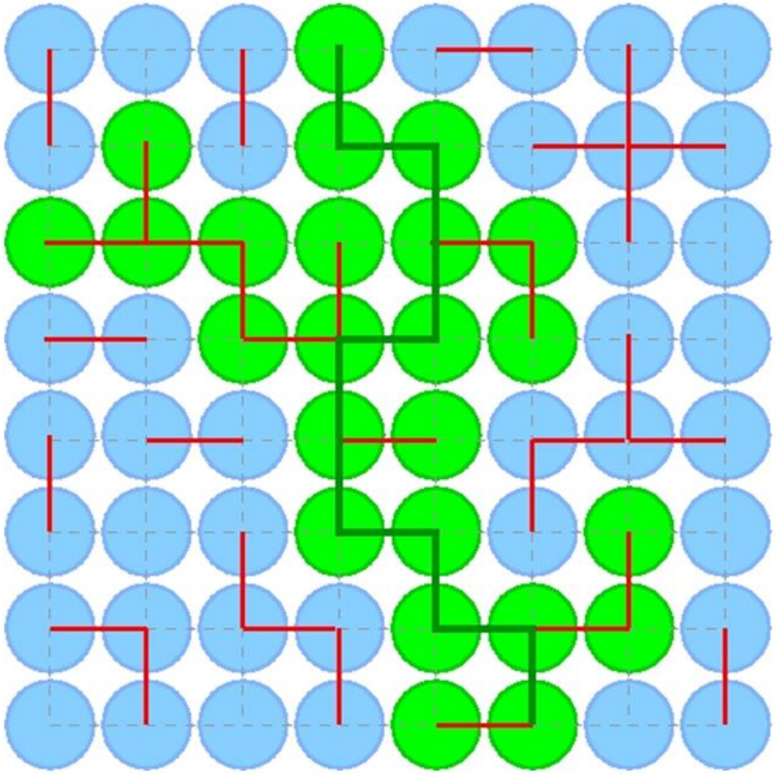


$$p^{O}(1-p)^{E}$$

Percolation model

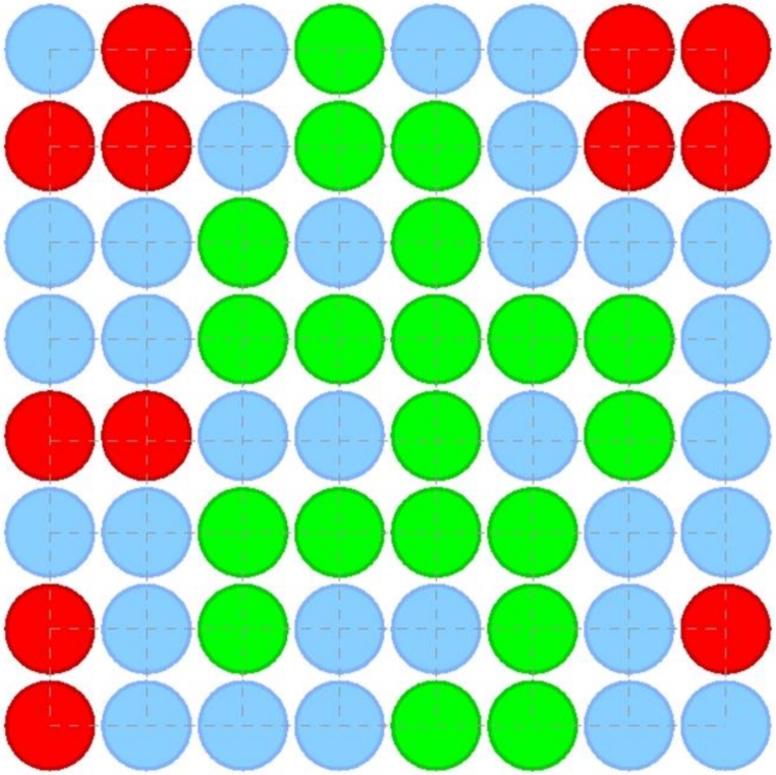
$$p^{O}(1-p)^{E}$$

Bonds



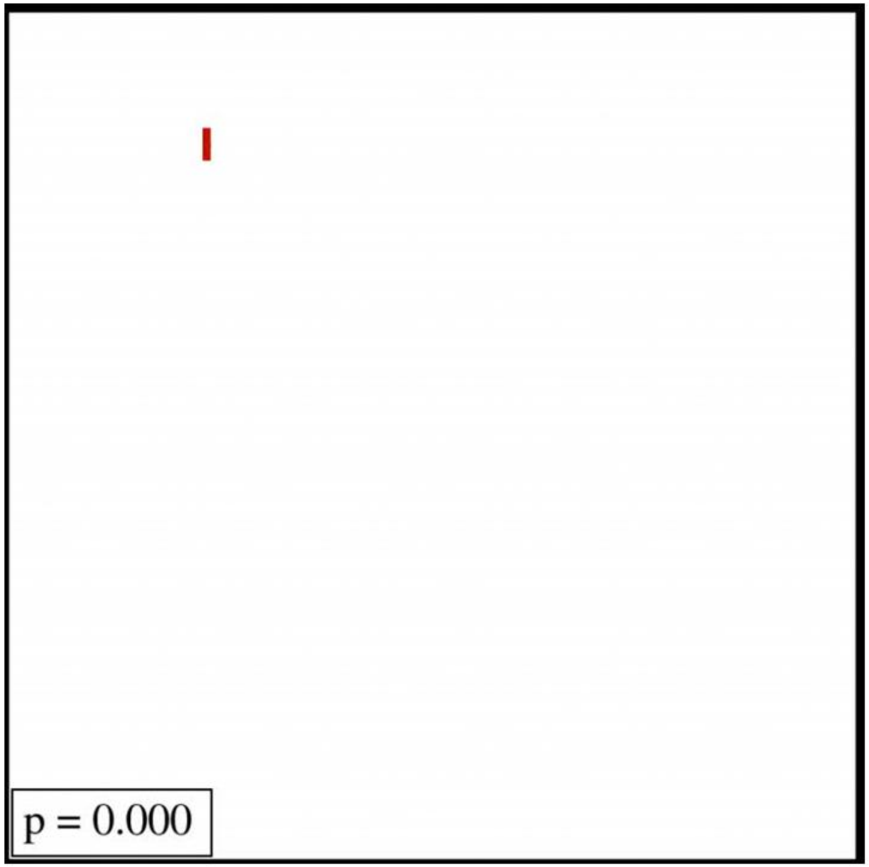
$$2^{N_{Bonds}}$$

Sites



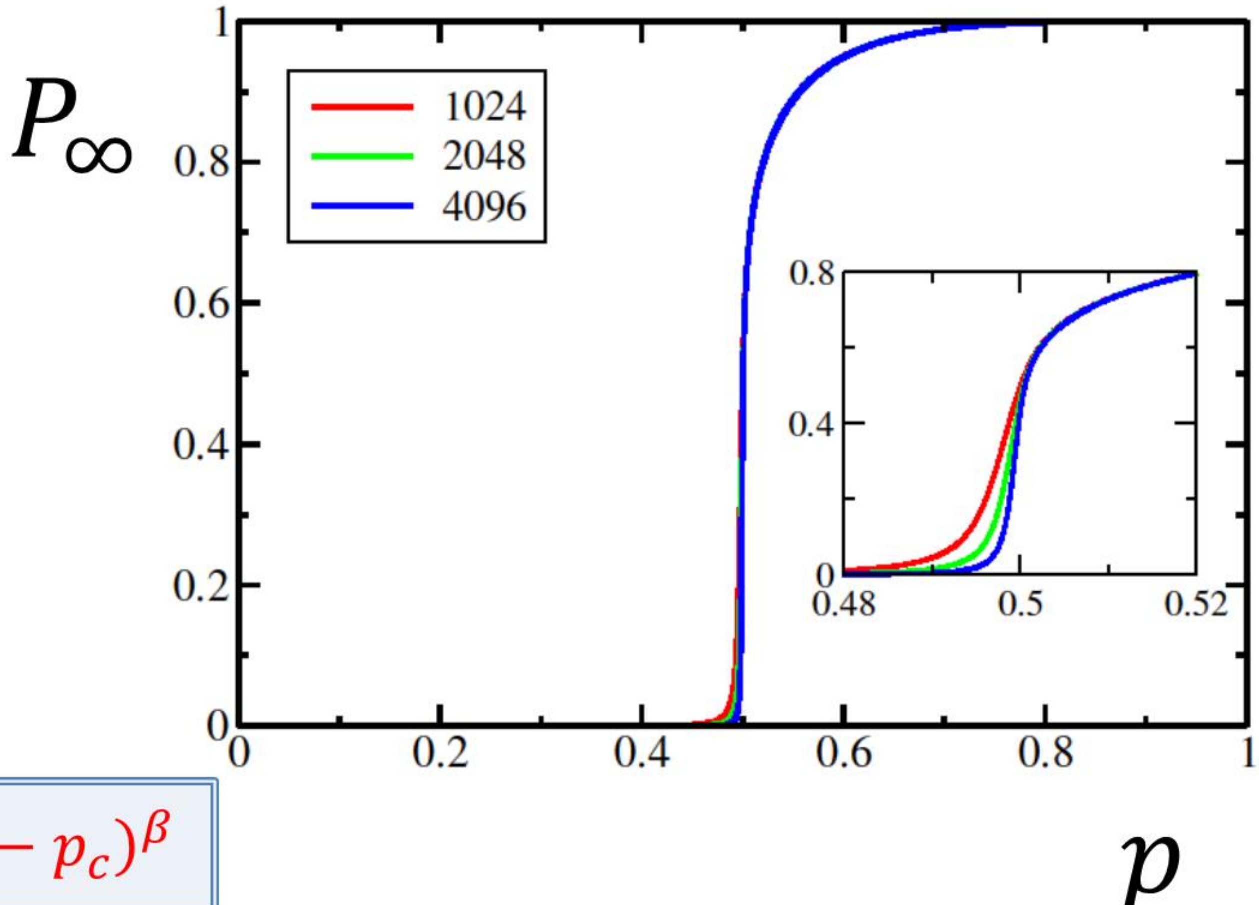
$$2^{N_{Sites}}$$

Percolation model



Percolation model
order parameter

$$P_\infty = \frac{s_{max}}{N}$$



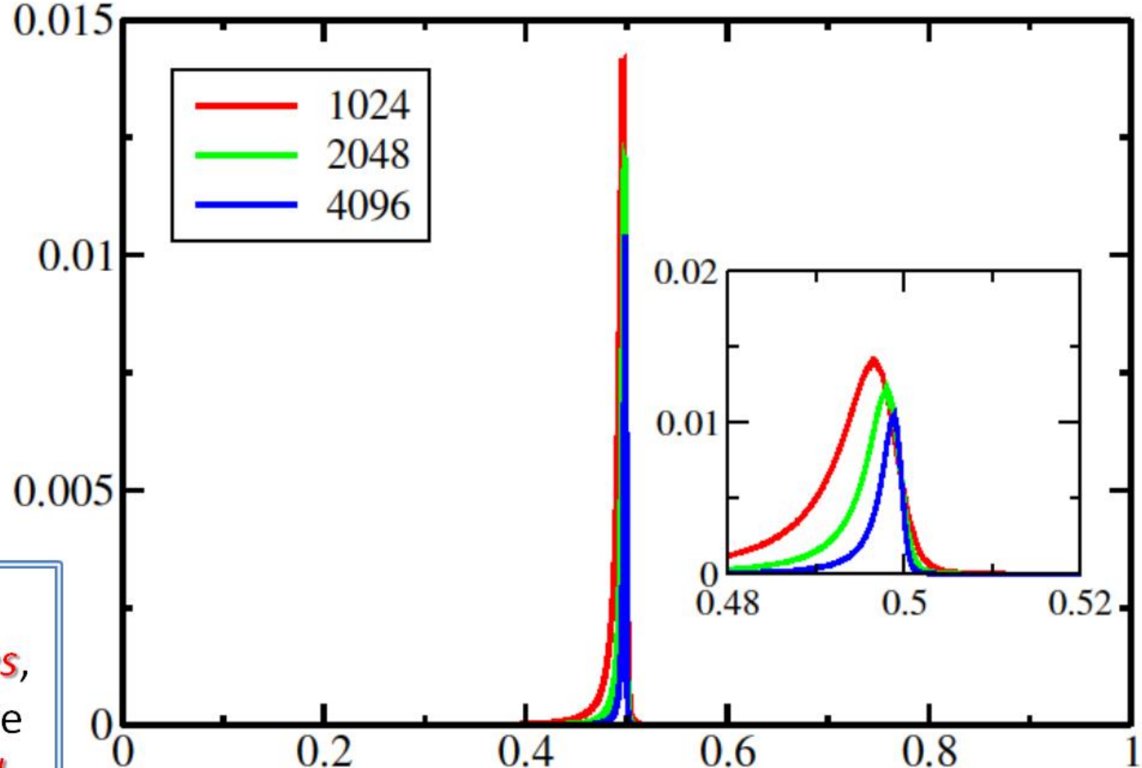
$$P_\infty \sim (p - p_c)^\beta$$

Percolation model

$$\chi = \frac{1}{N} \sum_{i \neq \max} s_i^2$$

fluctuations (mean cluster size)

$$\frac{\chi}{N}$$



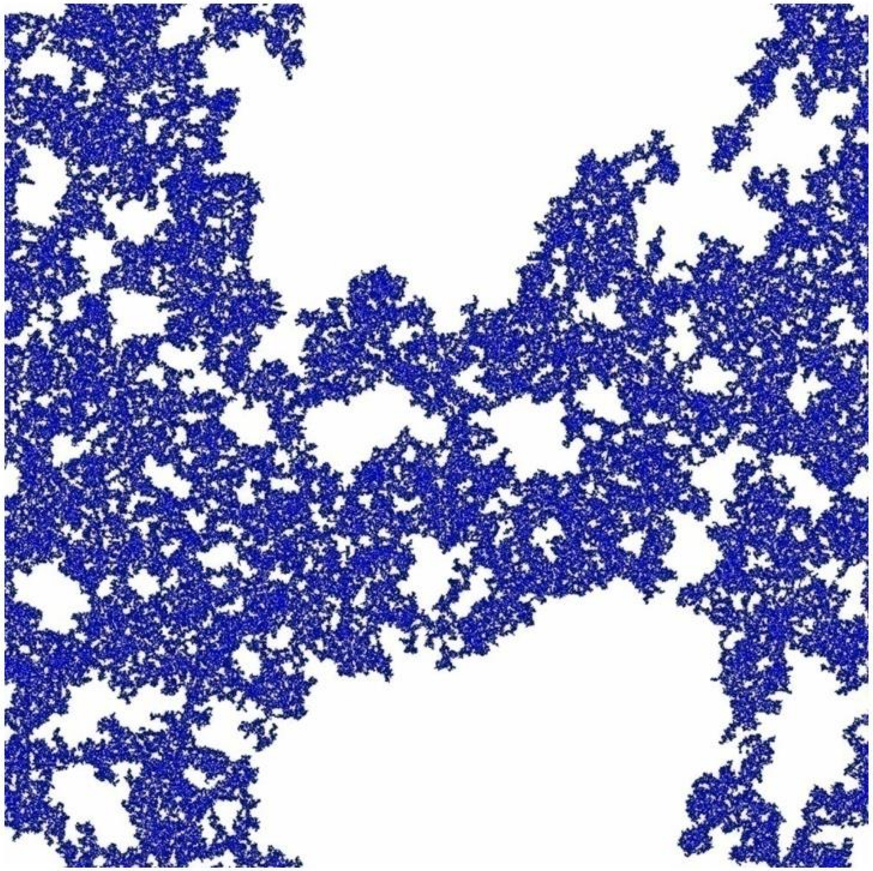
p

Mean cluster size when ***occupied sites***, and not clusters, are ***selected with equal probability***.

$\chi \sim (p_c - p)^{-\gamma}$

Percolation threshold

largest cluster: fractal dimension

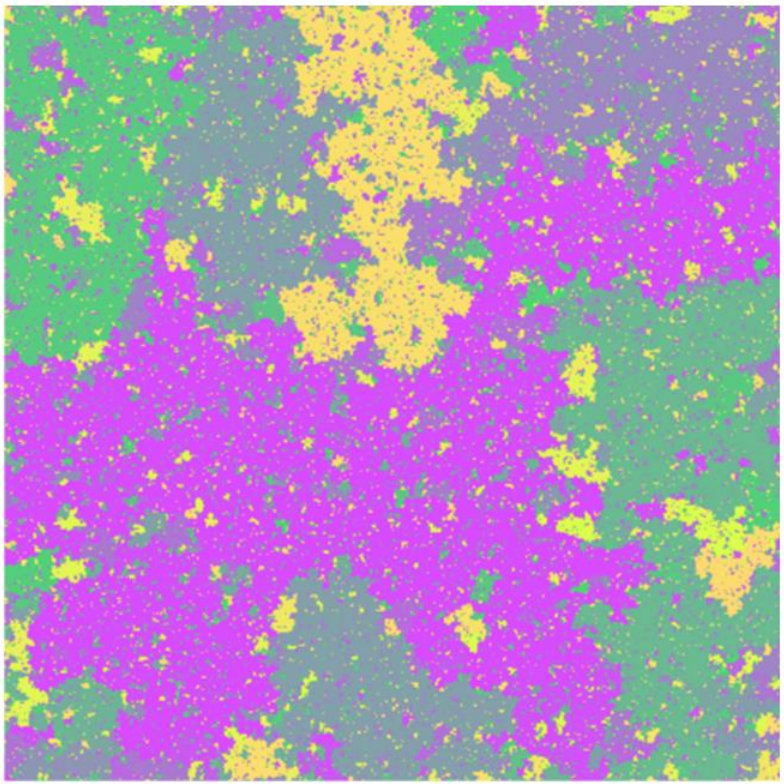


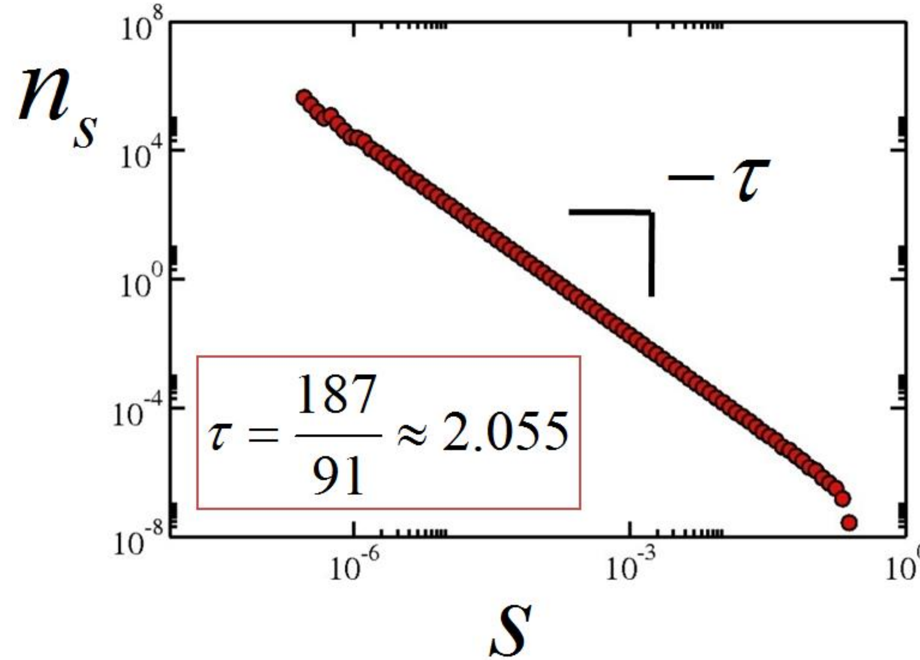
$$d_f = \frac{91}{48} \approx 1.896$$

Percolation threshold

cluster-size distribution

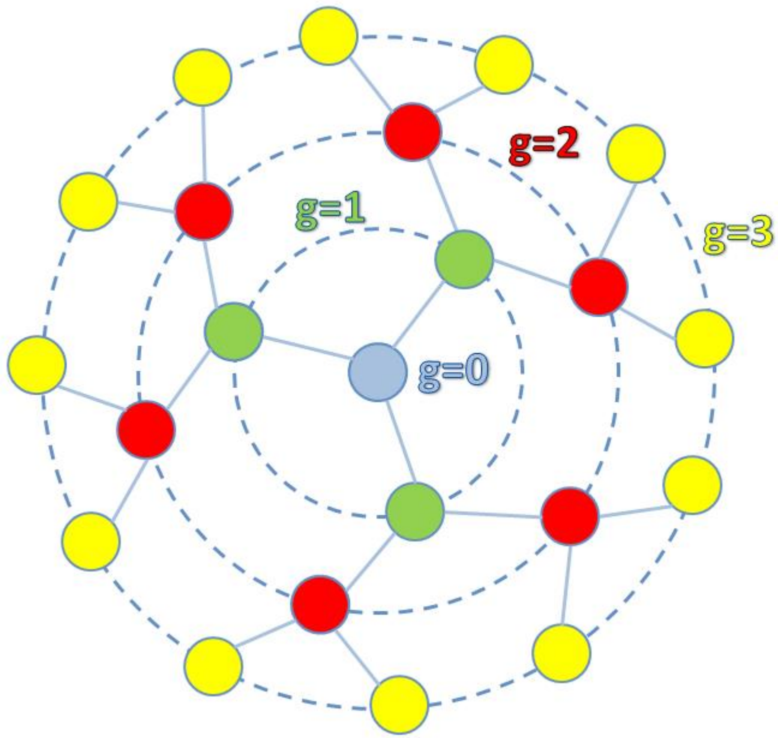
$$n_s \sim s^{-\tau}$$





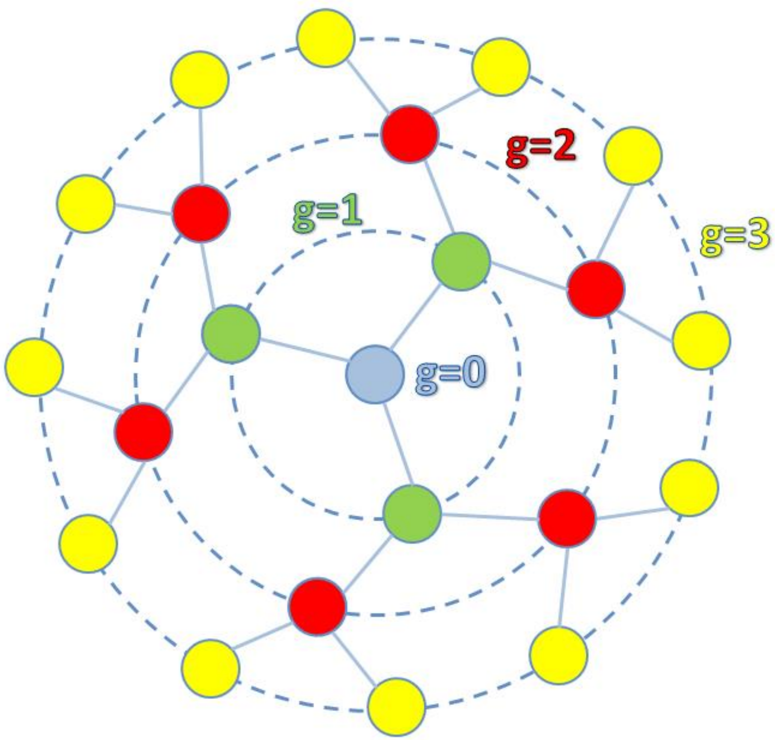
Site percolation on the Bethe lattice

mean-field (e.g. z=3)



Site percolation on the Bethe lattice

mean-field



$$p(z-1) \geq 1$$

$$\boxed{p_c = \frac{1}{z-1}}$$

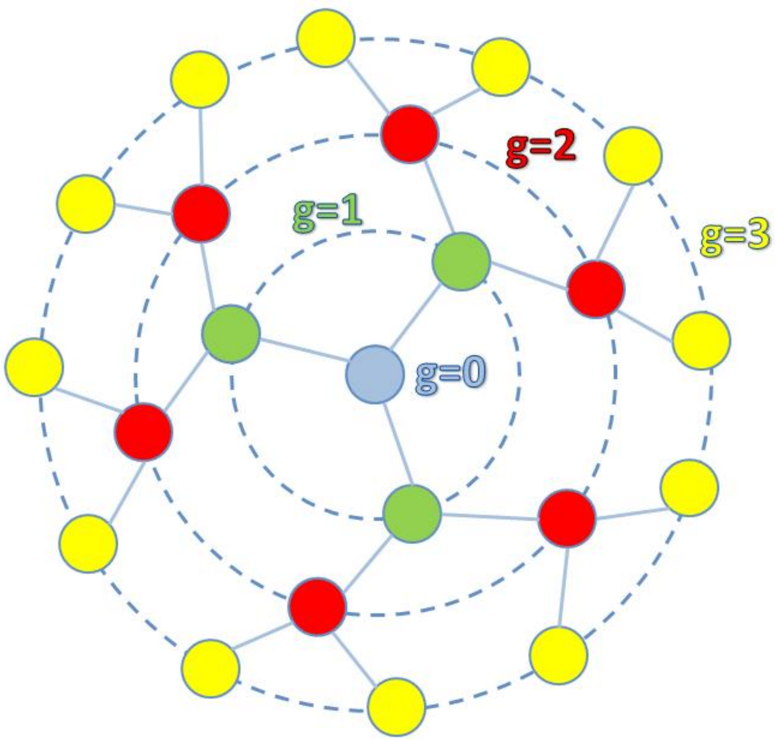
$z = 2$: $p_c = 1$ (1D)

$z > 2$: $p_c < 1$

Site percolation on the Bethe lattice

z=3

Q prob. site not connected to infinity through one branch.



$$Q = 1 - p + pQ^2$$

neighbor empty

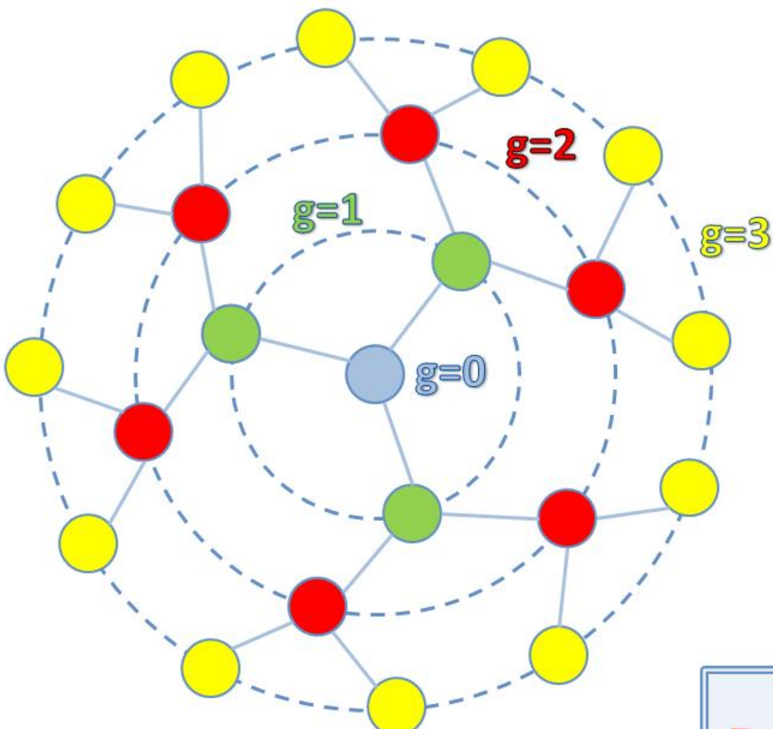
neighbor occupied

$$Q = 1$$

$$Q = \frac{1-p}{p}$$

Site percolation on the Bethe lattice

z=3 (order parameter)



P_∞ prob. site connected to infinity

$$p - P_\infty = pQ^3$$

Occupied but not connected to infinity

$$P_\infty = p(1 - Q^3)$$

$p < p_c$

$$P_\infty = 0$$

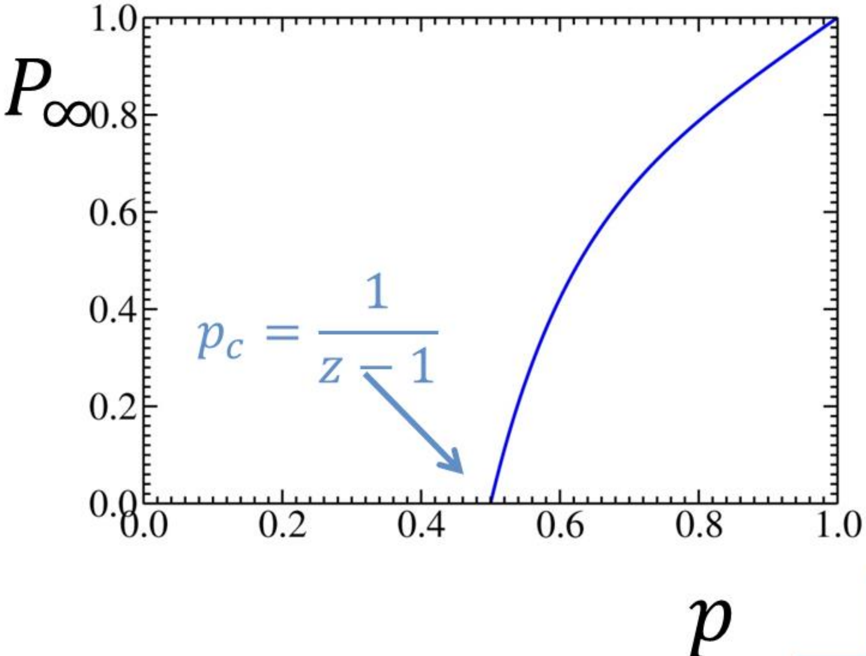
$p > p_c$

$$P_\infty = p\left[1 - \left(\frac{1-p}{p}\right)^3\right]$$

Site percolation on the Bethe lattice

z=3 (order parameter)

P_∞ prob. site connected to infinity



$$p - P_\infty = pQ^3$$

Occupied but not connected to infinity

$$P_\infty = p(1 - Q^3)$$

$$P_\infty \sim (p - p_c)^1$$

$p < p_c$

$$P_\infty = 0$$

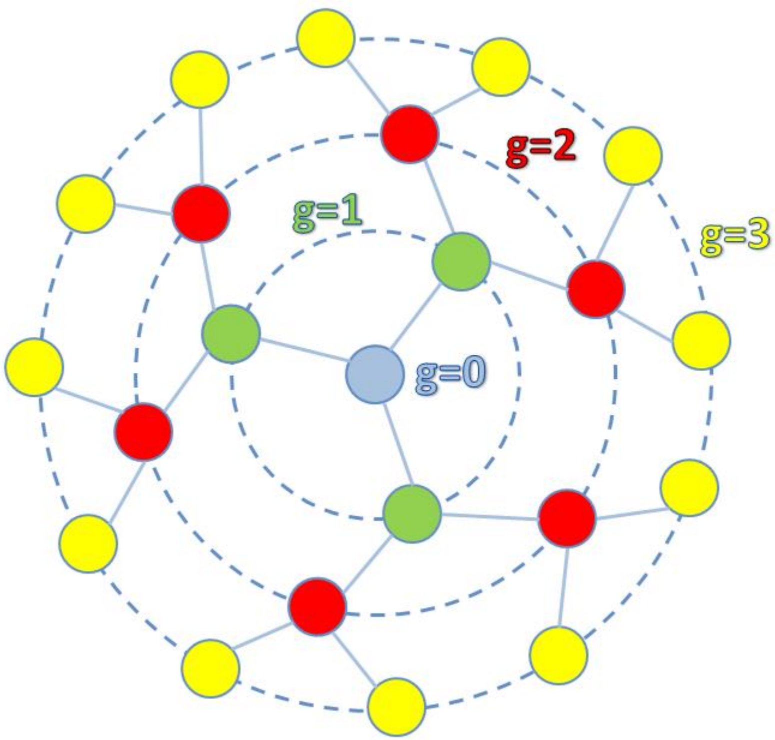
$p > p_c$

$$P_\infty = p\left[1 - \left(\frac{1-p}{p}\right)^3\right]$$

Site percolation on the Bethe lattice

z=3 (fluctuations)

T contribution to the mean cluster size for one branch.



$$T = (1-p)0 + p(1+2T)$$

neighbor empty

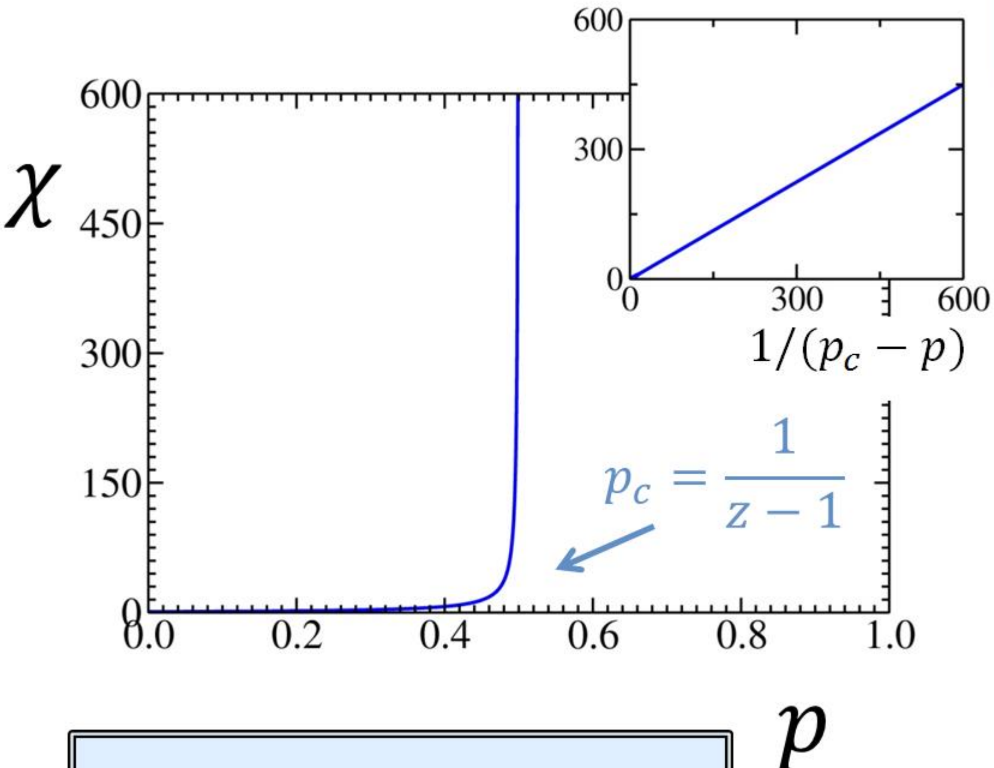
neighbor occupied

$$T = \frac{p}{1-2p}$$

Diverges for $p = p_c = \frac{1}{2}$

Site percolation on the Bethe lattice

z=3 (fluctuations)



$$\chi \sim (p_c - p)^{-1}$$

χ mean cluster size (fluctuations)

Mean cluster size for the occupied origin

$$\chi = (1 + 3T)$$

$$T = \frac{p}{1-2p}$$

$$p < p_c$$

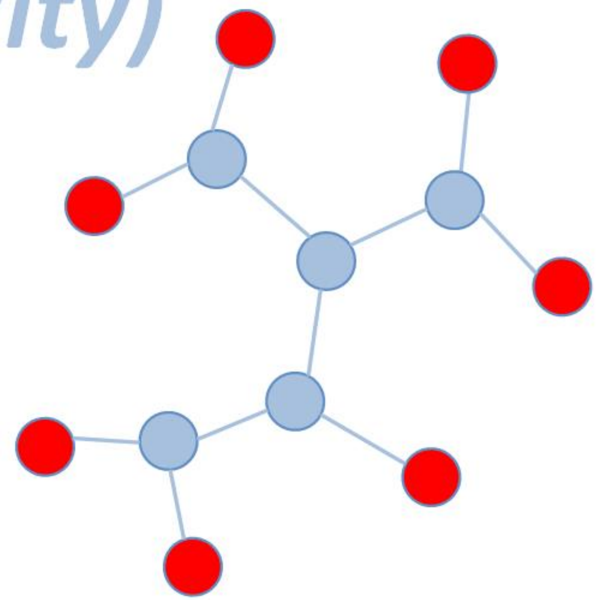
$$\chi = \frac{2(1+p)}{\frac{1}{2} - p}$$

Site percolation on the Bethe lattice

z=3 (cluster number density)

$$n(s,p) = \sum_{t=1}^{\infty} g(s,t)(1-p)^t p^s$$

degeneracy factor



size s, perimeter t

$$\boxed{t = 2 + s(z-2)}$$

$$n(s,p) = g[s, 2 + s(z-2)](1-p)^{2+s(z-2)} p^s$$

$$\boxed{n(s,p) = g(s, 2+s)(1-p)^{2+s} p^s}$$

$$\boxed{z = 3}$$

Site percolation on the Bethe lattice

z=3 (characteristic cluster size)

$$\frac{n(s,p)}{n(s,p_c)} = \left[\frac{1-p}{1-p_c}\right]^2 \left[\frac{(1-p)p}{(1-p_c)p_c}\right]^s$$

$$= \left[\frac{1-p}{1-p_c}\right]^2 \exp\left(s\ ln\left[\frac{(1-p)p}{(1-p_c)p_c}\right]\right)$$

$$= \left[\frac{1-p}{1-p_c}\right]^2 \exp(-s/s_\xi)$$

$$\boxed{s_\xi = -\frac{1}{\ln\left[\frac{(1-p)p}{(1-p_c)p_c}\right]}}$$

$$\boxed{z = 3}$$

$$\boxed{s_\xi \sim (p-p_c)^{-2}}$$

Site percolation on the Bethe lattice

mean-field exponents

$$\boxed{P_\infty \sim (p - p_c)^1}$$

$$P_\infty \sim (p - p_c)^\beta$$

$$\boxed{\beta = 1}$$

$$\boxed{\chi \sim (p_c - p)^{-1}}$$

$$\chi \sim (p_c - p)^{-\gamma}$$

$$\boxed{\gamma = 1}$$

$$\boxed{s_\xi \sim (p - p_c)^{-2}}$$

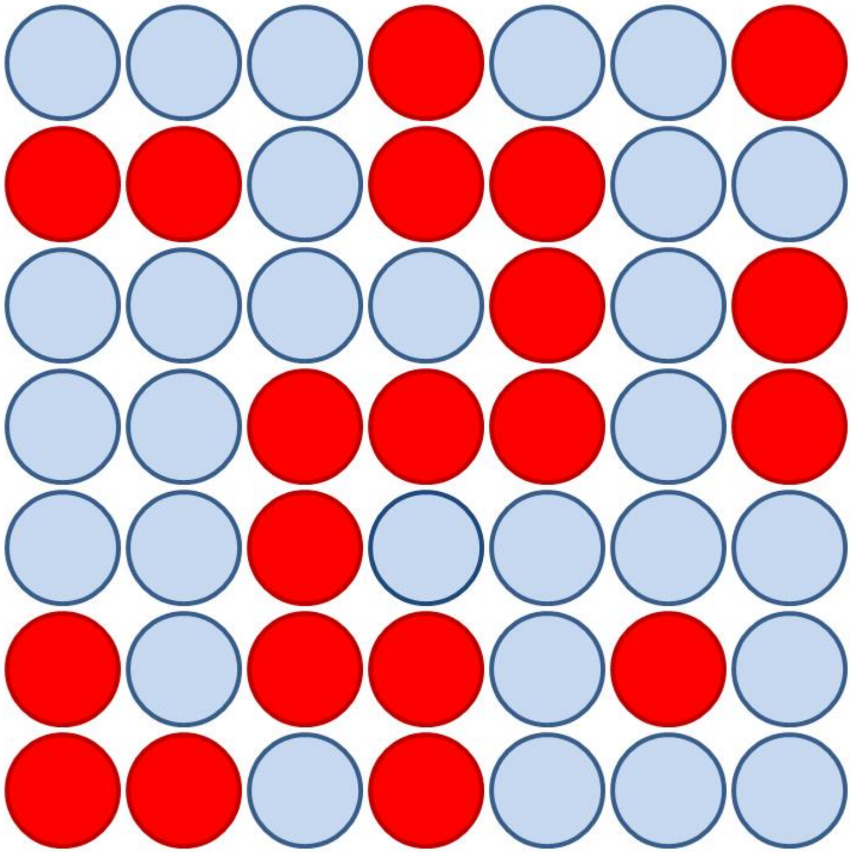
$$s_\xi \sim (p - p_c)^{-\frac{1}{\sigma}}$$

$$\boxed{\sigma = \frac{1}{2}}$$

Algorithms
Hoshen and Kopelman

k = 2

M(k) = 0

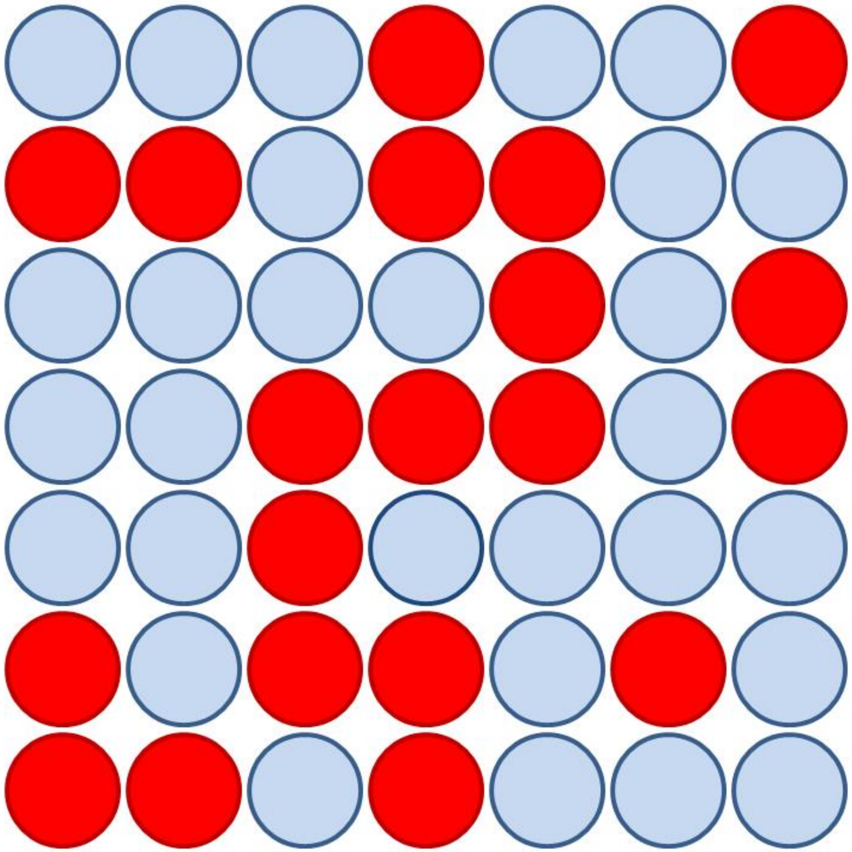


1. **start** from the site in the **left-bottom corner**;
2. **sweep** from **left to right bottom to top**;
3. **only** verify **left** and **bottom** neighbors.

J. Hoshen and R. Kopelman. *Phys. Rev. B* **14**, 3438 (1976)

Algorithms

Hoshen and Kopelman



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k	M(k)
2	0

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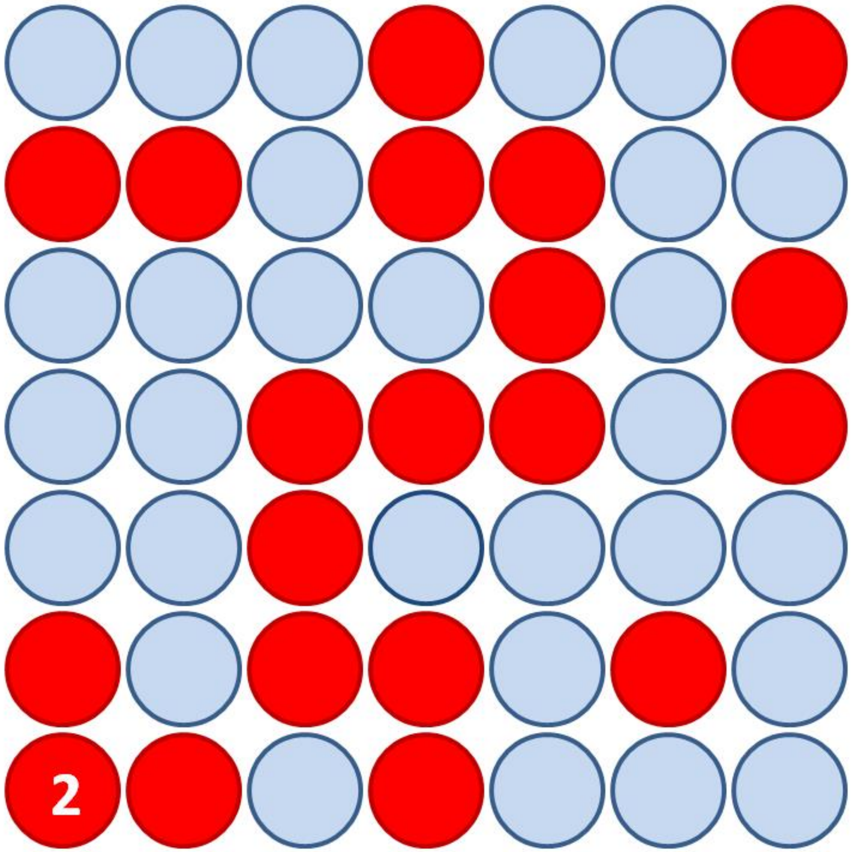
Algorithms

Hoshen and Kopelman



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k	M(k)
2	1

J. Hoshen and R. Kopelman. *Phys. Rev. B* **14**, 3438 (1976)

Algorithms

Hoshen and Kopelman

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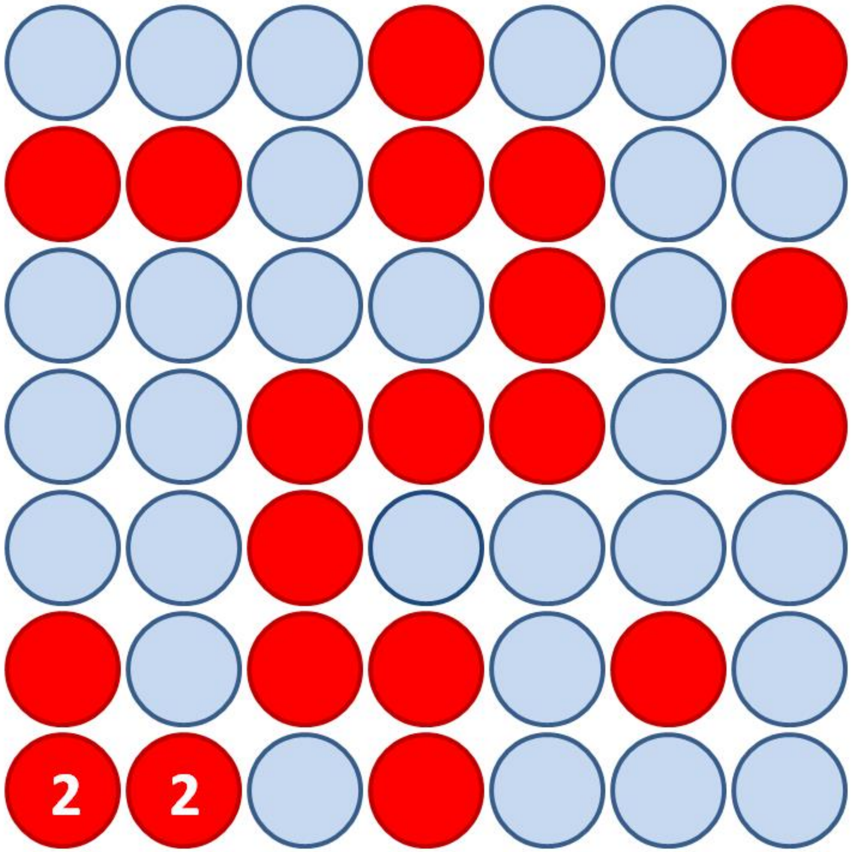
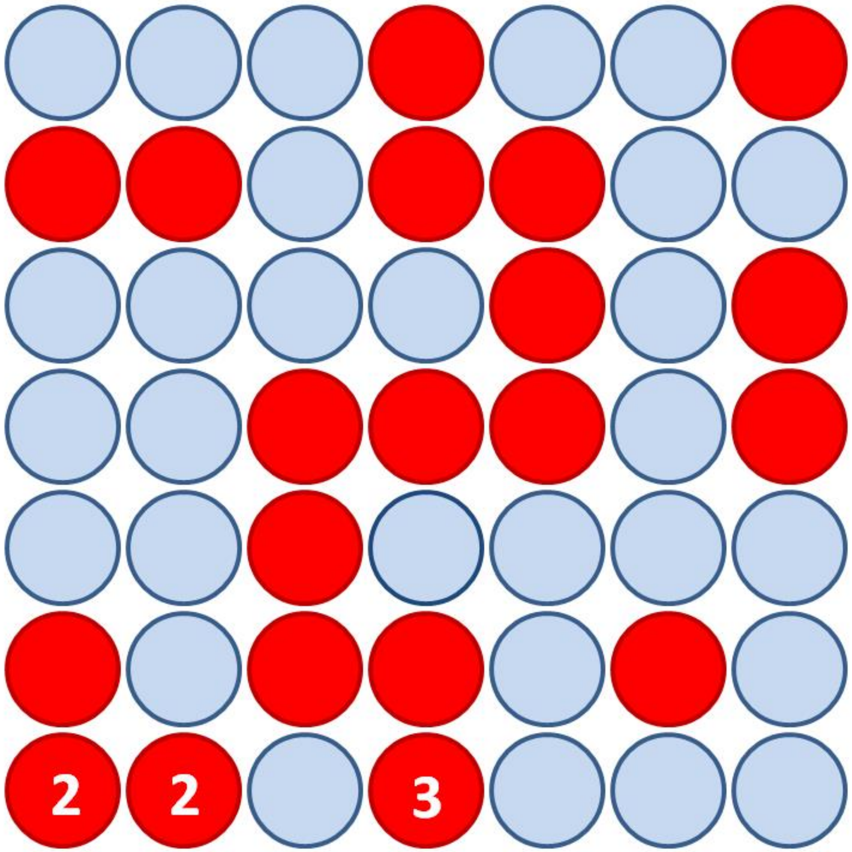


k	M(k)
2	2

J. Hoshen and R. Kopelman. *Phys. Rev. B* **14**, 3438 (1976)

Algorithms
Hoshen and Kopelman



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k	M(k)
2	2
3	1

J. Hoshen and R. Kopelman. *Phys. Rev. B* **14**, 3438 (1976)

Algorithms
Hoshen and Kopelman

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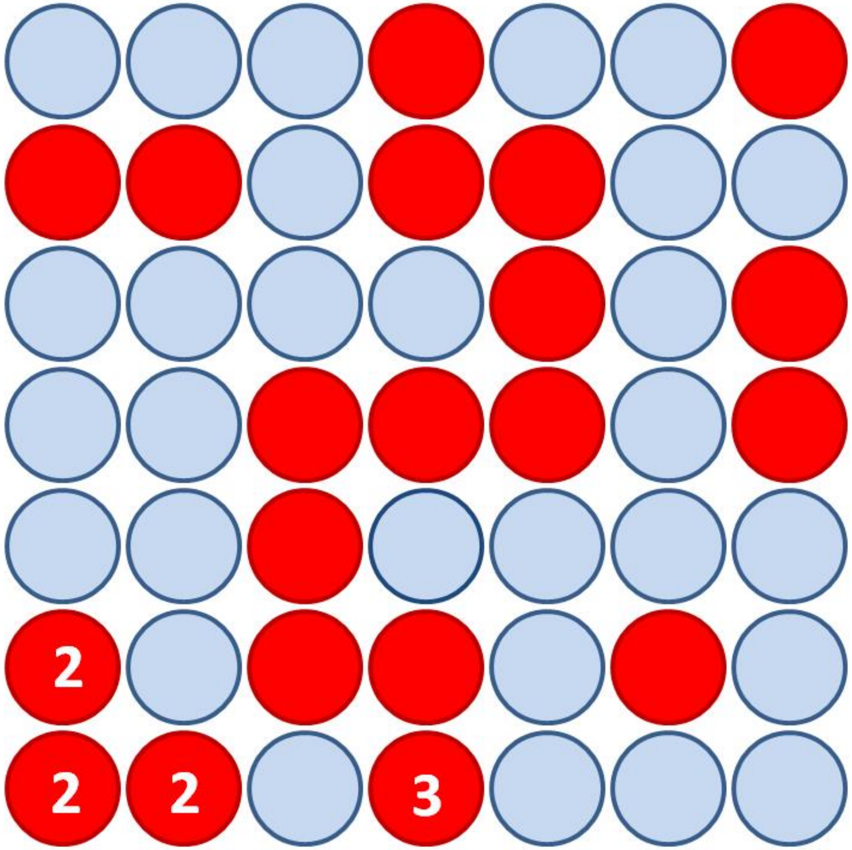

k	M(k)
2	3
3	1

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Algorithms

Hoshen and Kopelman

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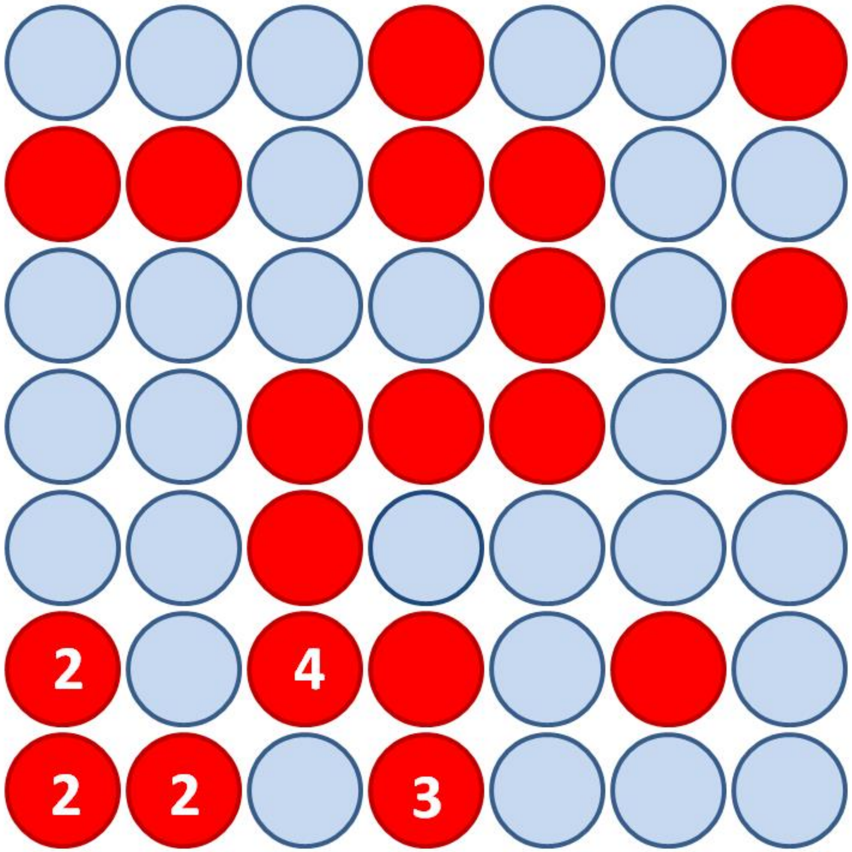

k	M(k)
2	3
3	1
4	1

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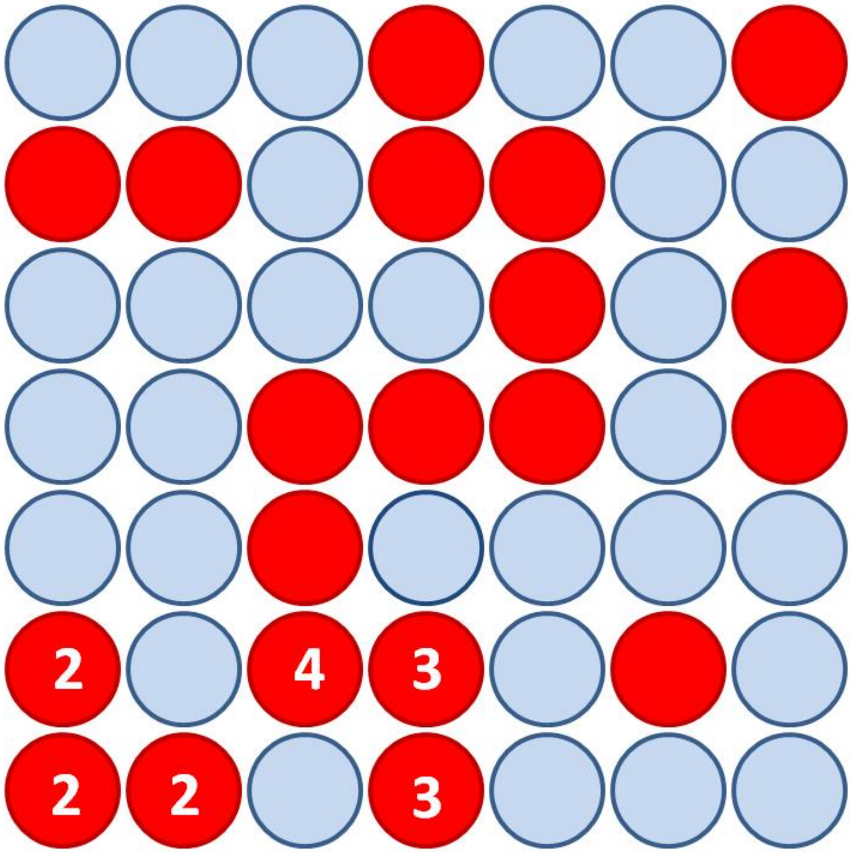
Algorithms
Hoshen and Kopelman



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3. **only** verify **left** and **bottom** neighbors.

k	M(k)
2	3
3	3
4	-3

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Algorithms

Hoshen and Kopelman

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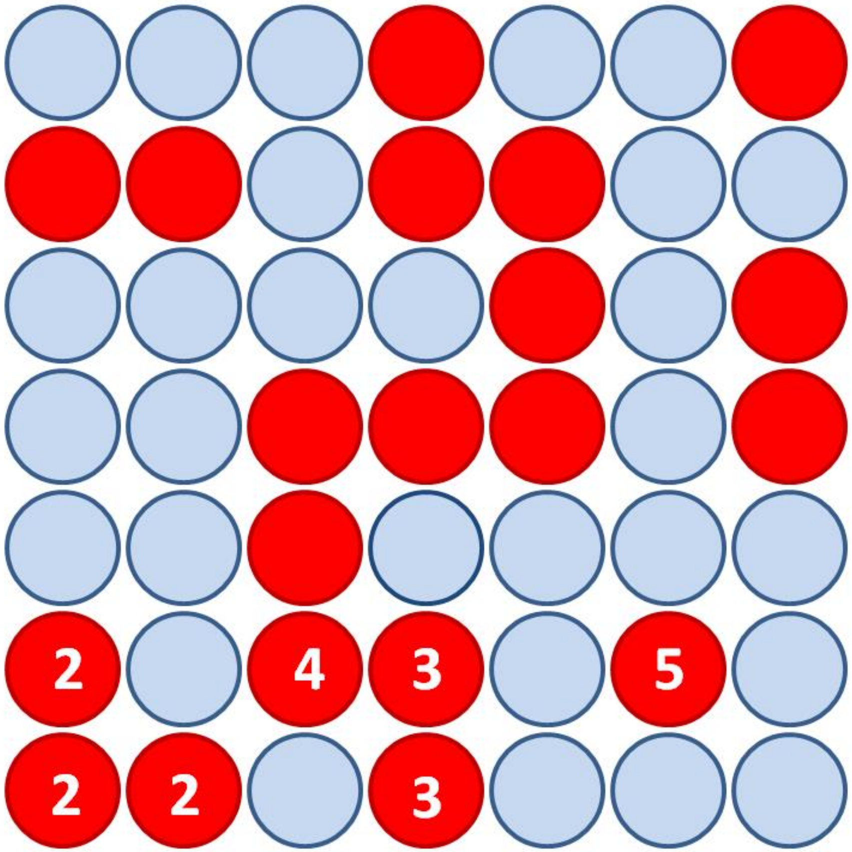

k	M(k)
2	3
3	3
4	-3
5	1

J. Hoshen and R. Kopelman. *Phys. Rev. B* **14**, 3438 (1976)

Algorithms
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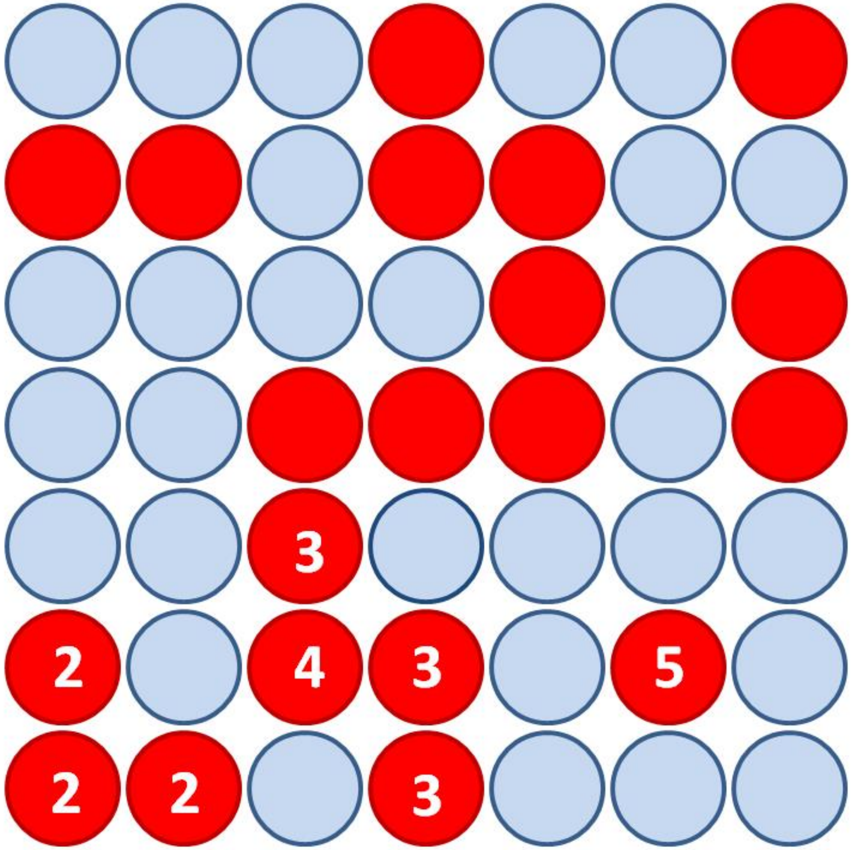

k	M(k)
2	3
3	4
4	-3
5	1

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Algorithms

Hoshen and Kopelman

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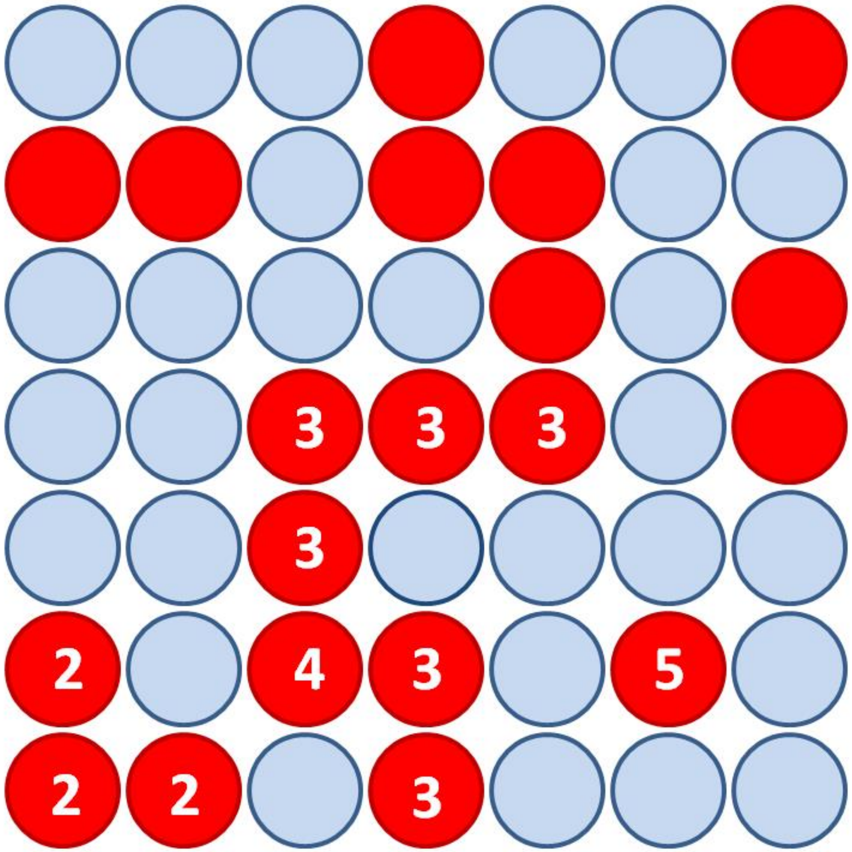

k	M(k)
2	3
3	7
4	-3
5	1

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Algorithms
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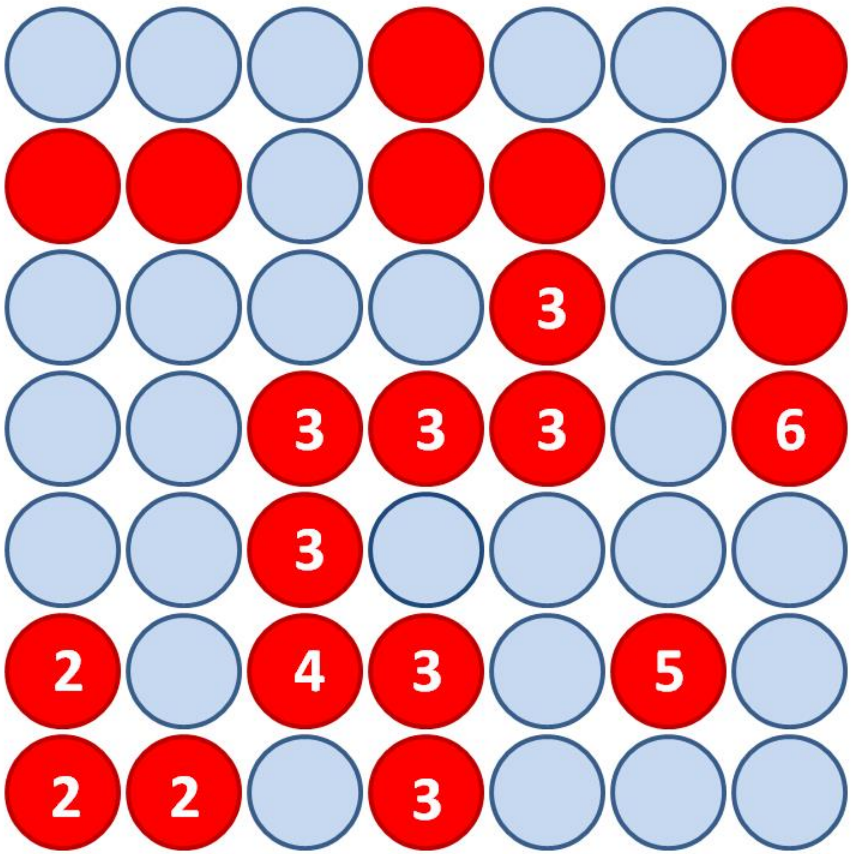

k	M(k)
2	3
3	8
4	-3
5	1
6	1

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Algorithms

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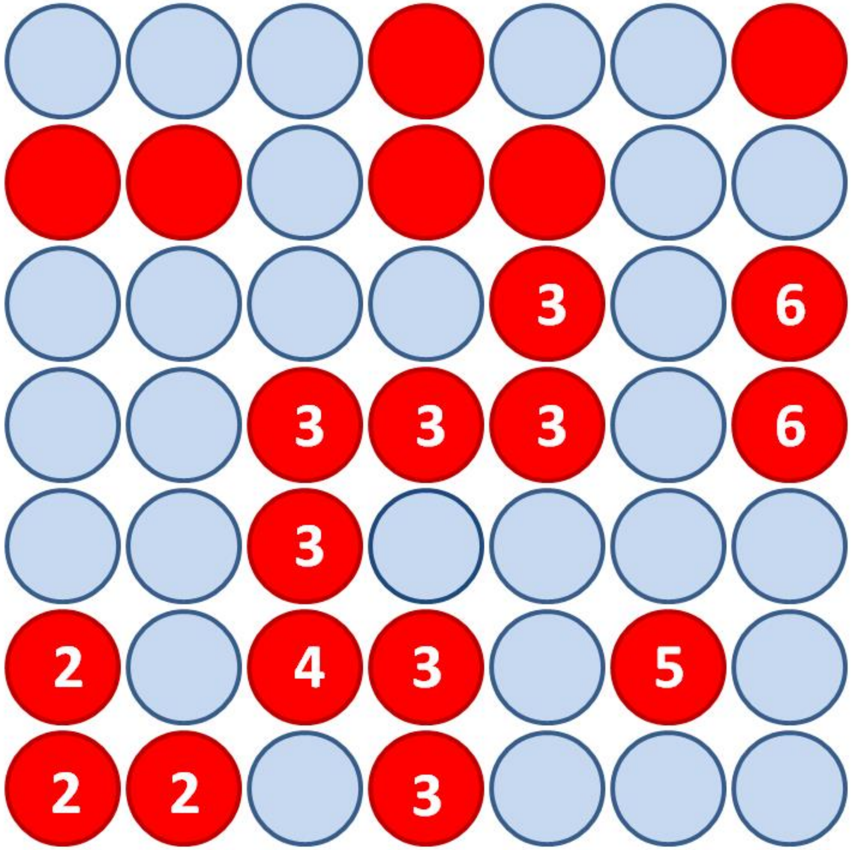

k	M(k)
2	3
3	8
4	-3
5	1
6	2

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Algorithms

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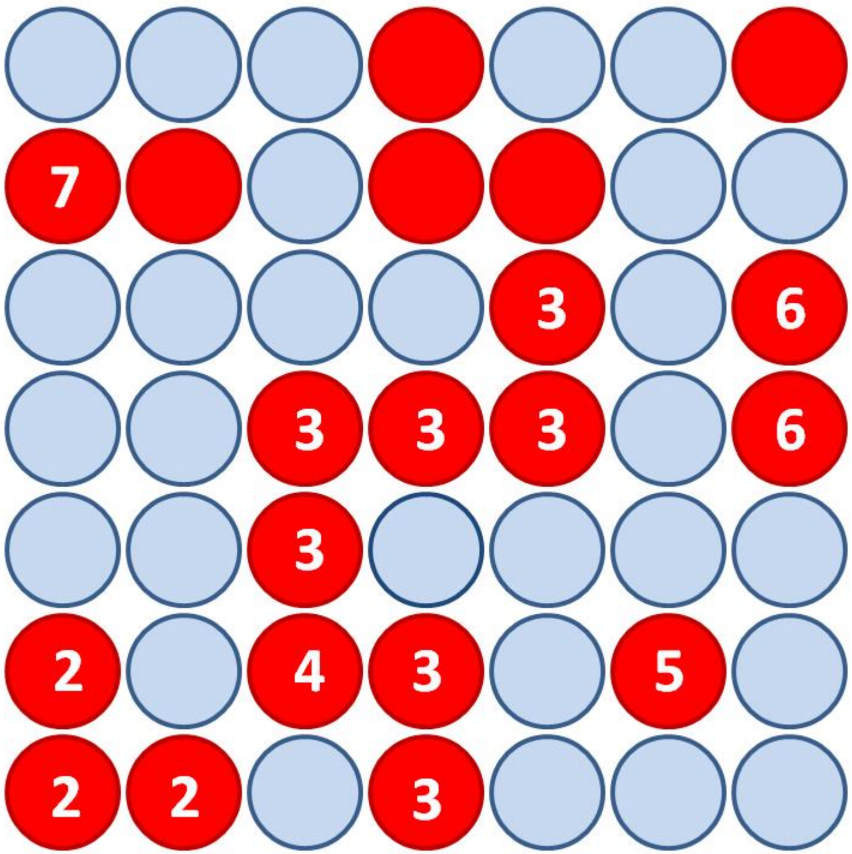

k	M(k)
2	3
3	8
4	-3
5	1
6	2
7	1

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Algorithms
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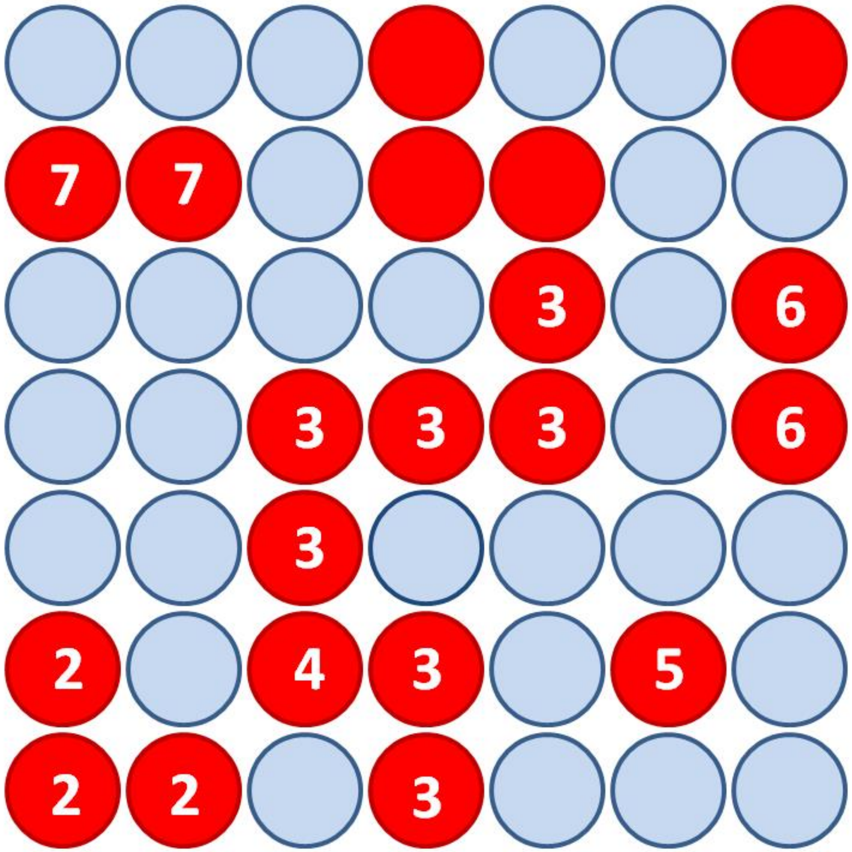

k	M(k)
2	3
3	8
4	-3
5	1
6	2
7	2

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Algorithms

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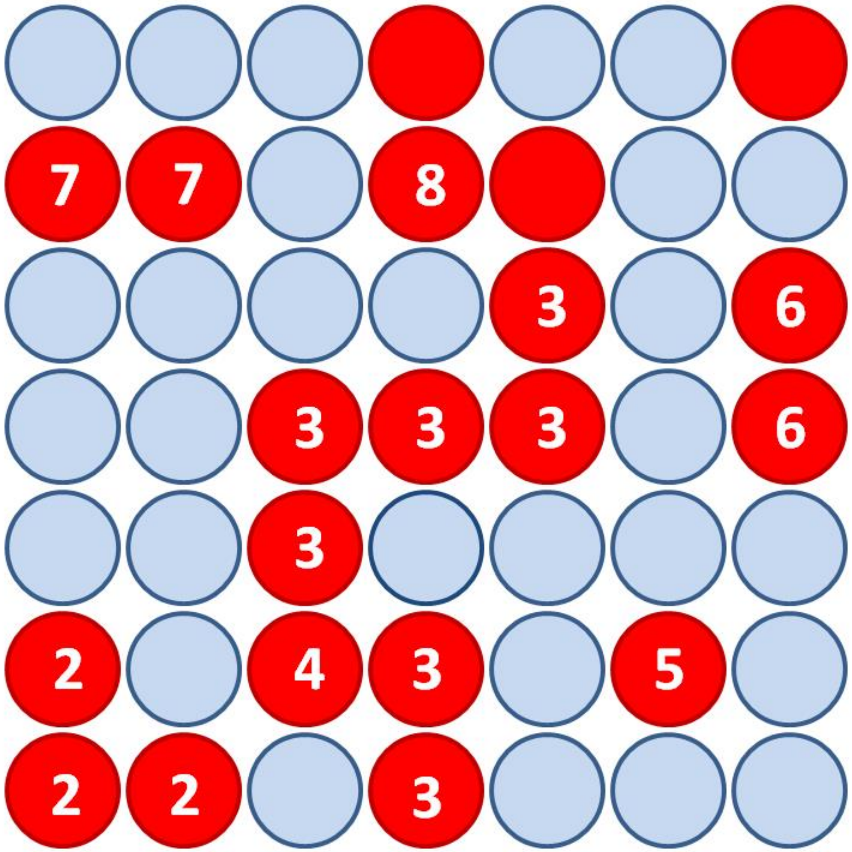

k	M(k)
2	3
3	8
4	-3
5	1
6	2
7	2
8	1

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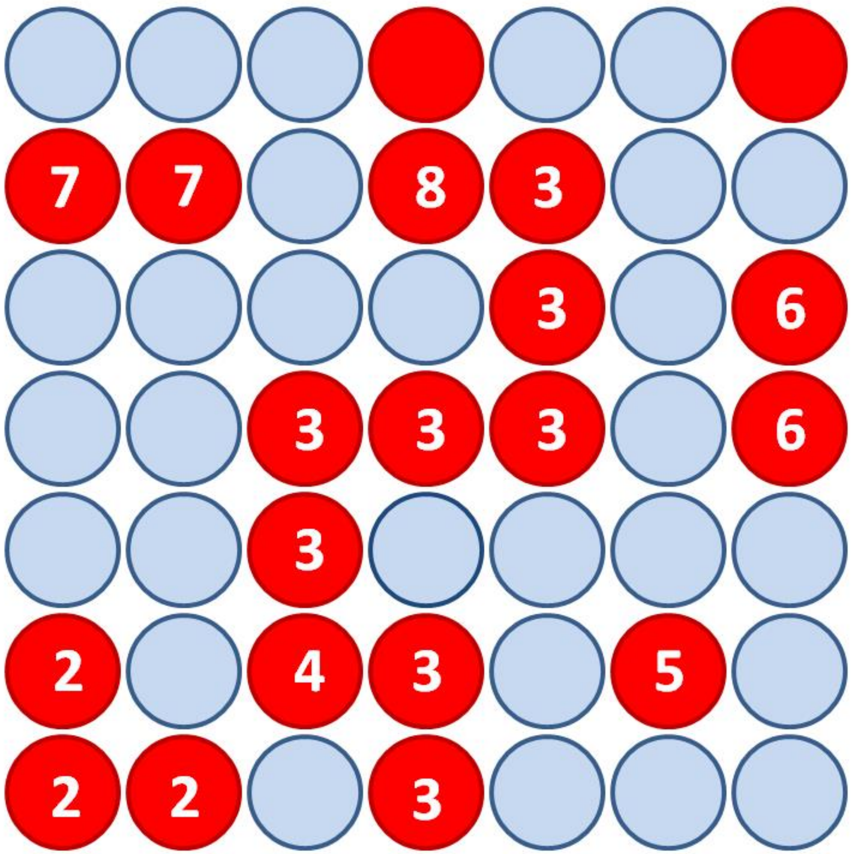

k	M(k)
2	3
3	10
4	-3
5	1
6	2
7	2
8	-3

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Algorithms
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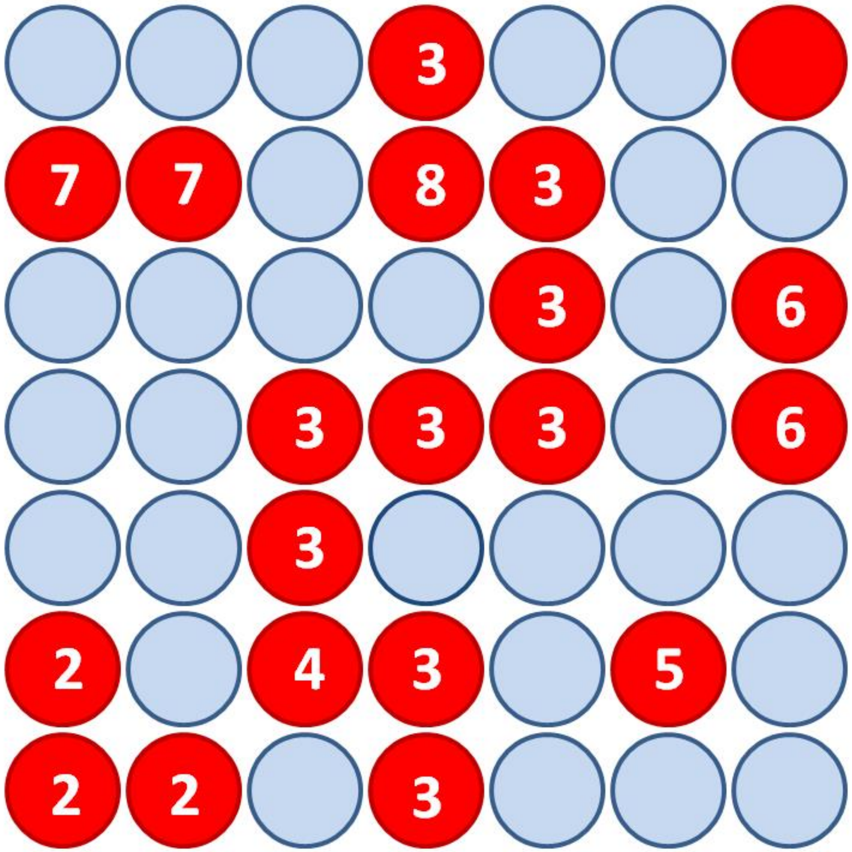

k	M(k)
2	3
3	11
4	-3
5	1
6	2
7	2
8	-3

J. Hoshen and R. Kopelman. *Phys. Rev. B* **14**, 3438 (1976)

Algorithms

Hoshen and Kopelman

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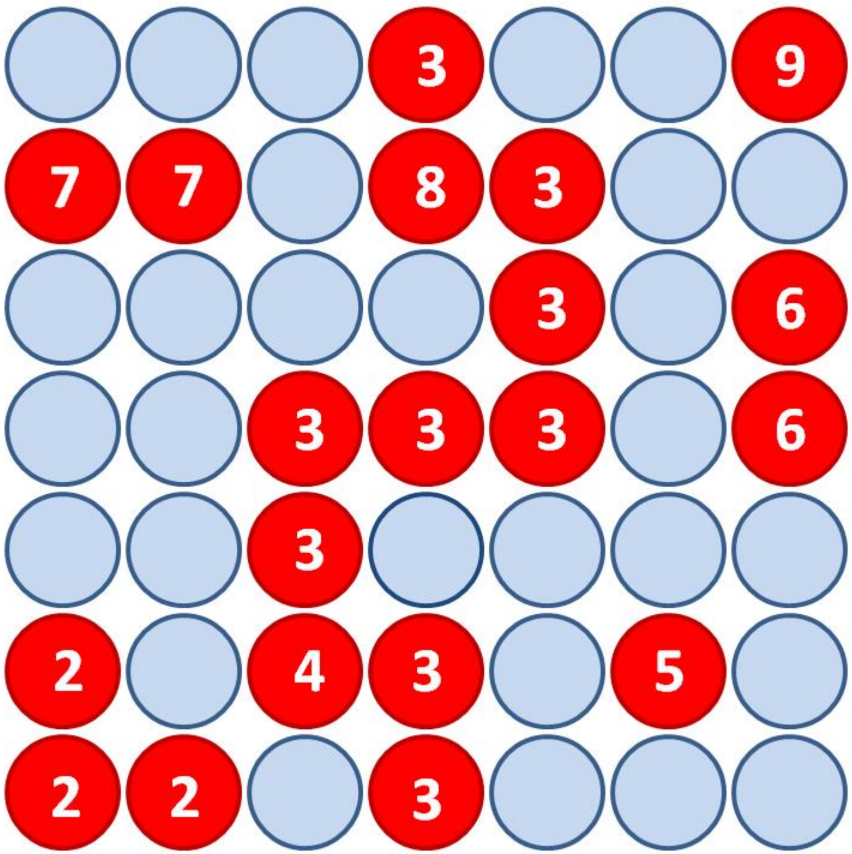
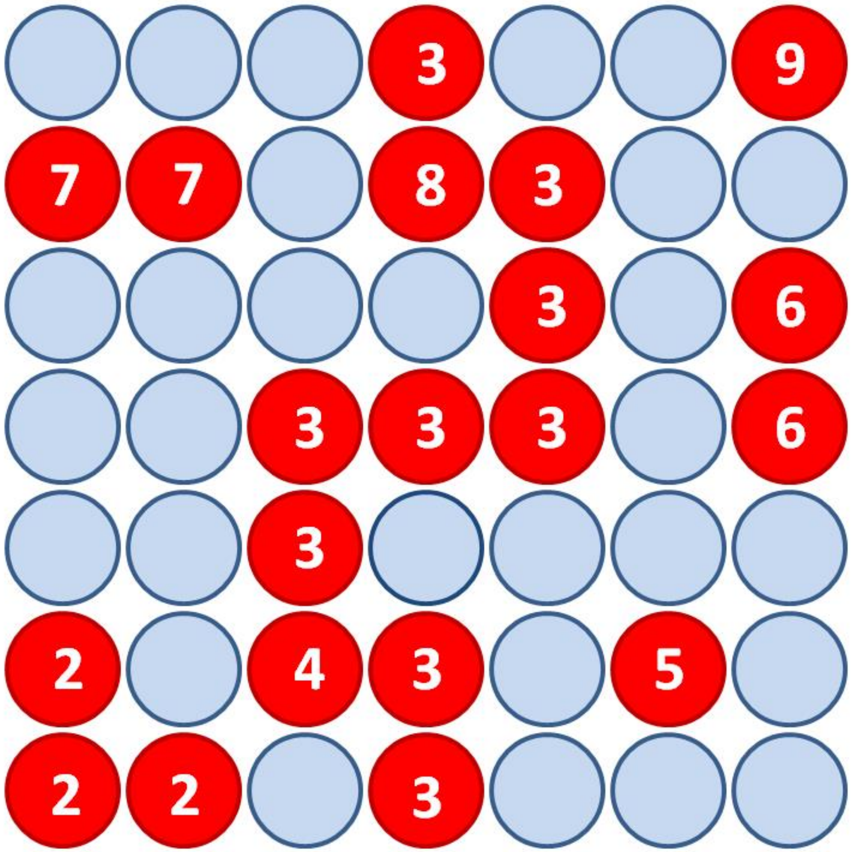


k	M(k)
2	3
3	11
4	-3
5	1
6	2
7	2
8	-3
9	1

J. Hoshen an ... 8 (1976)

Algorithms
Hoshen and Kopelman



k	M(k)
2	3
3	11
4	-3
5	1
6	2
7	2
8	-3
9	1

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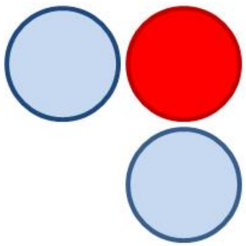
Algorithms

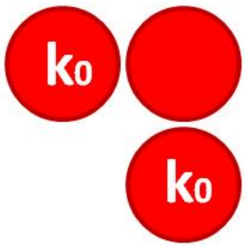
Hoshen and Kopelman



Isolated

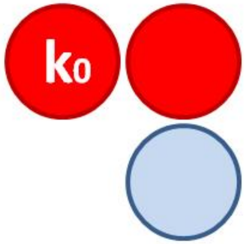
$k = k + 1$

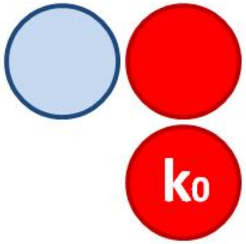
$M(k) = 1$



Two neighbor k_0:

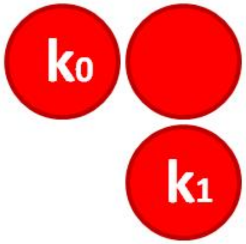
$M(\underline{k_0}) = M(\underline{k_0}) + 1$





One neighbor k_0:

$M(\underline{k_0}) = M(\underline{k_0}) + 1$

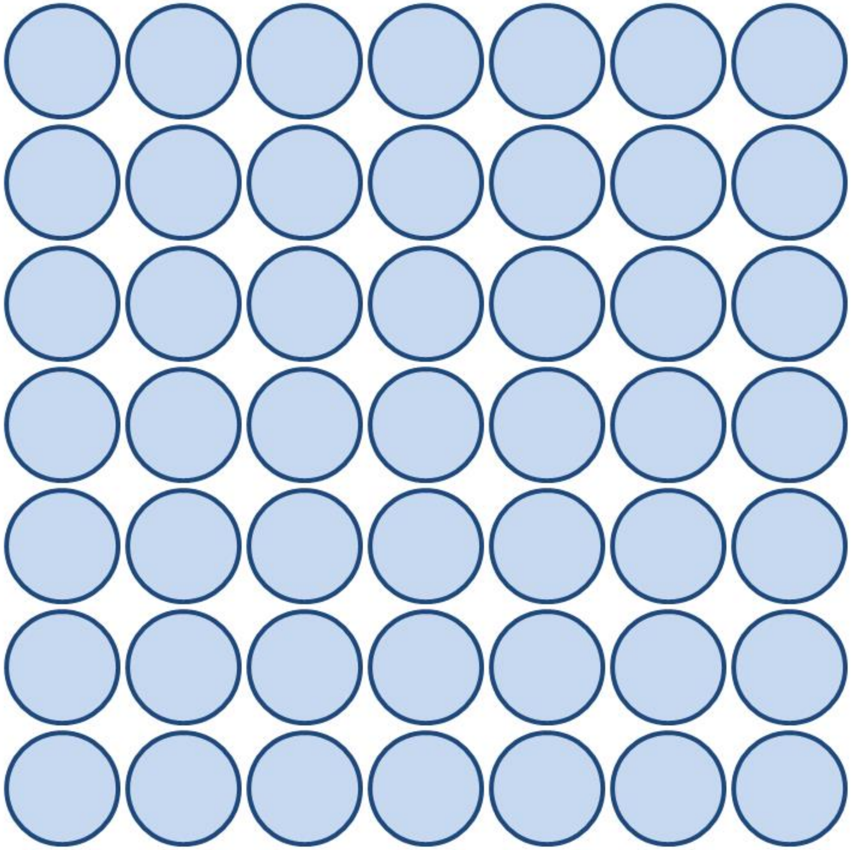


One neighbor k_0 and one neighbor k_1 :

$M(\underline{k_0}) = M(\underline{k_0}) + M(\underline{k_1}) + 1$

J. Hoshen and R. Kopelman. *Phys. Rev. B* **14**, 3438 (1976)

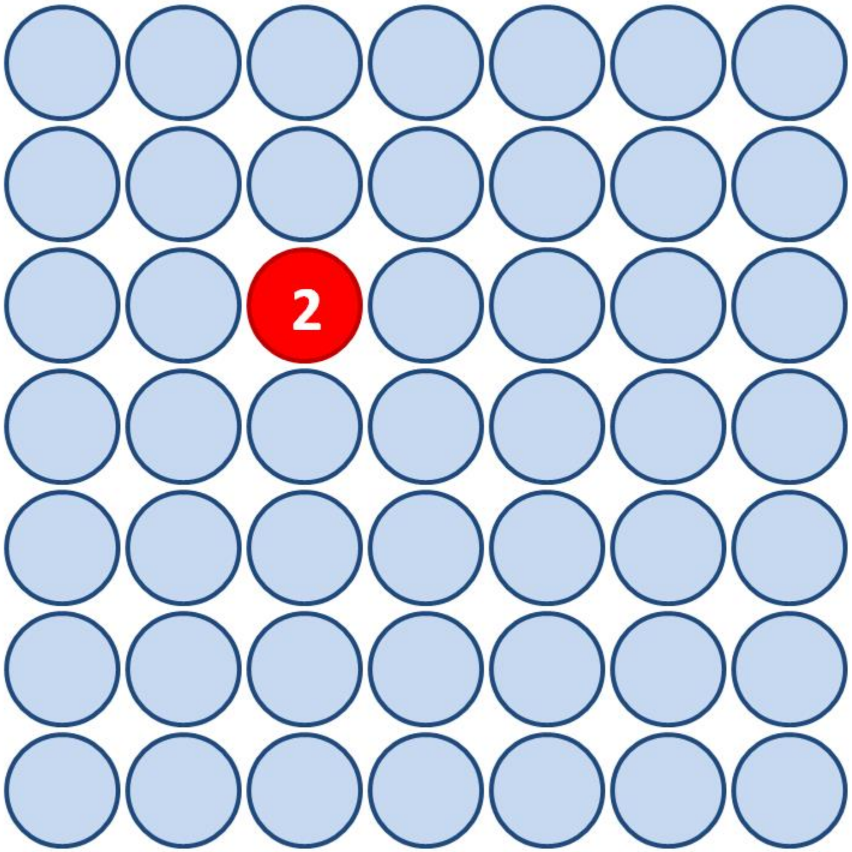
Algorithms
Newman and Ziff (microcanonical)



k	M(k)
2	0

M. E. J. Newman and R. M. Ziff. *Phys. Rev. Lett.* **85**, 4104 (2000)
M. E. J. Newman and R. M. Ziff. *Phys. Rev. E* **64**, 016706 (2001)

Algorithms
Newman and Ziff (microcanonical)

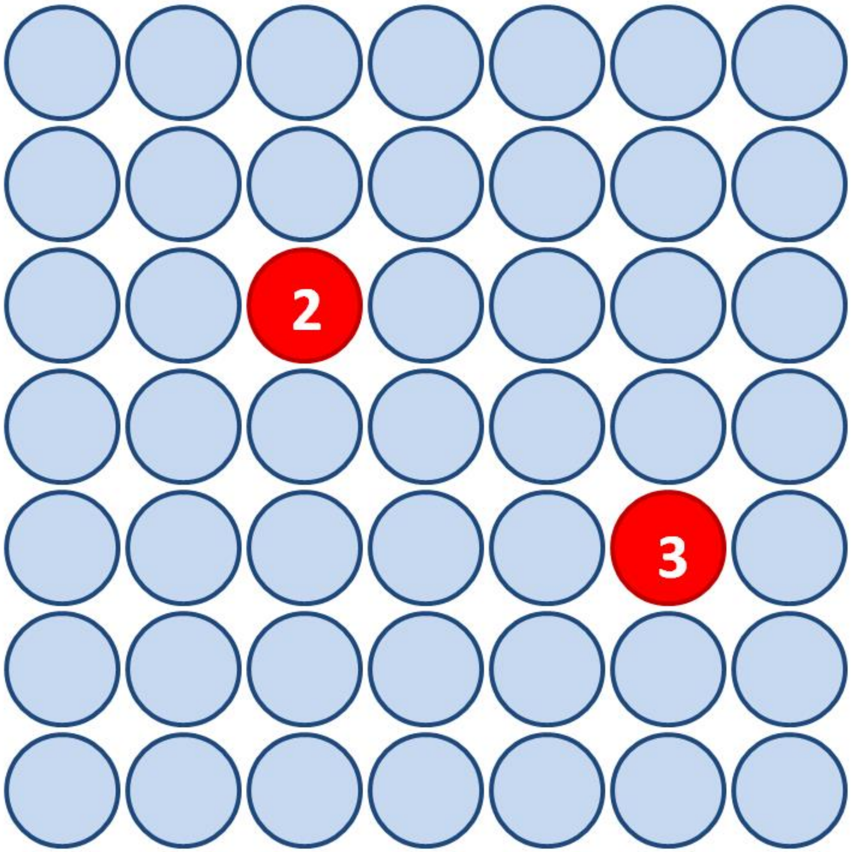


k	M(k)
2	1

M. E. J. Newman and R. M. Ziff. *Phys. Rev. Lett.* **85**, 4104 (2000)
M. E. J. Newman and R. M. Ziff. *Phys. Rev. E* **64**, 016706 (2001)

Algorithms

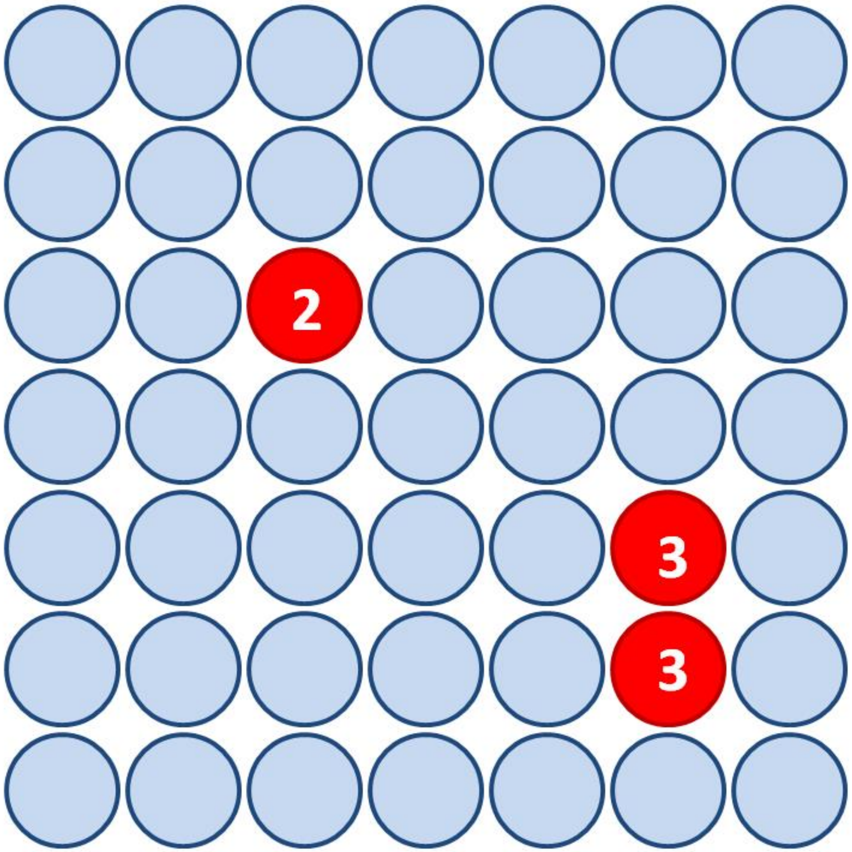
Newman and Ziff (microcanonical)



k	M(k)
2	1
3	1

M. E. J. Newman and R. M. Ziff. *Phys. Rev. Lett.* **85**, 4104 (2000)
M. E. J. Newman and R. M. Ziff. *Phys. Rev. E* **64**, 016706 (2001)

Algorithms
Newman and Ziff (microcanonical)

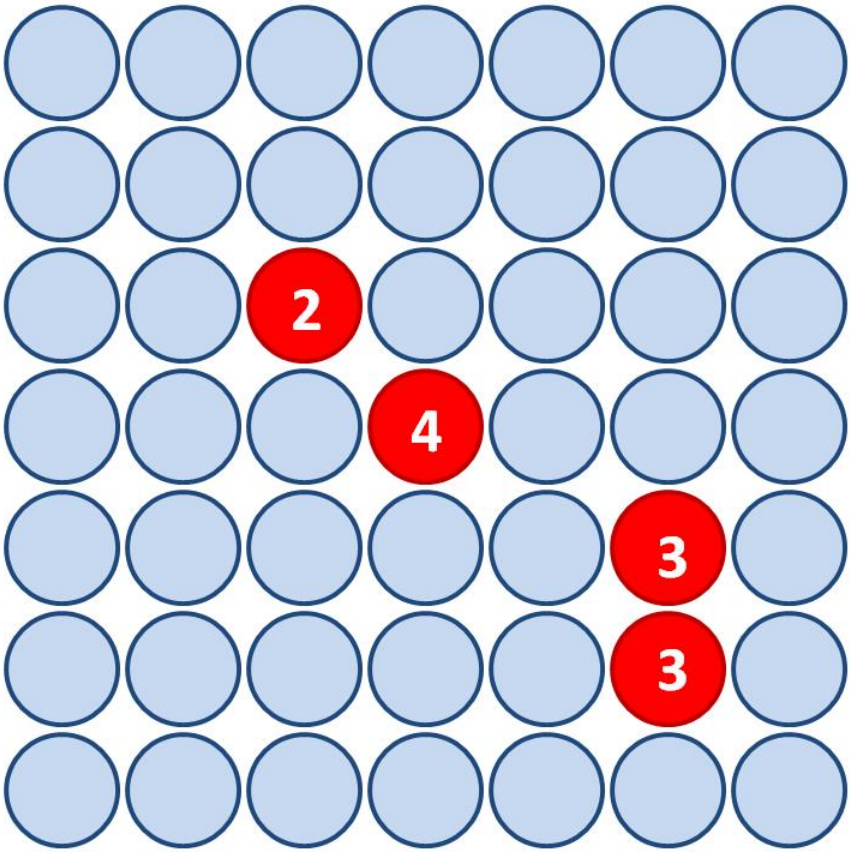


k	M(k)
2	1
3	2

M. E. J. Newman and R. M. Ziff. *Phys. Rev. Lett.* **85**, 4104 (2000)
M. E. J. Newman and R. M. Ziff. *Phys. Rev. E* **64**, 016706 (2001)

Algorithms

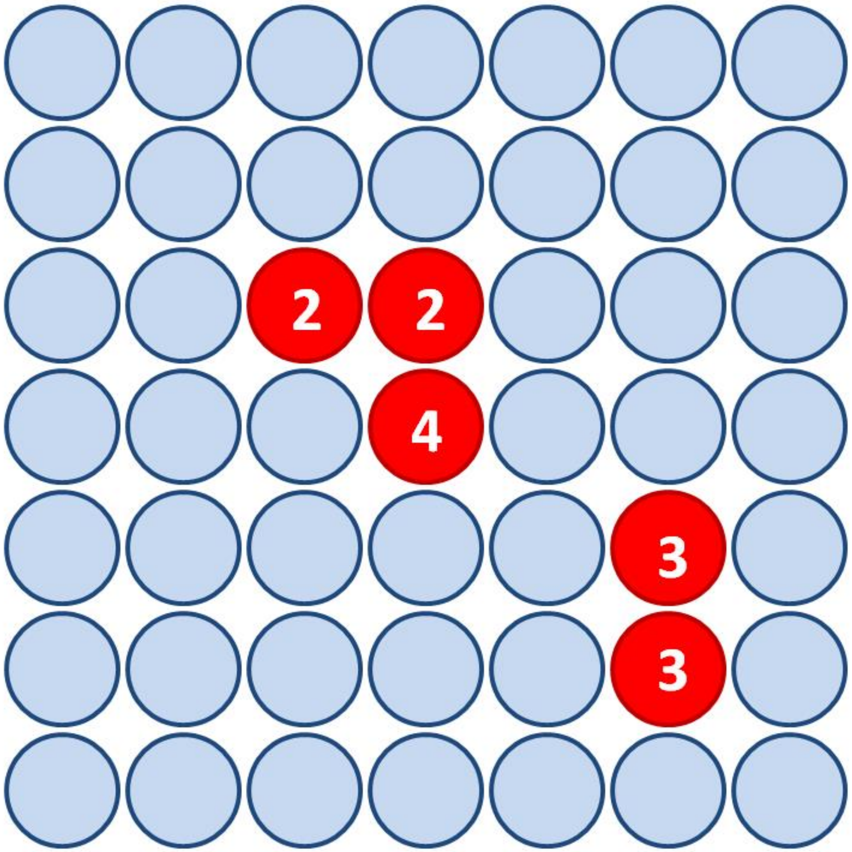
Newman and Ziff (microcanonical)



k	M(k)
2	1
3	2
4	1

M. E. J. Newman and R. M. Ziff. *Phys. Rev. Lett.* **85**, 4104 (2000)
M. E. J. Newman and R. M. Ziff. *Phys. Rev. E* **64**, 016706 (2001)

Algorithms
Newman and Ziff (microcanonical)



k	M(k)
2	3
3	2
4	-2

M. E. J. Newman and R. M. Ziff. *Phys. Rev. Lett.* **85**, 4104 (2000)
M. E. J. Newman and R. M. Ziff. *Phys. Rev. E* **64**, 016706 (2001)

Algorithms
Microcanonical vs canonical

Fixed number of occupied sites (n)

Fixed probability that a site is occupied (p)

$$B(N,n,p) = \binom{N}{n} p^n (1-p)^{N-n}$$

$B(N,n,p)$: probability that exactly n sites are occupied in a *canonical* configuration

$$Q(p) = \sum_{n=0}^{N} B(N,n,p) Q_n = \sum_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} Q_n$$

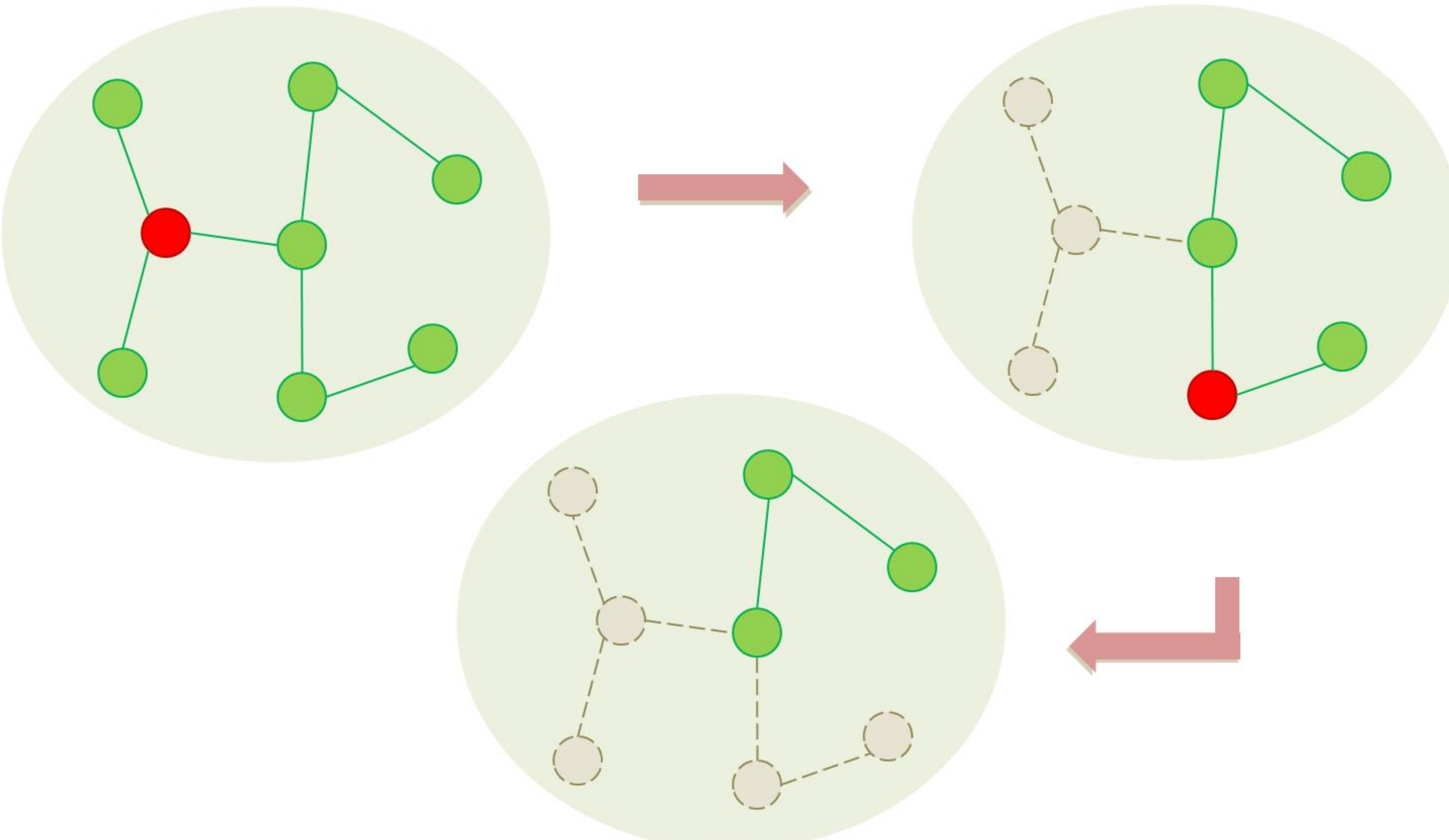
M. E. J. Newman and R. M. Ziff. *Phys. Rev. Lett.* **85**, 4104 (2000)

M. E. J. Newman and R. M. Ziff. *Phys. Rev. E* **64**, 016706 (2001)

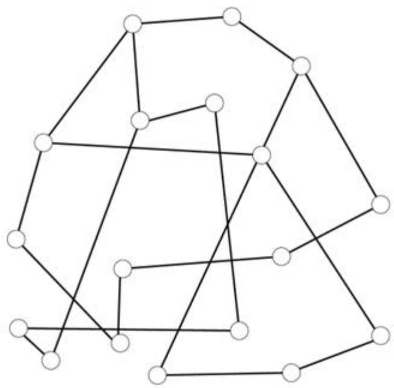
What is going on...

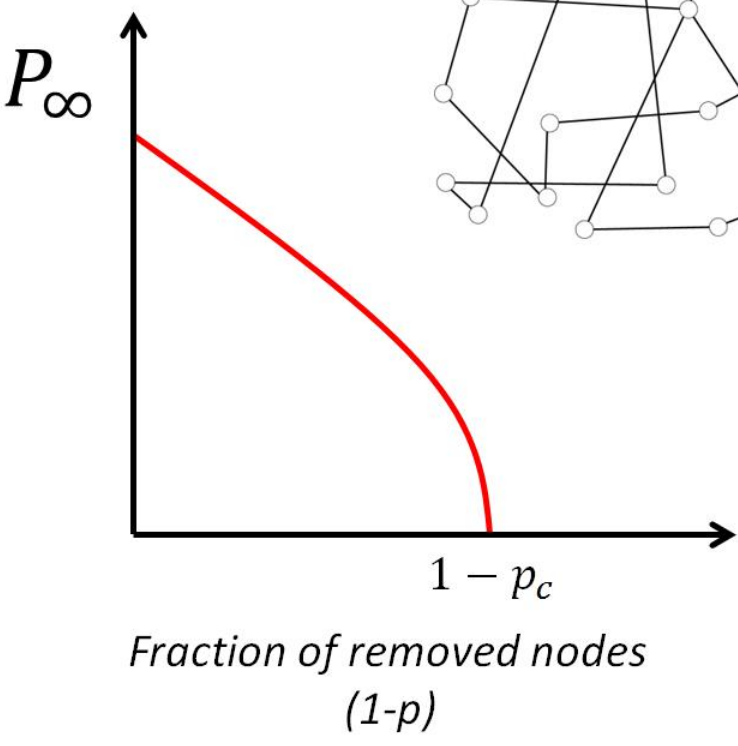
N. A. M. Araújo, P. Grassberger, B. Kahng, K. J. Schrenk, R. M. Ziff.
Recent Advances and Open Challenges in Percolation. arXiv:1404.5325.

The quest of global connectivity



Random Graph



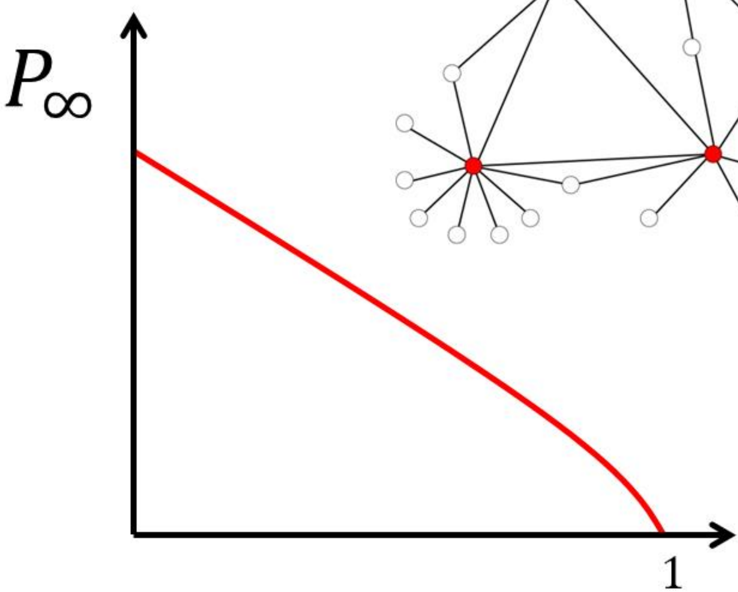


P_∞: Fraction of sites in the giant cluster

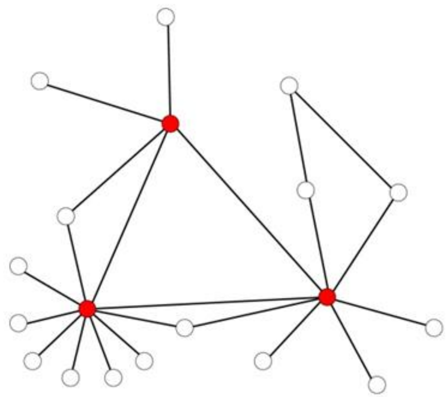
$$P_\infty \sim (p - p_c)^\beta$$

$$\boxed{p_c = \frac{1}{< k >}}$$

Scale-free Graph



Fraction of removed nodes
(1-p)



P_∞: Fraction of sites in the giant cluster

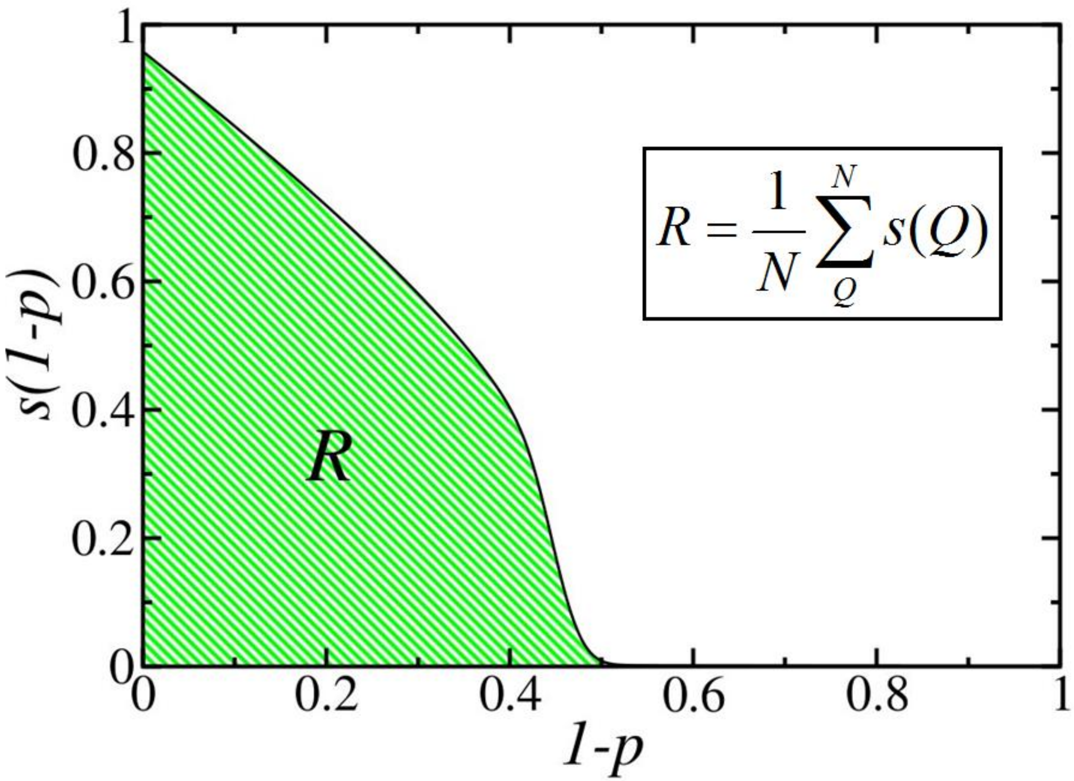
$$P_\infty \sim (p - p_c)^\beta$$

$$p_c = 0$$

Bottom line: Resilient to random attacks but vulnerable to targetted ones.

Robustness to failures or attacks

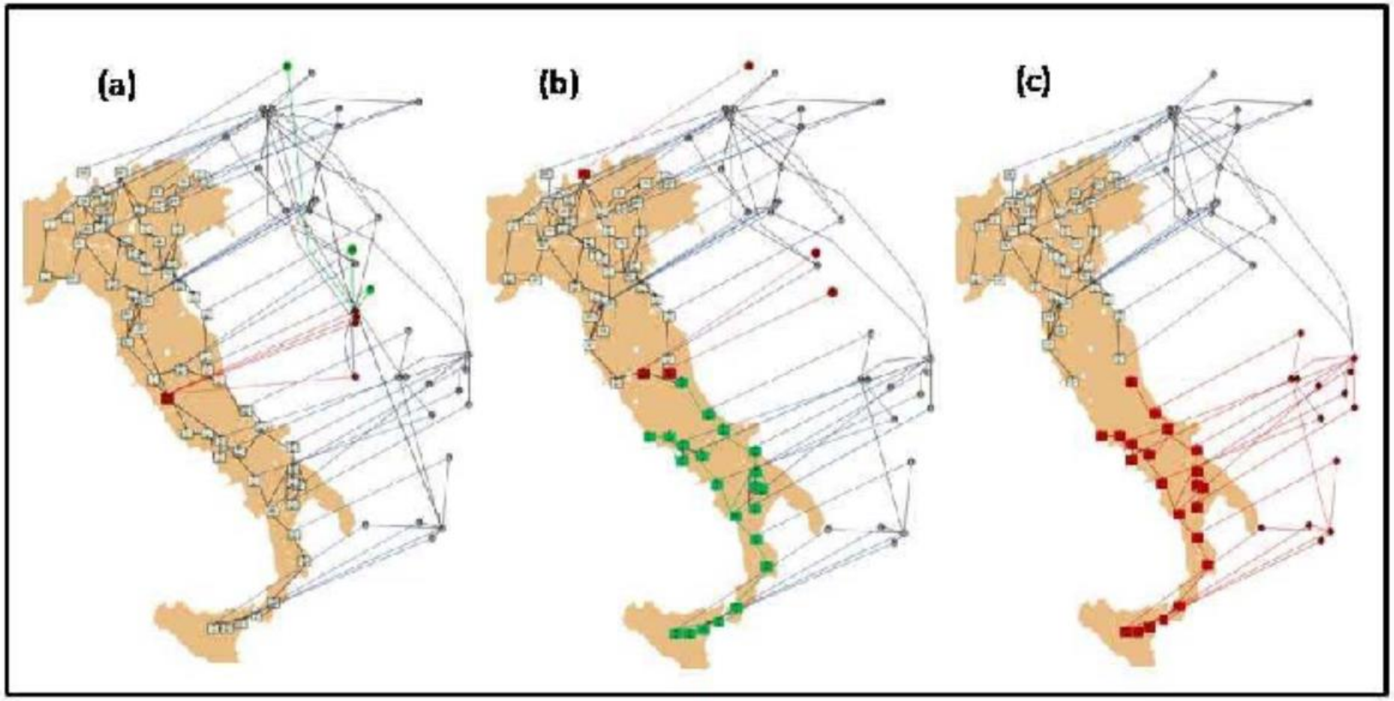
Largest cluster



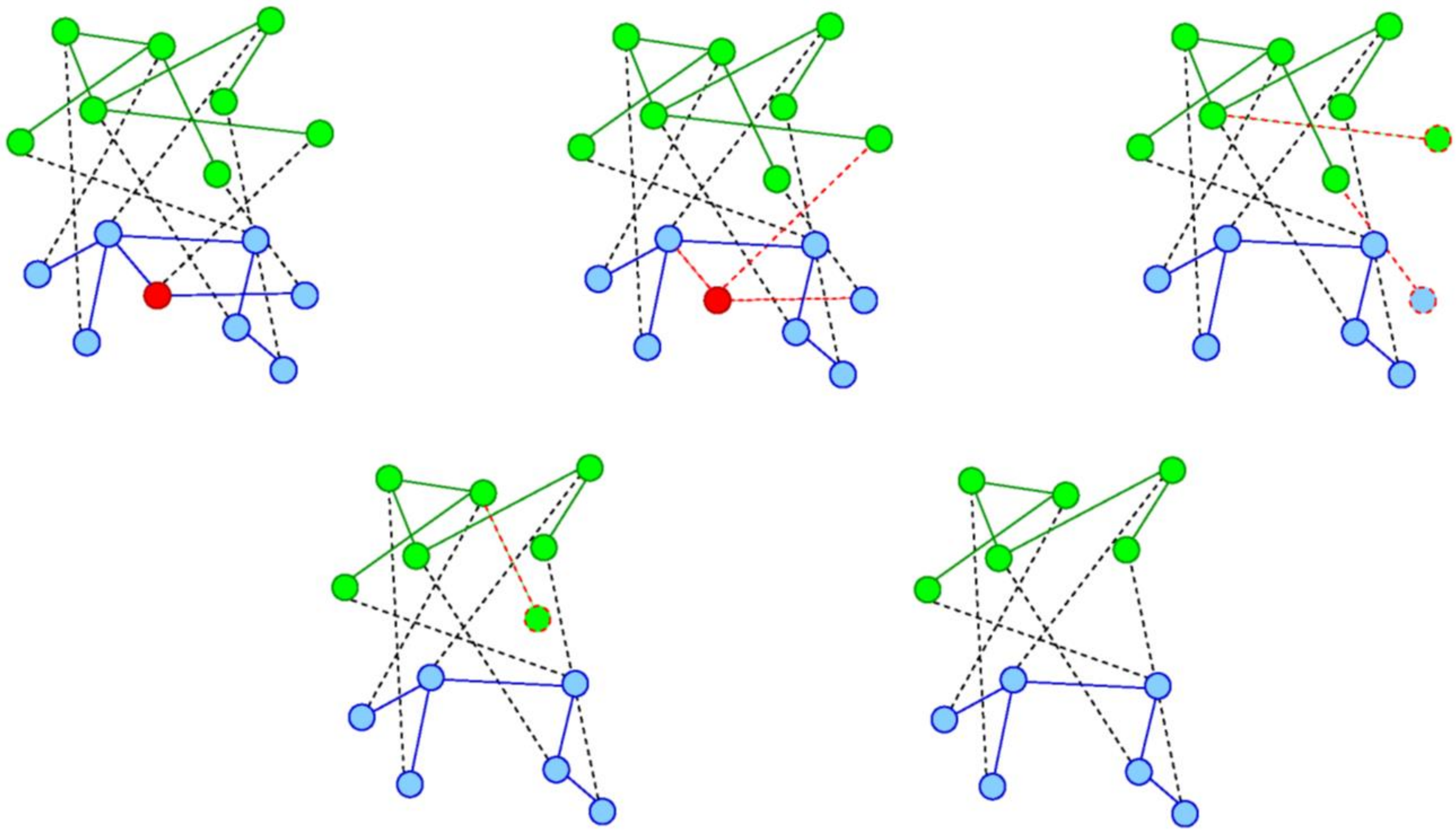
Fraction of attacked nodes

C. M. Schneider, A. A. Moreira, J. S. Andrade Jr., S. Havlin, and H. J. Herrmann. *PNAS* **108**, 3838 (2011)

2003 blackout in Italy and Switzerland

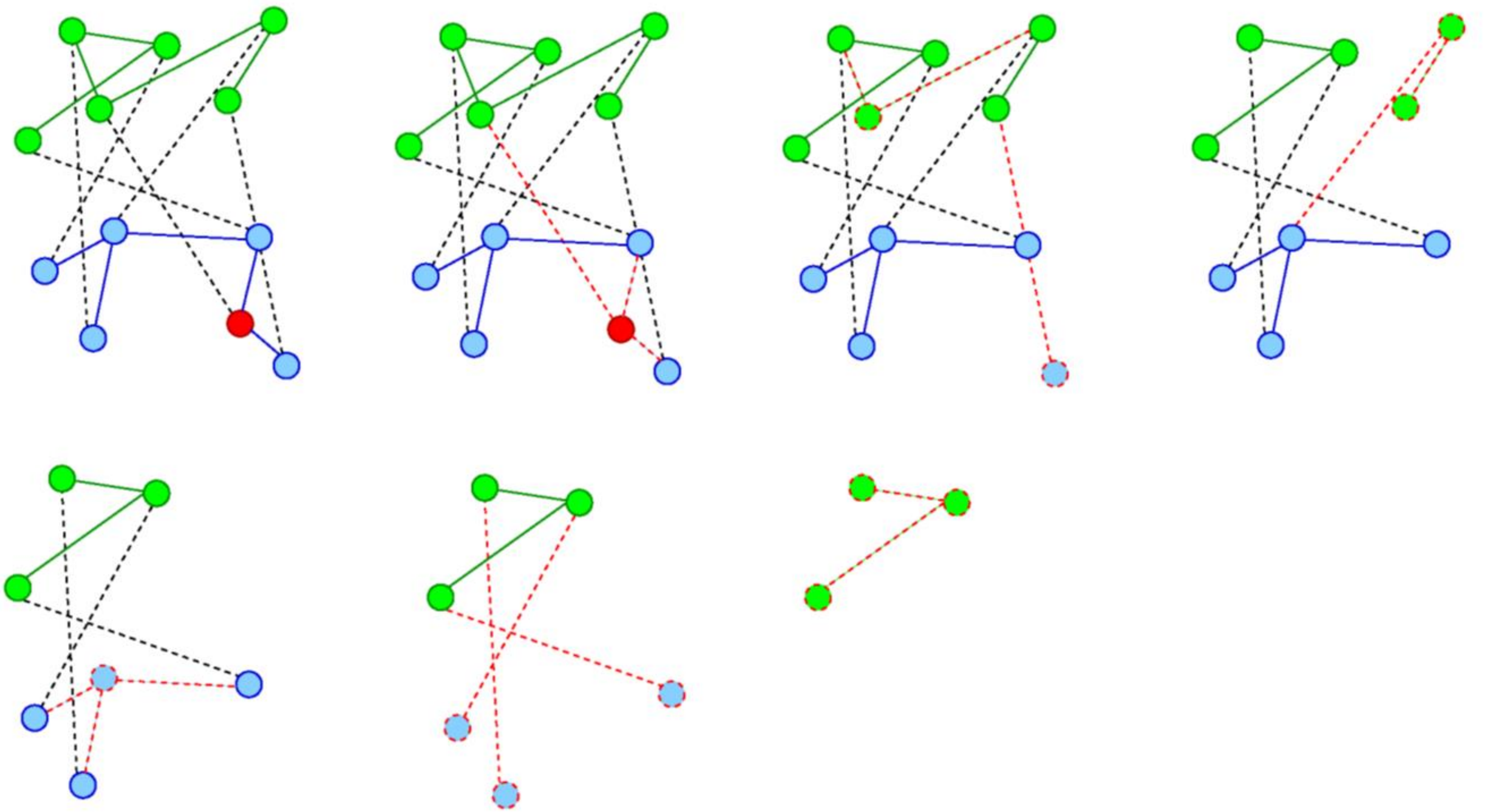


collapse of coupled networks



S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, S. Havlin. *Nature* **464**, 1025 (2010)

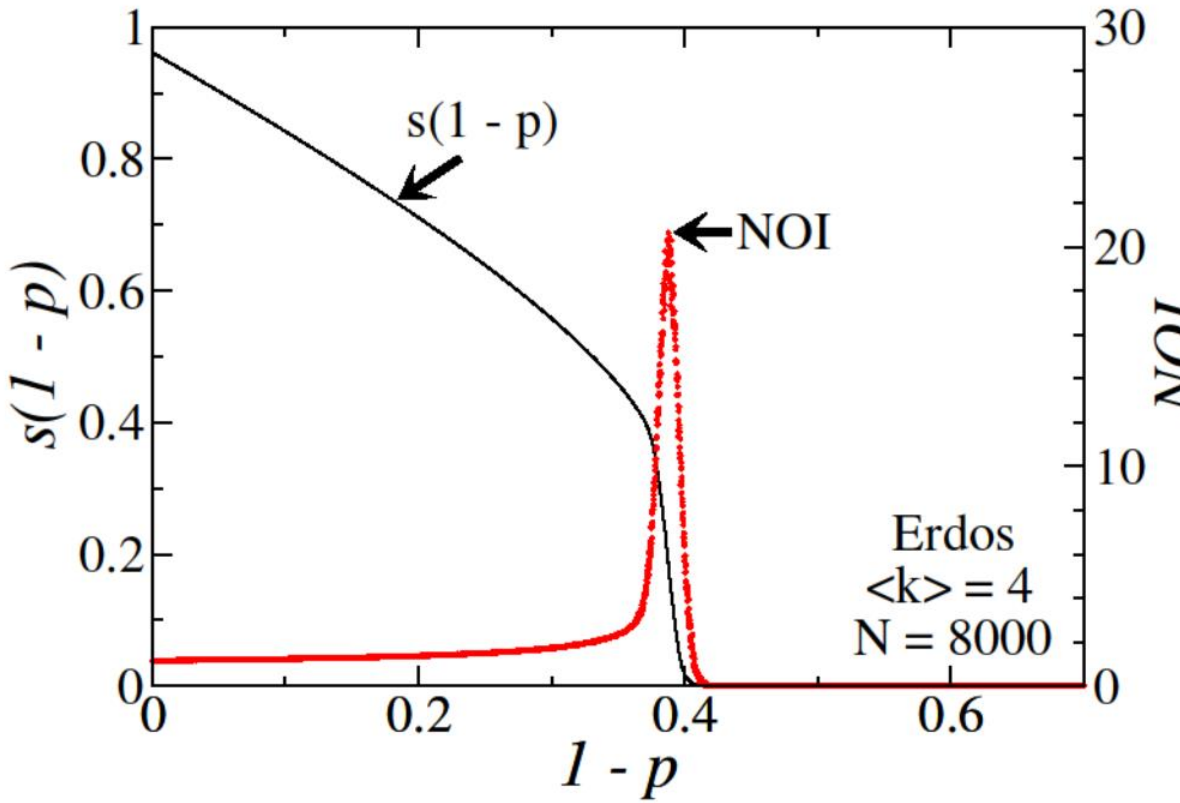
collapse of *coupled* networks



S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, S. Havlin. *Nature* **464**, 1025 (2010)

collapse of coupled networks

Largest cluster

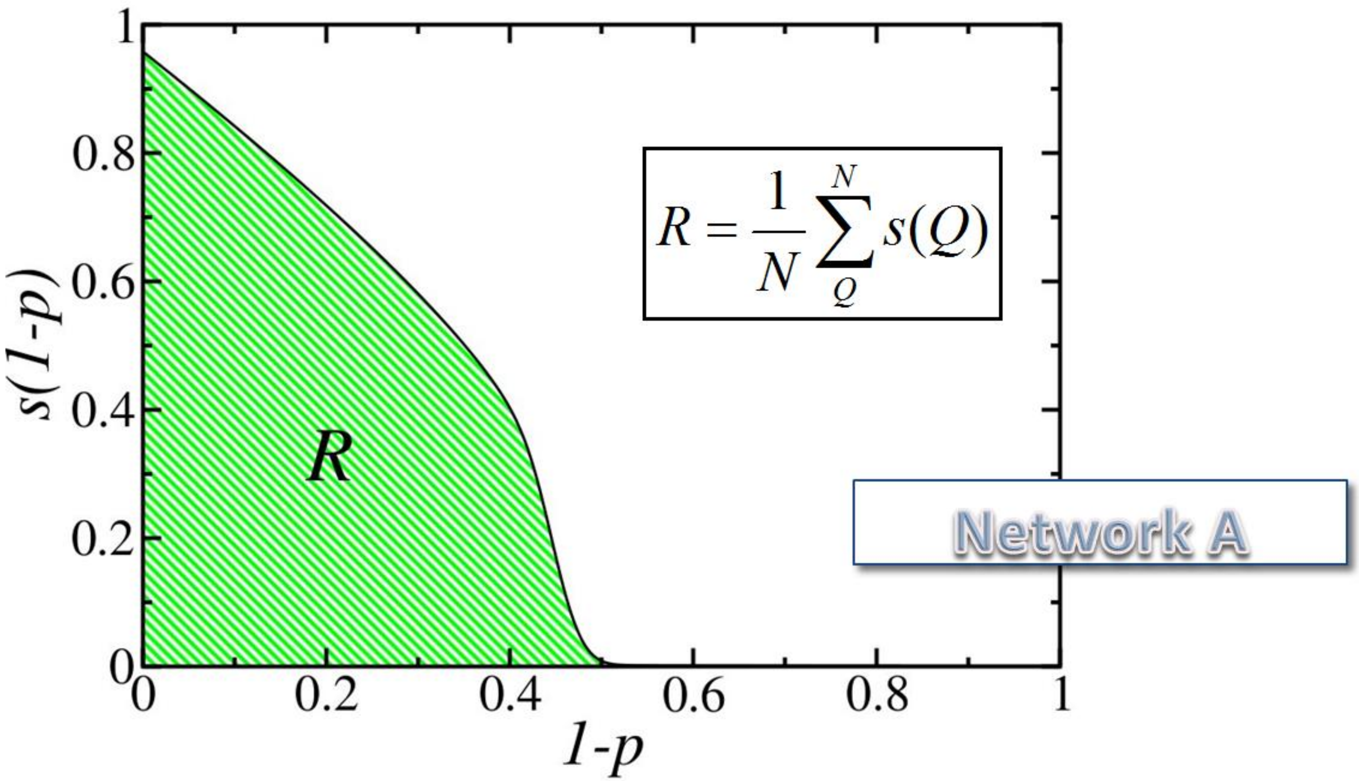


Number of iterations

Fraction of attacked nodes

robustness of coupled networks

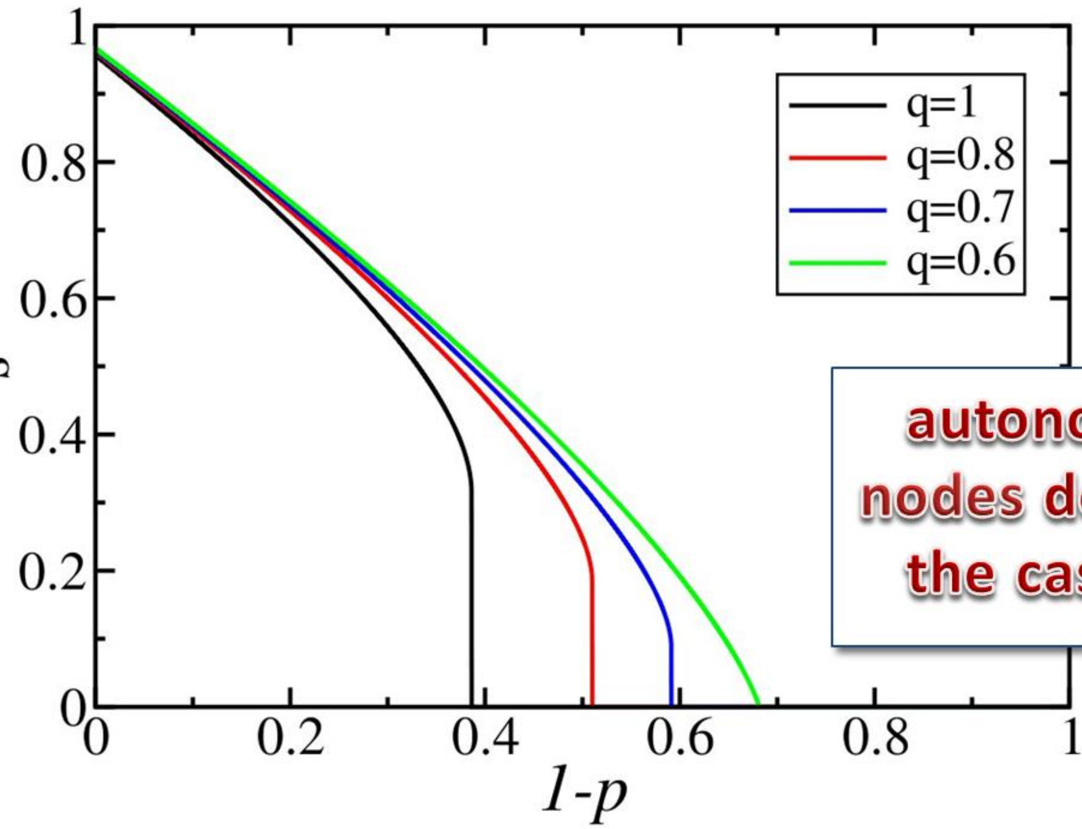
Largest cluster



Fraction of attacked nodes

C. M. Schneider, A. A. Moreira, J. S. Andrade Jr., S. Havlin, and H. J. Herrmann. *PNAS* **108**, 3838 (2011)

Mitigating risk by decoupling networks



Largest cluster

Fraction of attacked nodes

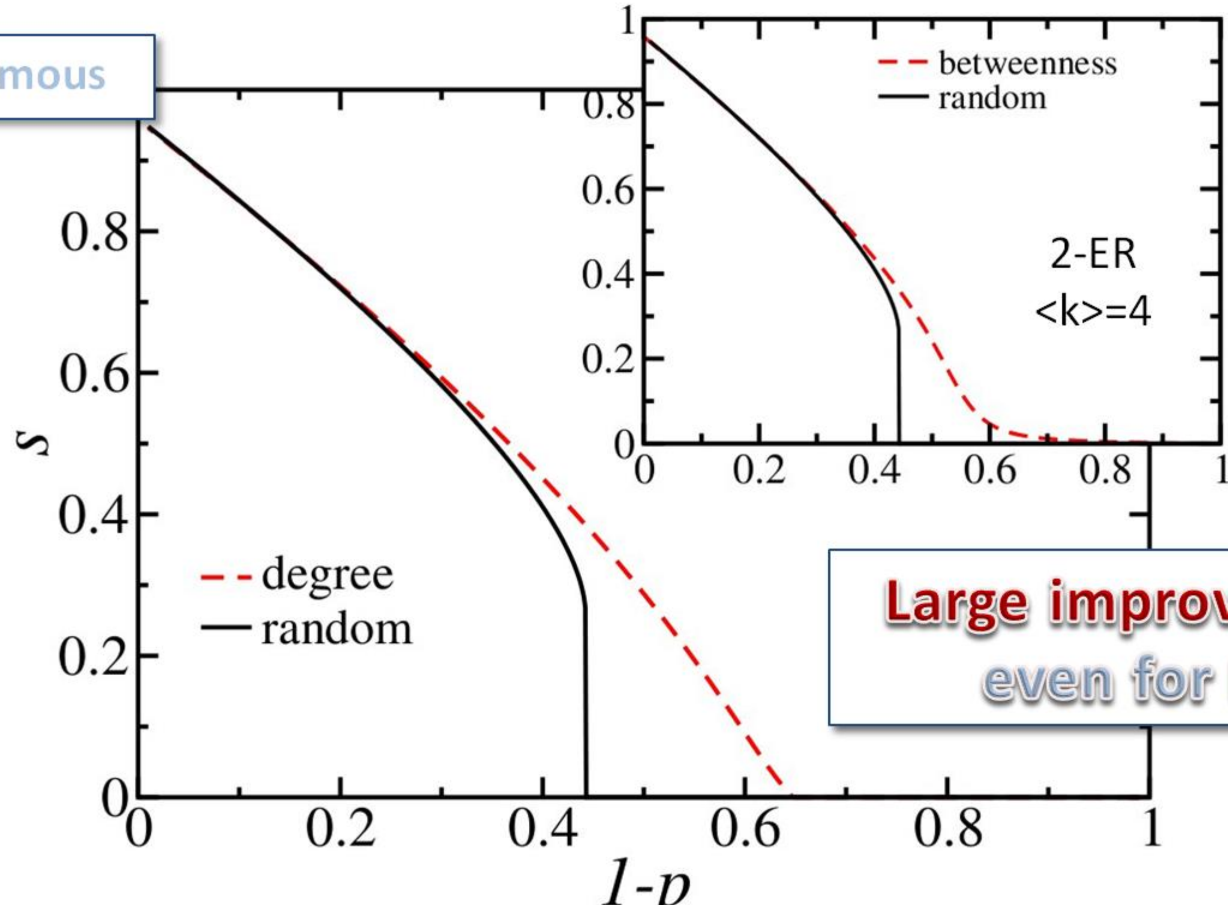
autonomous nodes downsize the cascade!

R. Parshani, S. V. Buldyrev, S. Havlin. *Phys. Rev. Lett.* **105**, 048701 (2010)

How to select autonomous nodes?

10% autonomous

Largest cluster



Large improvement
even for ER!

Fraction of attacked nodes