THE MASTER STABILITY FUNCTION

$$\dot{\bar{x}}_i = f(\bar{x}_i) - \sigma \sum_{j=1}^{N} A_{ij} \left[h(\bar{x}_j) - h(\bar{x}_i) \right] =$$

$$= f(\bar{x}_i) - \sigma \sum_{j=1}^{N} \mathcal{L}_{ij} \, h(\bar{x}_j)$$

$\{A_{ij}\}$ elements of the adjacency matrix

$\{\mathcal{L}_{ij}\}$ elements of the Laplacian matrix

$\bar{x}_i \in \mathbb{R}^m$ vector state of the network's node i

$f(\bar{x})$ function determining the local dynamics

$h(\bar{x})$ output function

Properties of the Laplacian matrix

a) zero row sum $\Rightarrow$ $\sum_i \mathcal{L}_{ij} = 0 \quad \boxed{\forall j}$

Consequence:

- Laplacian is diagonalizable (symmetric)
- Eigenvalues are real
- First eigenvalue is $\lambda_0 = 0$
- Corresponding eigenvector is

$$\bar{v}_0 = \frac{1}{\sqrt{N}} (1, 1, \ldots\ldots, 1)$$

- The set of eigenvectors $\bar{v}_1 \ldots \bar{v}_N$ forms a basis of the space transverse to $\bar{v}_0$

The solution

$$\bar{x}_1(t) = \bar{x}_2(t) = \bar{x}_3(t) = \dots = \bar{x}_N(t) = \bar{x}_s(t)$$

exists!

$\bar{x}_s(t)$ obeys $\boxed{\dot{\bar{x}}_s = f(\bar{x}_s)}$

- The synchronization manifold is ALONG $\bar{v}_0$

Stability

synchronization error

$$\delta \bar{x}_i = \bar{x}_i - \bar{x}_s$$

~~Too~~ The vector ~~vector~~ $\overline{\delta X} = \left(\delta \bar{x}_1, \delta \bar{x}_2, \ldots, \delta \bar{x}_N \right)$

~~se puede escribir como~~ can be written in the basis that diagonalizes $\mathcal{L}$

$$\boxed{\overline{\delta X} = \sum_{k=1}^{N} \bar{\eta}_k \otimes \bar{v}_k}$$

Linear stability analysis

a) Get the eqs. for each $\delta \bar{x}_i$ (linearizing Eq. 1 around $\bar{x}_s$)

b) Get the eq. for $\overline{\delta X}$

c) Project the equation for $\overline{\delta X}$ in the basis diagonalizing $\mathcal{L}$

$\Downarrow$

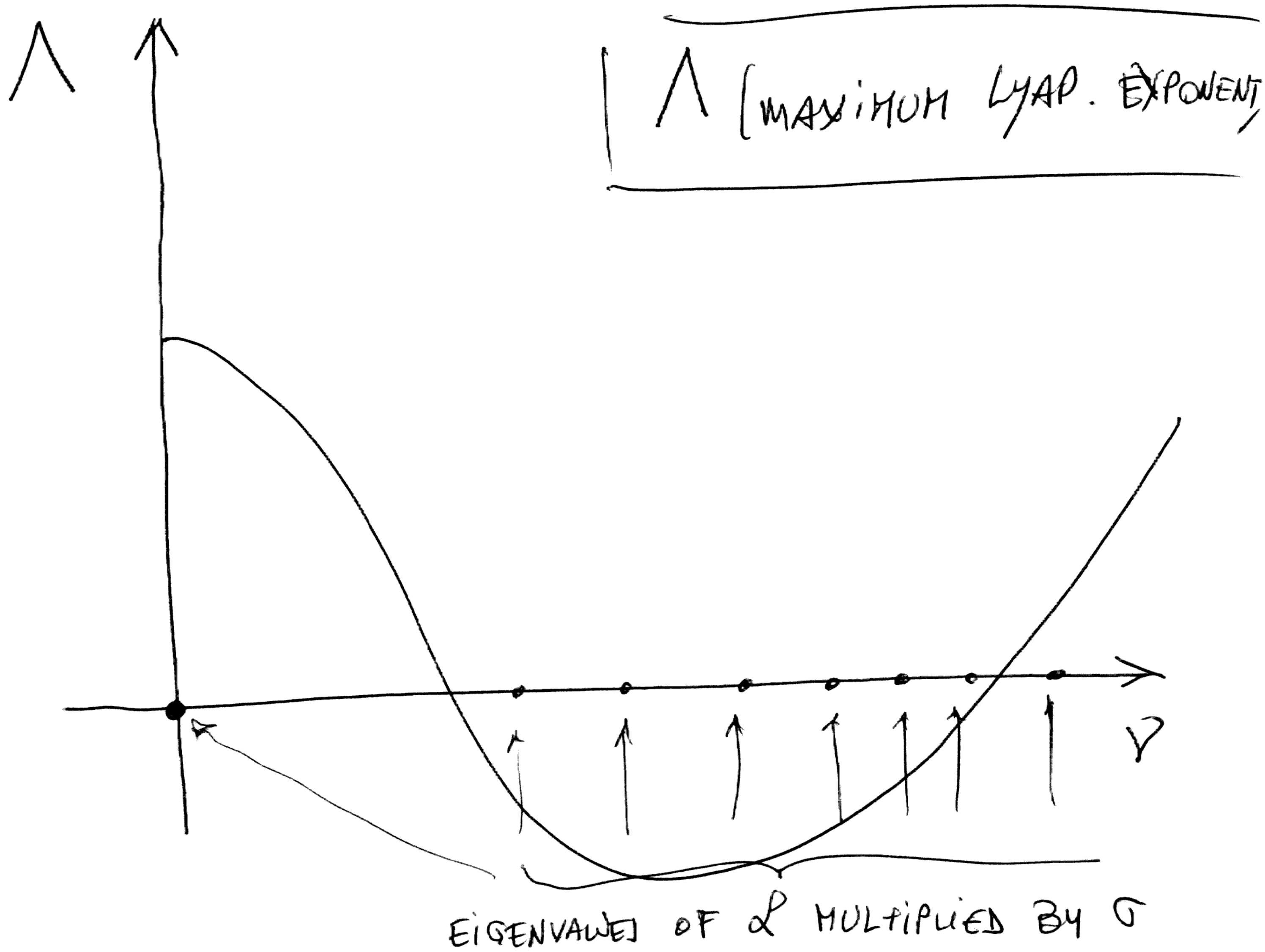
$$\boxed{\dot{\eta}_k = \left[Jf(\bar{x}_s) - \sigma \lambda_k Jh(\bar{x}_s) \right] \eta_k}$$

variational equations ...

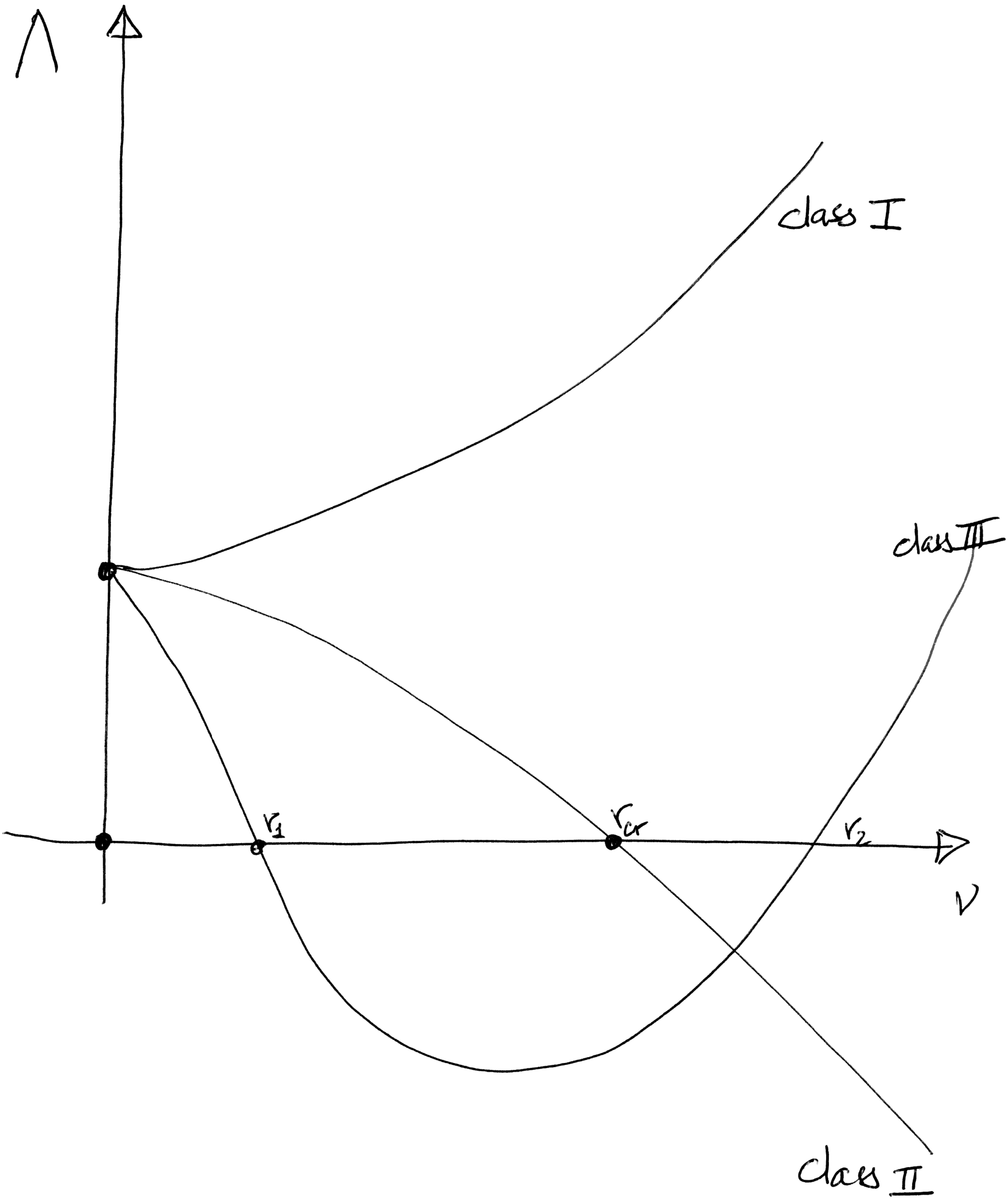
Parametrization

$$\begin{cases} \dot{\bar{x}}_s = f(\bar{x}_s) & \leftarrow \mathbb{R}^m, \text{ nonlinear} \\ \dot{\bar{\eta}} = \left[Jf(\bar{x}_s) - \nu\, Jh(\bar{x}_s) \right] \bar{\eta} \end{cases}$$

$\mathbb{R}^m$, parametric, linear



CLASSES



CONDITIONS

CLASS I $\rightarrow$ synchronization always unstable

CLASS II $\rightarrow$ synch. stable for $\boxed{\sigma > \frac{\nu_{cr}}{\lambda_2}}$!

The role of the topology is to renormalize the synch. threshold by means of the "second largest eigenvalue"

CLASS III $\Rightarrow$

$$\boxed{\begin{array}{c} \sigma > \frac{\nu_1}{\lambda_2} \\ \sigma < \frac{\nu_2}{\lambda_N} \end{array}}$$

not always possible ....

synchronizability

$$\boxed{r = \frac{\lambda_N}{\lambda_2} > 1}$$

Is that approach limited to Eq. 1?

NO

ONLY CONDITION IS TO HAVE A ZERO-ROW SUM, DIAGONALIZABLE, COUPLING MATRIX ...

Extensions provided for:

+ Weighted networks
+ Directed networks
+ Multilayer networks

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