The Master Stability Function

\[ \overline{x}_i = \sum_{j=1}^{N} A_{ij} \left[ h(x_j) - h(\overline{x}_i) \right] = \sum_{j=1}^{N} \alpha_{ij} h(\overline{x}_j) \]

- \{A_{ij}\} elements of the adjacency matrix
- \{\alpha_{ij}\} elements of the Laplacian matrix
- \overline{x}_i \in \mathbb{R}^m vector state of the network's node i
- \text{f(}x\text{)} function determining the local dynamics
- \text{h(}x\text{)} output function
Properties of the Laplacian matrix

a) Zero row sum $\implies \sum_i d_{ij} = 0$ [A. j]

Consequence:

- Laplacian is diagonalizable (symmetric)
- Eigenvalues are real
- First eigenvalue is $\lambda_0 = 0$
- Corresponding eigenvector is

$$\bar{v}_0 = \frac{1}{\sqrt{N}} \left( 1, 1, \ldots, 1 \right)$$

- The set of eigenvectors $\bar{v}_1, \ldots, \bar{v}_N$ forms a basis of the space transverse to $\bar{v}_0$
The solution exists!

\[ \bar{X}_1(t) = \bar{X}_2(t) = \bar{X}_3(t) = \ldots = \bar{X}_N(t) = \bar{X}_s(t) \]

\[ \bar{X}_s(t) \] obeys

\[ \bar{X}_s = f(\bar{x}_s) \]

- The synchronization manifold is along \( \bar{V}_o \)
Stability

synchronization error

\[ \delta \bar{x}_i = \bar{x}_i - \bar{x}_s \]

The vector \( \delta \bar{X} \) can be written in the basis that diagonalizes \( \tilde{L} \)

\[ \delta \bar{X} = \sum_{k=1}^{N} \bar{\eta}_k \otimes \bar{v}_k \]
Linear stability analysis

a) Get the eqs. for each $S\tilde{x}_i$ (linearizing Eq. 1 around $\bar{x}_S$)

b) Get the eq. for $\bar{S}X$

c) Project the equation for $\bar{S}X$ in the basis diagonalizing $L$

\[
\dot{\eta}_k = \left[ Jf(\bar{x}_S) - \sum_k Jh(\bar{x}_S) \right] \eta_k
\]

Variational equations...
Parametrization

\begin{align*}
\dot{x}_s &= f(x_s) \quad \text{\(\mathbb{R}^m\), nonlinear} \\
\eta &= [J_f(x_s) - \nu J^h(x_s)] \eta \\
\mathbb{R}^m \quad \text{parametric, linear}
\end{align*}

\[ \Lambda \text{ Maximum Lyap. Exponent} \]

\[ \text{Eigenvalej of } \lambda \text{ multiplied by } \sigma \]
Conditions

Class I \rightarrow \text{Synchronization} \quad \text{always unstable}

Class II \rightarrow \text{synch. stable for} \quad \sigma > \frac{\nu_2}{\lambda_2}

The role of the topology is to renormalize the synch. threshold by means of the "second largest eigenvalue".

Class III \rightarrow

\begin{cases}
\sigma > \frac{\nu_2}{\lambda_2} \\
\sigma < \frac{\nu_2}{\lambda_N} \\

\text{not always possible} \quad \ldots \quad \text{synchronizability}
\end{cases}

\quad r = \frac{\lambda_N}{\lambda_2} > 1
Is that approach limited to Eq. 1?

NO

Only condition is to have a zero-row sum, diagonalizable, coupling matrix...

Extensions provided for:

+ Weighted networks
+ Directed networks
+ Multi-layer networks

...