Some multiplex dynamics that I find interesting

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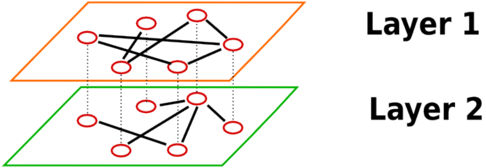
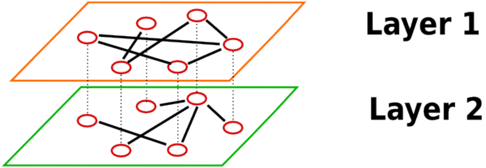
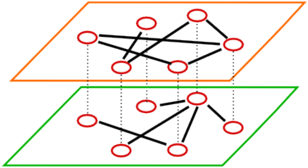
SUPERDIFFUSION


Layer 1
Layer 2



DIFFUSION EQUATION

$$\frac{dx_i^{[\alpha]}}{dt} = D^{[\alpha]} \sum_j a_{ij}^{[\alpha]} (x_j^{[\alpha]} - x_i^{[\alpha]}) + D_x (x_i^{[\beta]} - x_i^{[\alpha]})$$



Layer 1

Layer 2

DIFFUSION EQUATION

$$\frac{dx_i^{[\alpha]}}{dt} = \boxed{D^{[\alpha]} \sum_j a_{ij}^{[\alpha]} (x_j^{[\alpha]} - x_i^{[\alpha]})} + \boxed{D_x (x_i^{[\beta]} - x_i^{[\alpha]})}$$

INTRA-LAYER **INTER-LAYER**

$$\frac{dx_i^{[\alpha]}}{dt} = D^{[\alpha]} \sum_j a_{ij}^{[\alpha]} (x_j^{[\alpha]} - x_i^{[\alpha]}) + D_x (x_i^{[\beta]} - x_i^{[\alpha]})$$

$$\boxed{\dot{\boldsymbol{x}} = -\boldsymbol{\mathcal{L}} \dot{\boldsymbol{x}}}$$

$$\mathcal{L} = \begin{pmatrix} D^{[\alpha]} L^{[\alpha]} + D_x I & -D_x I \\ -D_x I & D^{[\beta]} L^{[\beta]} + D_x I \end{pmatrix}$$

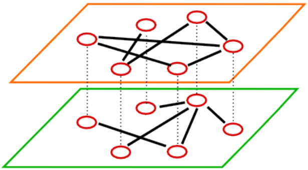
Where $L^{[\alpha]}, L^{[\beta]}$ are the Laplacians of the two layers

$$\dot{x} = -\mathcal{L}\dot{x}$$

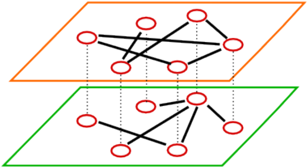
DIFFUSION TIME-SCALE

$$\tau = \frac{1}{\lambda_{\min}}$$

where $\lambda_{\min}$ is the smallest non-zero eigenvalue of $\mathcal{L}$

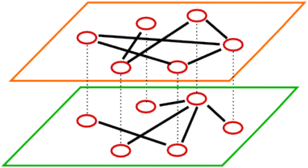


D_x



D_x

$D_x \simeq 0 \qquad \lambda_{\min} = min(\lambda_2^{\alpha}, \lambda_2^{\beta})$

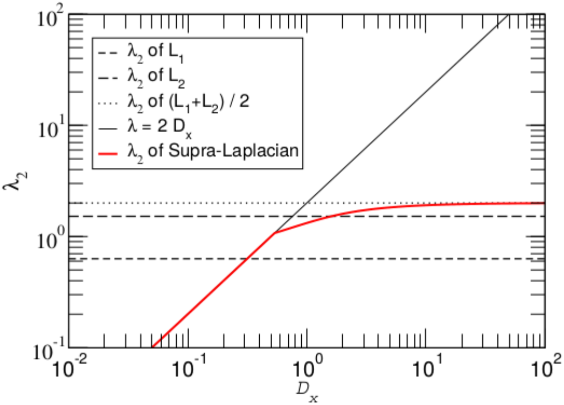


D_x

$$D_x \simeq 0 \qquad \lambda_{\min} = min(\lambda_2^\alpha, \lambda_2^\beta)$$

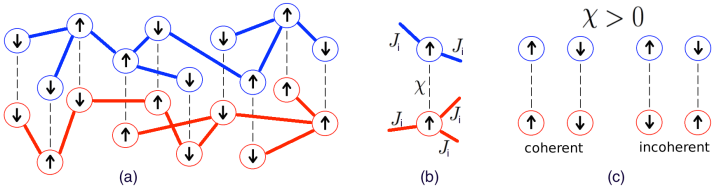
$$D_x \to \infty$$

$$\lambda_{\min} = \frac{\lambda_s}{2} \geq \frac{\lambda_2^\alpha + \lambda_2^\beta}{2} \geq min(\lambda_2^\alpha, \lambda_2^\beta)$$

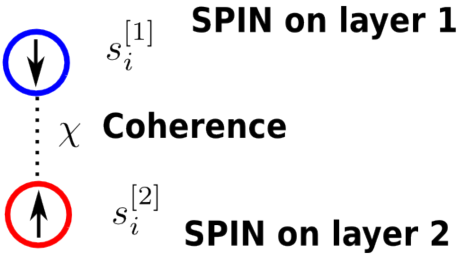


λ2 of L1
λ2 of L2
λ2 of (L1+L2) / 2
λ = 2 Dx
λ2 of Supra-Laplacian
λ2
Dx
10^-2
10^-1
10^0
10^1
10^2

ISING MODEL

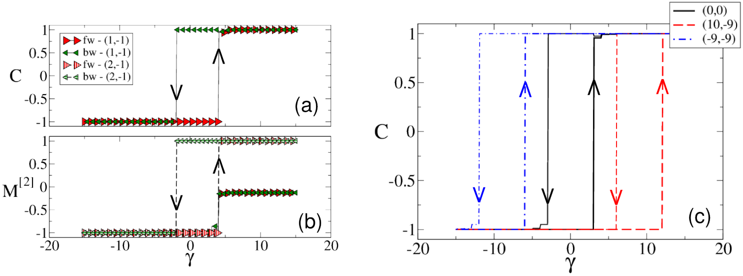


χ > 0
J_i
χ
coherent
incoherent
(a)
(b)
(c)

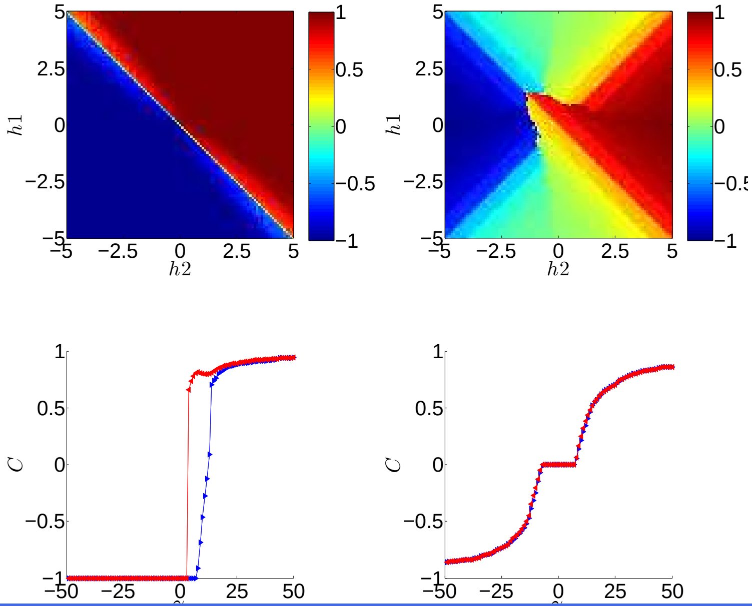


$$F_i^{[\alpha]} = J_i \sum_{j=1}^{N} a_{ij}^{[\alpha]} s_j^{[\alpha]} + \gamma \frac{\chi_i}{J_i} \sum_{\beta \neq \alpha}^{M} s_i^{[\beta]} + h^{[\alpha]}$$

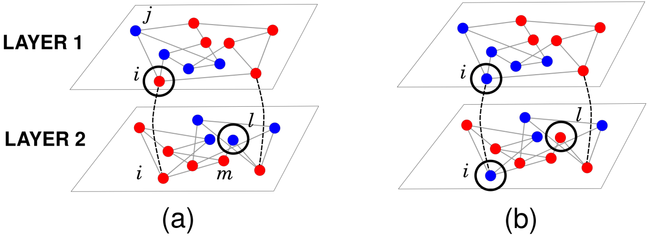
Relative weight of coherecnce

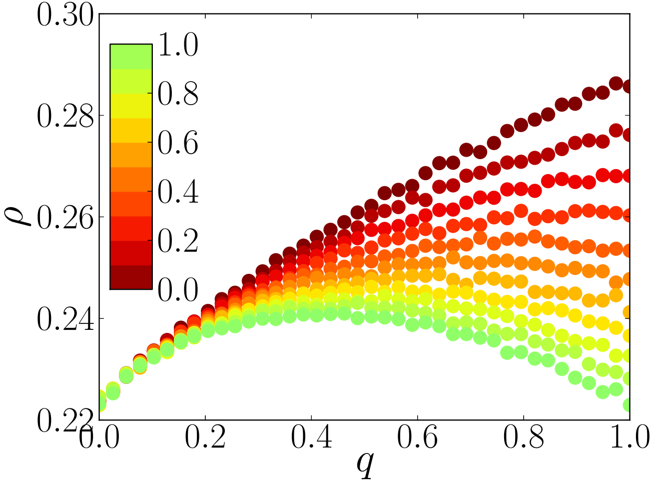


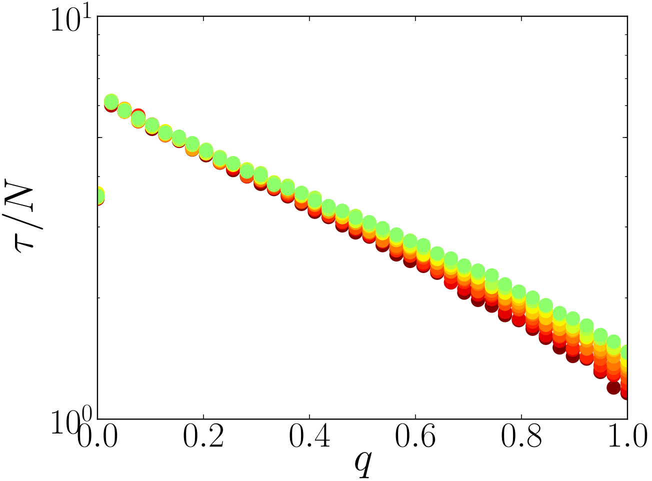
fw - (1,-1)
bw - (1,-1)
fw - (2,-1)
bw - (2,-1)
(a)
(b)
(c)
C
$M^{[2]}$
γ
(0,0)
(10,-9)
(-9,-9)


h1
h2
C
5
2.5
0
−2.5
−5
1
0.5
−0.5
−1
−50
−25
25
50

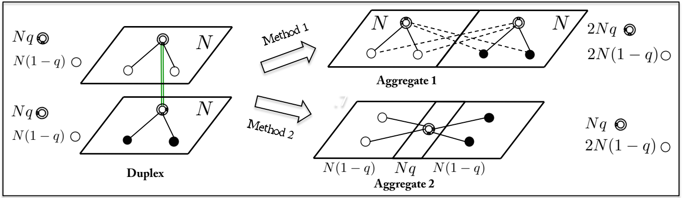
VOTER MODEL


LAYER 1
LAYER 2
j
i
i
l
m
(a)
i
i
l
(b)

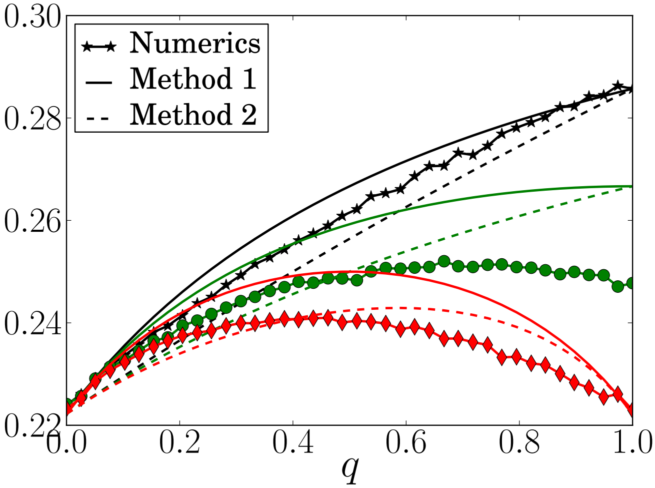

0.30
0.28
0.26
0.24
0.22
ρ
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
q

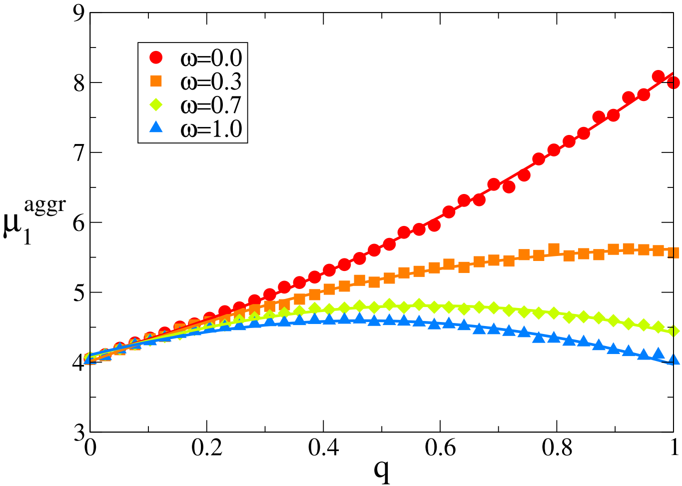


10^1
10^0
τ/N
0.0
0.2
0.4
0.6
0.8
1.0
q

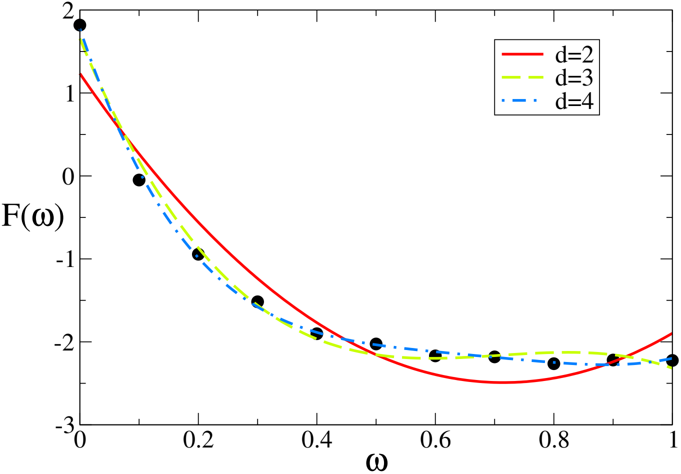


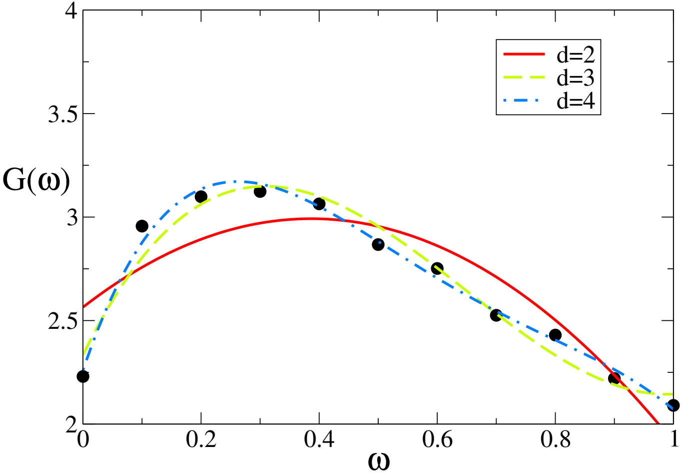
Nq
$N(1-q)$
N
Nq
$N(1-q)$
N
Duplex
Method 1
Method 2
N
N
$2Nq$
$2N(1-q)$
Aggregate 1
Nq
$2N(1-q)$
$N(1-q)$ Nq $N(1-q)$
Aggregate 2


Numerics
Method 1
Method 2
0.30
0.28
0.26
0.24
0.22
0.0
0.2
0.4
0.6
0.8
1.0
q



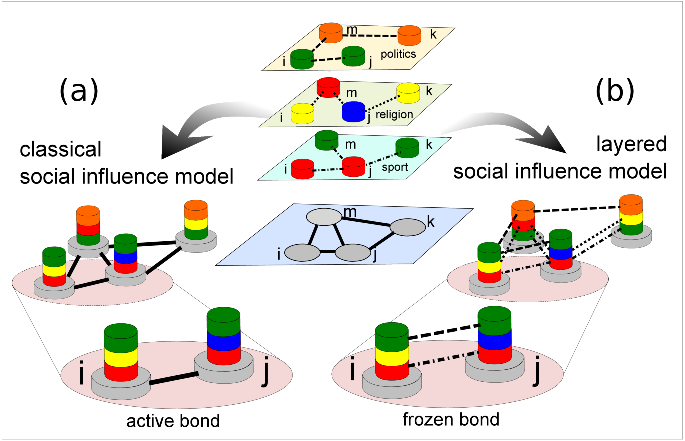
ω=0.0
ω=0.3
ω=0.7
ω=1.0
μ_1^{aggr}
q
9
8
7
6
5
4
3
0
0.2
0.4
0.6
0.8
1

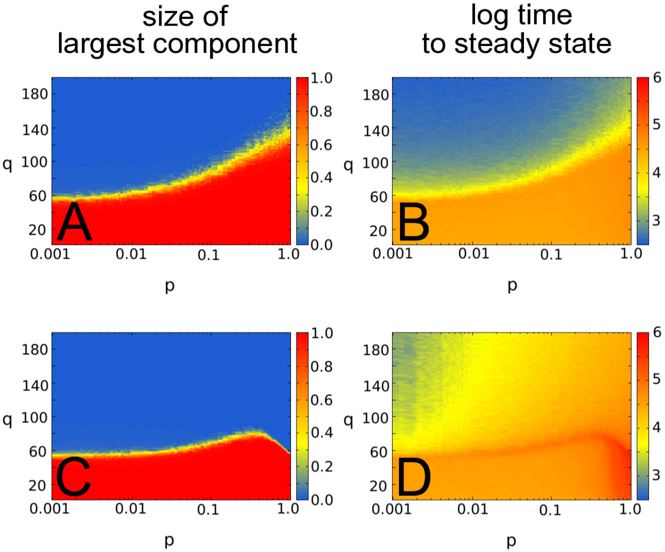

d=2
d=3
d=4
F(ω)
ω
2
1
0
-1
-2
-3
0
0.2
0.4
0.6
0.8
1

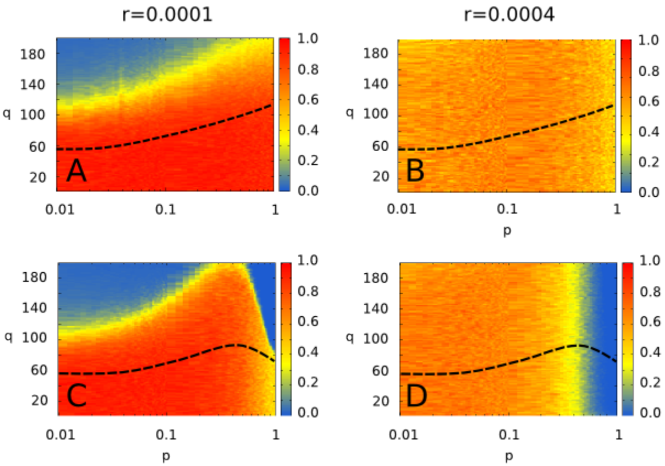


d=2
d=3
d=4
G(ω)
ω
4
3.5
3
2.5
2
0
0.2
0.4
0.6
0.8
1

AXELROD MODEL

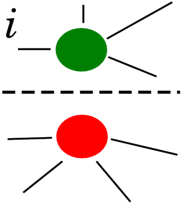

(a)
(b)
classical
social influence model
layered
social influence model
politics
religion
sport
m
k
i
j
active bond
frozen bond

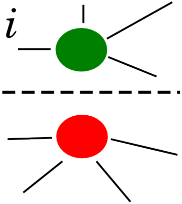

size of
largest component
log time
to steady state
A
B
C
D
q
p
180
140
100
60
20
0.001
0.01
0.1
1.0
0.0
0.2
0.4
0.6
0.8
3
4
5
6


r=0.0001
r=0.0004
A
B
C
D
q
p
180
140
100
60
20
0.01
0.1
1
1.0
0.8
0.6
0.4
0.2
0.0

TURING PATTERNS

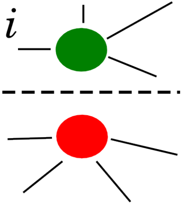
You could have had here a nice slide with hawks and hares, and another slide with some of the beautiful patterns formed by the skin of fish and other animals BUT I was actually brutally forced to go out for a dessert yesterday night, and since I don't like desserts, we ended up drinking cerveza nacional and talking of Yom Kippur, knowledge, atheism, Monty Python, and other amenities, while a few of us tried to explain to the waiter that waffels and bananas cannot stay in the same plate......then I fell asleep...


i



Activator: $u_i(t)$

Inhibitor: $v_i(t)$



Activator: $u_i(t)$

Inhibitor: $v_i(t)$

$$\frac{du(t)}{dt} = F\left(u(t), v(t)\right) + \sigma^{[1]} L^{[1]} u(t)$$
$$\frac{dv(t)}{dt} = G\left(u(t), v(t)\right) + \sigma^{[2]} L^{[2]} v(t)$$

Linear Stability:

$$J = \begin{pmatrix} L^{[1]} + f_u I & f_v I \\ g_u I & \sigma\left(L^{[2]} + g_v I\right) \end{pmatrix}$$

Linear Stability:

$$J = \begin{pmatrix} L^{[1]} + f_u I & f_v I \\ g_u I & \sigma \left(L^{[2]} + g_v I \right) \end{pmatrix}$$

$$\lambda_1 < \lambda_2 < \ldots < \lambda_N = \lambda_M$$

Linear Stability:

$$J = \begin{pmatrix} L^{[1]} + f_u I & f_v I \\ g_u I & \sigma \left(L^{[2]} + g_v I \right) \end{pmatrix}$$

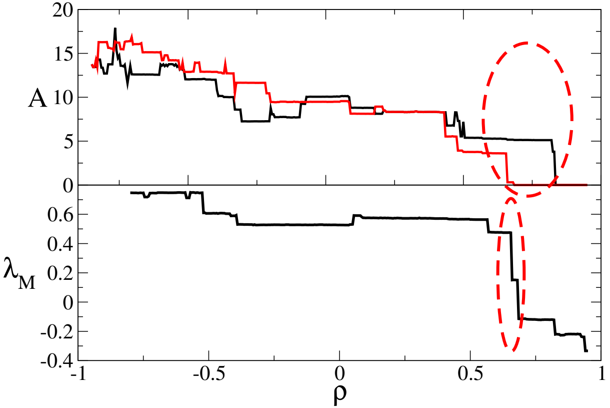
$$\lambda_1 < \lambda_2 < \ldots < \lambda_N = \lambda_M$$

Amplitude:

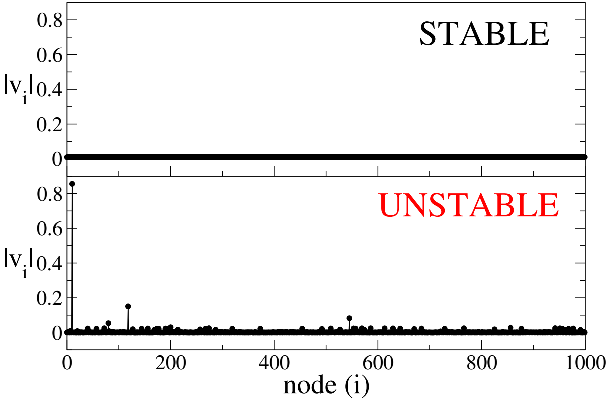
$$A = \sqrt{\sum_i (u_i - \overline{u})^2 + (v_i - \overline{v})^2}$$

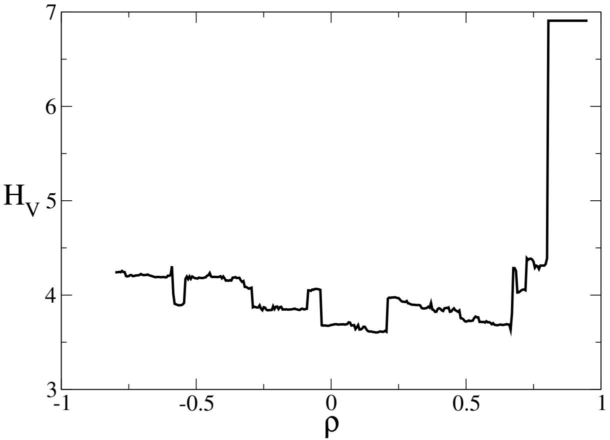
What if we tune inter-layer correlations?

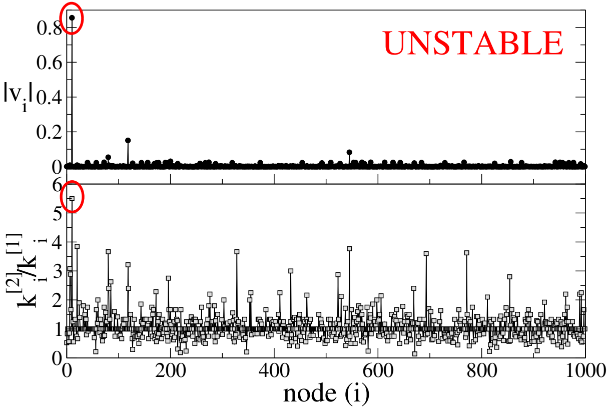
$$\rho_{\alpha,\beta} = \frac{\sum_i \left(R_i^{[\alpha]} - \overline{R^{[\alpha]}}\right)\left(R_i^{[\beta]} - \overline{R^{[\beta]}}\right)}{\sqrt{\sum_i \left(R_i^{[\alpha]} - \overline{R^{[\alpha]}}\right)^2 \sum_j \left(R_j^{[\beta]} - \overline{R^{[\beta]}}\right)^2}}$$

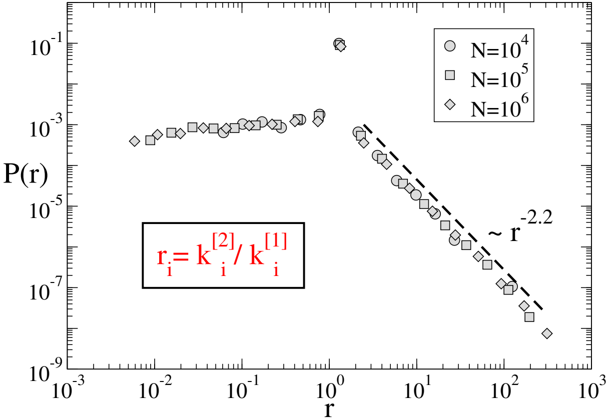


20
15
A 10
5
0
0.6
0.4
λ_M 0.2
0
-0.2
-0.4
-1
-0.5
0
0.5
1
ρ


STABLE
UNSTABLE
|v_i|
0.8
0.6
0.4
0.2
0
200
400
600
800
1000
node (i)


7
6
5
4
3
-1
-0.5
0
0.5
1
H_V
ρ


UNSTABLE
$|v_i|$
0.8
0.6
0.4
0.2
0
$k_i^{[2]}/k_i^{[1]}$
6
5
4
3
2
1
0
0
200
400
600
800
1000
node (i)

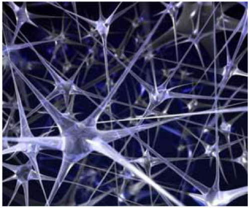


P(r)
r
N=10^4
N=10^5
N=10^6
~ r^-2.2
r_i = k_i^[2] / k_i^[1]

SYNCHRONISATION

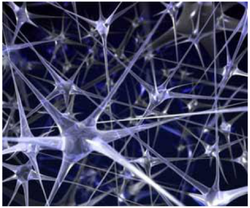
The "multiplex" brain

The "multiplex" brain

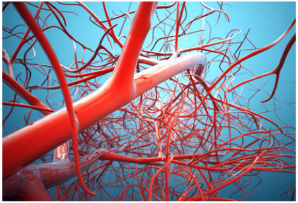


Neurons (Activity)

The "multiplex" brain

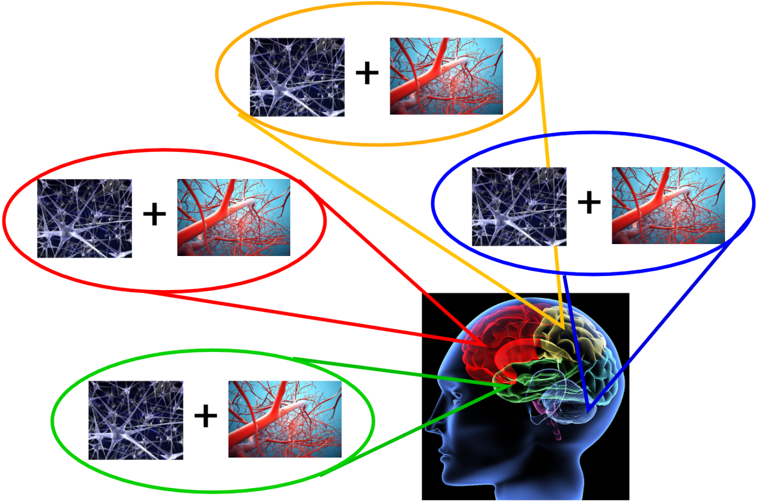


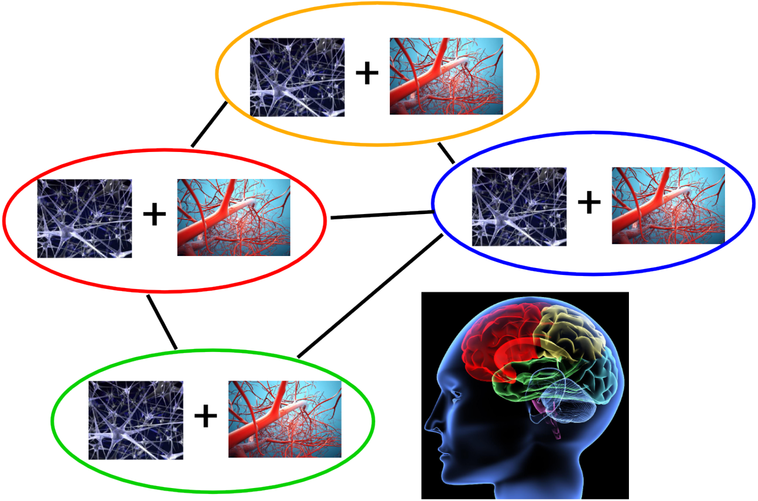
+

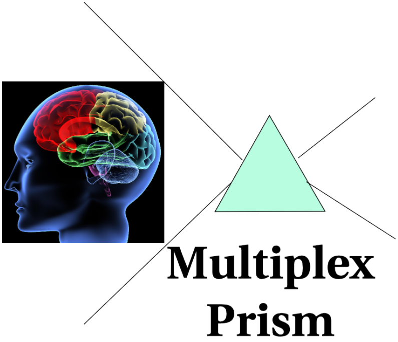


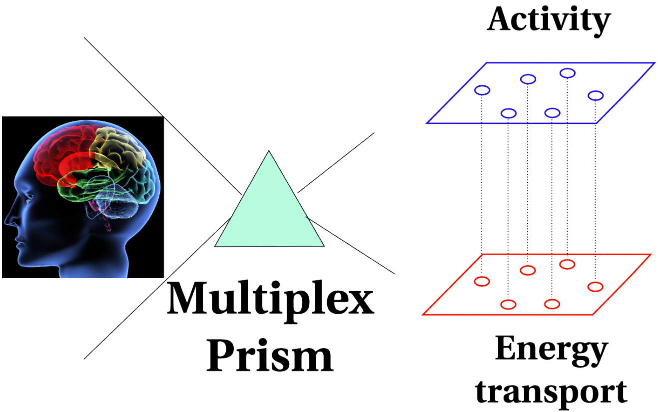
Neurons (Activity)

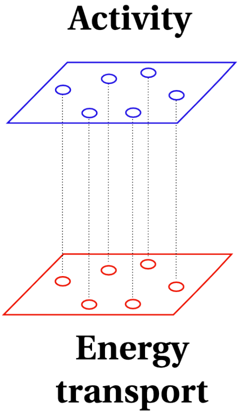
Blood vessels (Energy)



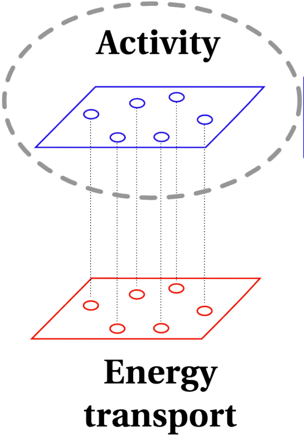



Multiplex
Prism


Activity
Multiplex
Prism
Energy
transport



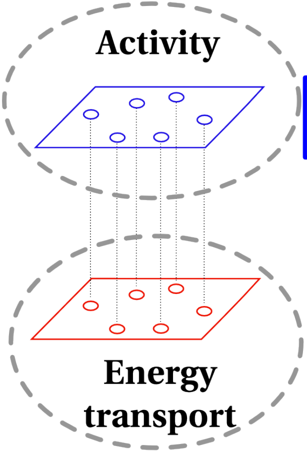
V. Nicosia, P. S. Skardal, V. Latora, A. Arenas, *under review*



Kuramoto Dynamics

$$\dot{\varphi}_i(t) = \omega_i + \lambda \sum_j a_{ij} \sin(\varphi_j - \varphi_i)$$

V. Nicosia, P. S. Skardal, V. Latora, A. Arenas, *under review*



Kuramoto Dynamics

$$\dot{\varphi}_i(t) = \omega_i + \lambda \sum_j a_{ij} \sin(\varphi_j - \varphi_i)$$

$$p_{i\to j} \propto e_{ij} f_j$$

Biased Random Walk

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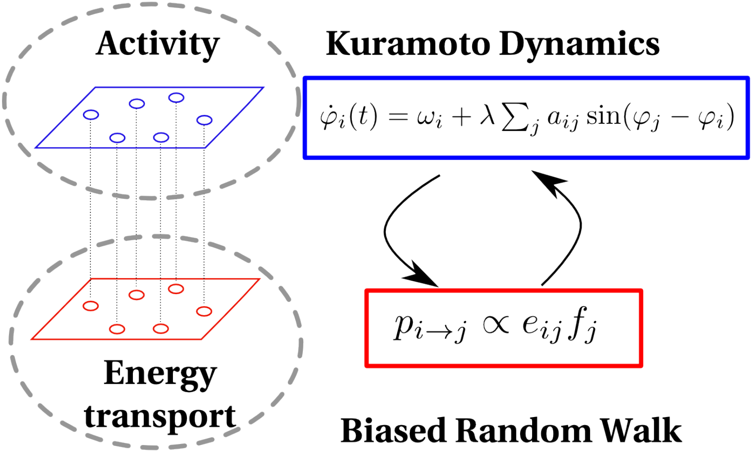


V. Nicosia, P. S. Skardal, V. Latora, A. Arenas, *under review*

$$\dot{\varphi}_i(t) = \omega_i + \lambda \sum_j a_{ij} \sin(\varphi_j - \varphi_i)$$

(more energy-> higher frequency)

$$\omega_i \propto p_i$$

V. Nicosia, P. S. Skardal, V. Latora, A. Arenas, *under review*

$$\dot{\varphi}_i(t) = \omega_i + \lambda \sum_j a_{ij} \sin(\varphi_j - \varphi_i)$$

(more energy-> higher frequency)

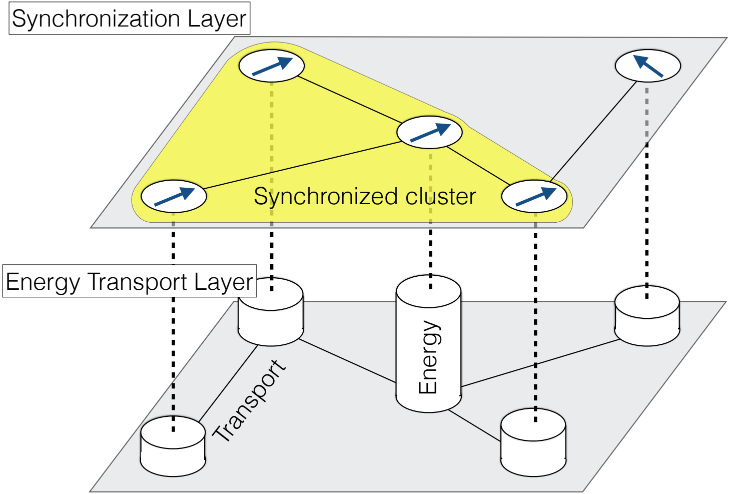
$$\omega_i \propto p_i$$

$$p_{i \to j} \propto e_{ij} f_j$$

(More synapses -> more blood)

$$f_j = k_j^{\alpha}$$

V. Nicosia, P. S. Skardal, V. Latora, A. Arenas, *under review*



V. Nicosia, P. S. Skardal, V. Latora, A. Arenas, *under review*

Kuramoto Dynamics

$$\dot{\varphi}_i(t) = \omega_i + \lambda \sum_j a_{ij} \sin(\varphi_j - \varphi_i)$$

Biased Random Walk

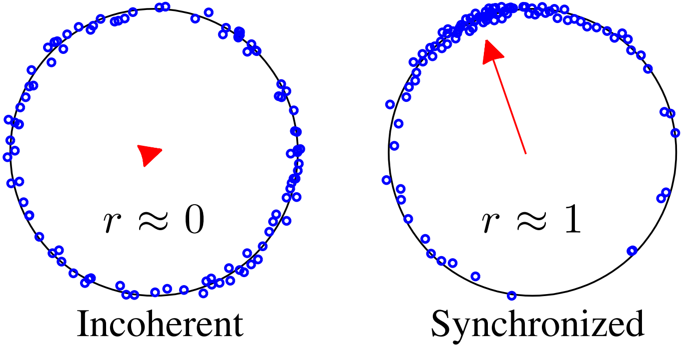
$$p_{i \to j} \propto e_{ij} f_j$$

Node state: (φ_i, p_i)

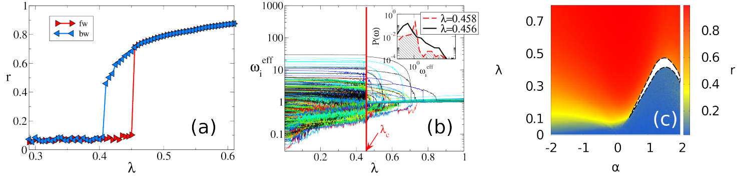
φ_i **phase (activity)**

p_i **fraction of walkers (energy)**

V. Nicosia, P. S. Skardal, V. Latora, A. Arenas, *under review*



$r \approx 0$
Incoherent
$r \approx 1$
Synchronized



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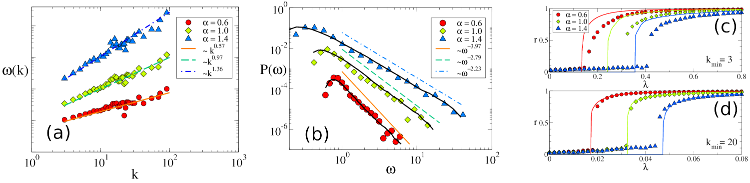

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- **IN SOME CASES** multiplex dynamics exhibit original new physics, which is **GENUINELY (due to the) MULTIPLEX**

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