A "Stringy" View of QFT Amplitudes

This set of lectures provides a simple introduction to the Cachazo-He-Yuan (CHY) formalism, and discuss some of its main applications as well as recent understanding.

- Lecture 1.
  - CHY: a formalism for scattering amplitudes of <u>massless</u> particles
    - orthogonal to the approach using Feynman diagrams
    - originally proposed at tree level.
      (there are also extentions to loop level)
  - Scattering amplitude A only depends on asymptotic data $\{k, \epsilon, \dots\}$
    it encodes particle dynamics & interactions in Minkowski in its analytic structure.
  - Unitary evolution:

$$U^\dagger U = \mathbb{1}, \quad U = \mathbb{1} + iT \quad \Rightarrow \quad -i(T - T^\dagger) = T^\dagger T$$

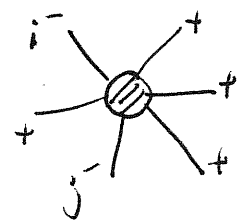
Insert completeness relation, $\mathbb{1} = \sum_\phi |\phi\rangle\langle\phi|$, ϕ: asymptotic states

$\Rightarrow$ tree level: A is meromorphic

- pole at $s=0$ for some Mandelstam variable s
- $A \xrightarrow{s \to 0} A_L \frac{1}{s} A_R$ + subleading

  - Feynman diagrams makes the above completely manifest in terms of propagators of virtual particles
    - this notion is essentially tied to the bulk of spacetime (explicitly) i.e., presence of ~~local~~ local quantum field
    - however: 1) inefficient for practical computations
      2) tension with gauge redundancy.

- Not surprisingly, people encountered many "surprises".
- Parke, Taylor studied 6-point scattering of gluons in the maximally-helicity-violating sector (1986)



If we color decompose

$$\mathcal{A}_n = \mathrm{tr}(T_1 T_2 \cdots T_n)\, \underbrace{A_n[12\cdots n]}_{\text{partial amplitude}} + \text{permutations}$$

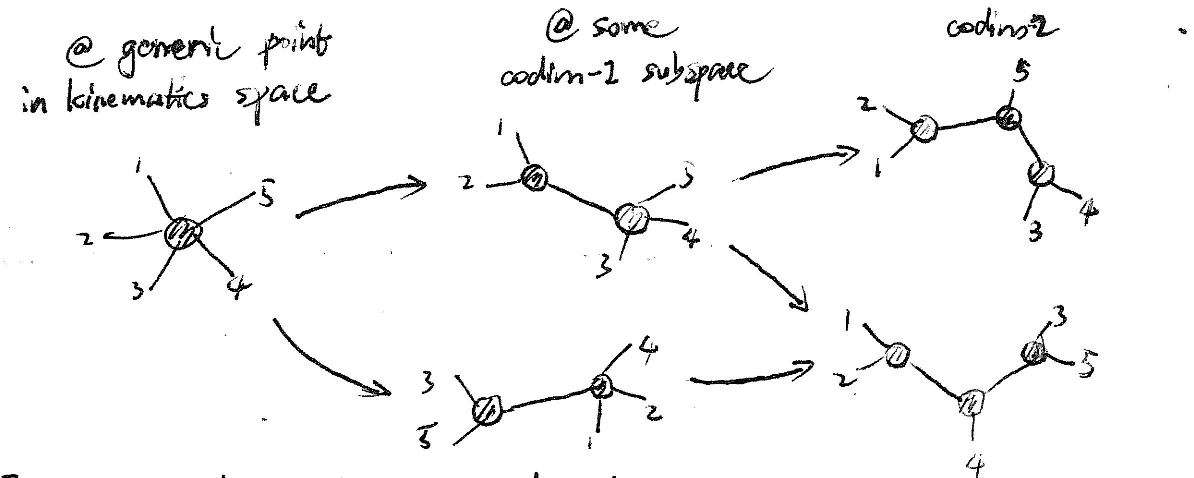
Then

$$\left|A_6[123456]\right|^2 = \frac{\left((k_i-k_j)^2\right)^4}{(k_1-k_2)^2(k_2-k_3)^2\cdots(k_6-k_1)^2} .$$

Anybody can immediately conjecture on its generalization.

Q : Why? Any way to make this obvious?

- Some "combinatorial" data universal to scattering



For n particles, stop at codim-(n-3).

- Unitarity is encoded at the singularities, which provide ~~data~~ "boundary data" for A at generic point.
  a la Cauchy theorem ($\Rightarrow$ modern on-shell techniques : BCFW, ...)

- Q: Any other space that canonically contains these "combinatorial" data?

  If yes, we can in principle

  1) build a model in that space;

  2) map it "physical observable" to physical space.

- Well-known candidate from string:

  space of n-punctured Riemann spheres. $\mathcal{M}_{0,n}$

  parametrize $\{z_1, z_2, \cdots, z_n\} / SL(2,\mathbb{C}) : z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}$, $\begin{vmatrix}\alpha & \beta \\ \gamma & \delta\end{vmatrix} = 1$.

  Factorization $\Longleftrightarrow$ degeneration

  in terms of the inhomogeneous coordinates, a subset of z's collide.

- Back to Parke-Taylor formula

  With spinor-helicity variables

  $$k_{\alpha\dot\alpha} \equiv k_\mu \sigma^\mu_{\alpha\dot\alpha}, \quad \det(k) = 0 \Rightarrow k_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$$

  $$\langle ij \rangle \equiv \epsilon^{\alpha\beta} \lambda_{i,\alpha} \lambda_{j,\beta}$$

  we have: $$A_n^{\text{MHV}}[12\cdots n] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}.$$

  In particular, we are able to identify

  $$\lambda_\alpha = \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad z = \frac{k^1 + i k^2}{k^0 + k^3}$$

  (Here z parametrize the actual celestial sphere)

  Then: $$A_n^{\text{MHV}}[12\cdots n] \propto \frac{1}{(z_1 - z_2)(z_2 - z_3) \cdots (z_n - z_1)}.$$

This looks very much like some correlator on a sphere.

$\Rightarrow$ a "string theory" that leads to YM amplitudes?

However, the punctured sphere configuration is determined by kinematics : no "summation over history"

$\sim$ path integral localized $\Rightarrow$ "topological"?

$\Rightarrow$ twistor string models [Witten '03, etc]

- We will not follow historical development nor construction of such models.

Instead, we will directly explore the idea of an underlying S^2.

* a full-fledged quantum theory is much more
* in practice we only need a few ingredients
* we will try some other view point later.

- A quick thought about our goal: schematically

$$A(\{k,\epsilon\}) = \sum_{z(k)} I(\{z\},\{k,\epsilon\})\Big|_{z=z(k)}$$

map analogous to that in Parke-Taylor

$$= \underbrace{\int dz\, \delta(z - z(k))}_{\text{universal}} \underbrace{I(\{z\},\{k,\epsilon\})}_{\text{theory specific}}$$

scattering equations

- Scattering equations:
  - seek for the map:
    * k_a^μ is naturally local data tied to z_a
    * construct a one-form: $\omega^\mu = \sum_{a=1}^{n} \frac{k_a^\mu}{z - z_a} dz$

$$k_a^\mu = \oint_{|z-z_a|=\epsilon} \omega^\mu$$

momentum conservation $\Leftrightarrow$ residue theorem

* further construct a quadratic differential

$$Q \equiv \omega \cdot \omega = \sum_{1 \le a < b \le n} \frac{2 k_a \cdot k_b}{(z-z_a)(z-z_b)} dz^2$$

(no double pole as k_a is massless on-shell)

Q has to be constrained since $z_a = z_a(\{k\})$

- Check the behavior of Q

  * 3-point : $Q \equiv 0$

  * 4-point : fix $\{z_1, z_3, z_4\} = \{0, 1, \infty\}$

    require the correspondence

$$\begin{cases} k_1 \cdot k_2 \to 0 & \Rightarrow \quad z_2 \to 0 \\ k_1 \cdot k_3 \to 0 & \Rightarrow \quad z_2 \to \infty \\ k_1 \cdot k_4 \to 0 & \Rightarrow \quad z_2 \to 1 \end{cases}$$

    simplest possibility : $z_2 = -\dfrac{k_1 \cdot k_2}{k_1 \cdot k_3}$

    substitute this into $Q \Rightarrow Q \equiv 0$!

- At any n point, it is then natural to impose

$$Q \equiv 0 .$$

Numerator of Q is of degree $n-3$, hence this solves all $\{z_a\}$.

This leads to the scattering equations.

- Cachazo-He-Yuan version :

$$E_a \equiv \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{k_a \cdot k_b}{z_a - z_b} = 0 , \qquad \forall a$$

There are 3 relations : $\sum_{a=1}^{n} z_a^i E_a \equiv 0$, $i = 0, 1, 2$

- Dolan-Goddard verison :

$$\sum_{\substack{S \subset \{1, \cdots, n\} \\ |S| = i}} k_S^2 z_S = 0 , \qquad i = 2, 3, \cdots, n-2$$

$$k_S = \sum_{a \in S} k_a , \quad z_S = \prod_{a \in S} z_a .$$

- Solution counting : $(n-3)!$ always
  - DG : gauge-fix $z_n \to \infty$, Bezout's theorem.
  - CHY : take soft limit $k_n \to 0$
    - $\{E_1, \dots, E_{n-1}\}$ reduce to those for $(n-1)$-point scattering.
    - $E_n = 0$ yields $n-3$ solutions of z_n for each solution of $\{z_1, \dots, z_{n-1}\}$ from above.
    - exactly one solution at 4 points.
- Why does this map meet our expectation in general?

  Explore a singular kinematics, e.g., $k_I^2 \to 0$, $k_I = k_1 + k_2 + \cdots + k_m$.

  Reparametrize
$$z_a = \begin{cases} \xi / u_a , & a \le m \\ v_a / \xi , & a > m \end{cases}$$

  Expand $E_{a>m}$:
$$E_{a>m} = \xi \frac{k_a \cdot k_I}{v_a} + \xi \sum_{\substack{b>m \\ b \neq a}} \frac{k_a \cdot k_b}{v_a - v_b} + O(\xi^3) = 0$$

  Sum up the above:
$$\sum_{a>m} E_a \Rightarrow -\frac{1}{2} k_I^2 + \xi^2 \sum_{\substack{a>m \\ b \le m}} \frac{k_a \cdot k_b}{v_a \cdot u_b} + O(\xi^4) = 0$$

  $\Rightarrow \exists$ solution such that $\xi \to 0$ as $k_I^2 \to 0$.
- We call these singular solutions,
  - $\{z_1, \dots, z_m\}$ collide
  - the original sphere degenerates into two, containing $\{z_1, \dots, z_m\}$ and $\{z_{m+1}, \dots, z_n\}$ respectively.
- In general, there are $(m-2)! \times (n-m-2)!$ such singular solutions.

Q1 : what are the ~~reg~~ remaining solutions responsible for?

Q2 : why Parke-Taylor formula seems to just depend on one solution?

We will answer these later.

— Lecture 2.

— CHY formalism:

$$A_n = \underbrace{\int \frac{d^n z_a}{\text{vol. } SL(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{\substack{d\mu_n \\ \text{universal} \\ \text{(any massless particles,} \\ \text{any spacetime dim)}}} \underbrace{I_n(\{z\},\{k,\epsilon\})}_{\text{theory dependent}}$$

$*$ $E_a \equiv \sum_{\substack{b=1 \\ \neq a}}^{n} \frac{k_a \cdot k_b}{z_a - z_b} = 0$, provides a map: kinematics $\longrightarrow$ $\{z_a\}$
1 $\quad$ $(n-3)!$

$*$ vol. $SL(2,\mathbb{C}) \equiv (z_i - z_j)(z_j - z_k)(z_k - z_i)\, dz_i\, dz_j\, dz_k$
for any choice $\{i, j, k\}$

$*$ $\prod_a{}' \equiv \frac{1}{(z_{i'}-z_{j'})(z_{j'}-z_{k'})(z_{k'}-z_{i'})} \prod_{\substack{a=1 \\ \notin \{i',j',k'\}}}^{n}$
for any choice $\{i', j', k'\}$

— under the action of $SL(2,\mathbb{C})$: $z_a \mapsto \frac{\alpha z_a + \beta}{\gamma z_a + \delta}$

$$d\mu_n \longmapsto d\mu_n \prod_{a=1}^{n} (\gamma z_a + \delta)^{-4}$$

A_n has to be invariant $\Longrightarrow$ universal constraint on I_n:

$$I_n \longmapsto I_n \prod_{a=1}^{n} (\gamma z_a + \delta)^4 \qquad \text{(weight 4)}$$

— Straightforward computation:

$$A_n = \sum_{i=1}^{(n-3)!} I_n / J_n \Big|_{\text{soln. } i}, \qquad J_n \equiv \frac{\left| \left(\frac{\partial \{E_a\}}{\partial \{z_b\}} \right)^{\hat{i}'\hat{j}'\hat{k}'}_{\hat{i}\hat{j}\hat{k}} \right|}{(z_i - z_j)(z_j - z_k)(z_k - z_i)(z_{i'} - z_{j'})(z_{j'} - z_{k'})(z_{k'} - z_{i'})}$$

$\hat{i}$: delete i^{th} row/column

— Unitarity

Consider factorization channel that separates n into n_L and n_R particles.

$$A_n = \int d\xi^2\, \xi^{2(n_L - n_R - 3)} \delta(\xi^2 F - k_I^2)\, d\mu_L\, d\mu_R\, I_n + \text{subleading}$$

regular solutions only contribute here.

We need

$$I_n \longrightarrow \xi^{2(-n_L + n_R + 2)} I_L I_R + \text{subleading}.$$

- Bi-adjoint Φ^3
  - simplest ~~stat~~ situation $I = I(\{z\})$.

    Recall Parke-Taylor factor:

$$C[12\cdots n] = \frac{1}{(z_1-z_2)(z_2-z_3)\cdots(z_n-z_1)} .$$

  - "double-partial amplitudes":

$$m[\alpha|\beta] = \int d\mu_n \; C[\alpha]\, C[\beta]$$

  - examples:

$$m[1234|1234] = -\left(\frac{1}{s_{12}} + \frac{1}{s_{23}}\right)$$

$$m[1234|1243] = \frac{1}{s_{12}}$$

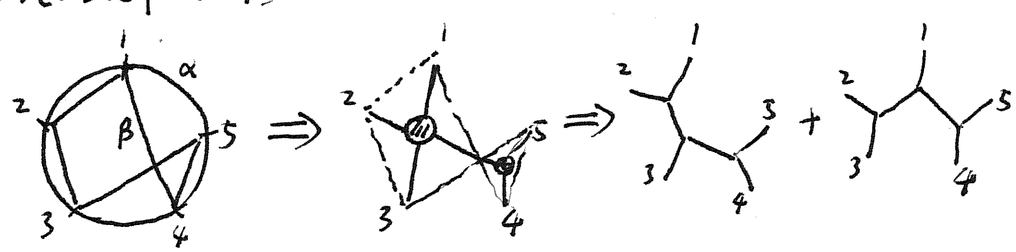
$$m[12345|12354] = -\left(\frac{1}{s_{12}} + \frac{1}{s_{23}}\right)\frac{1}{s_{45}}$$

$$m[12345|13254] = \frac{1}{s_{23}}\frac{1}{s_{45}}$$

$$m[12345|13524] = 0$$

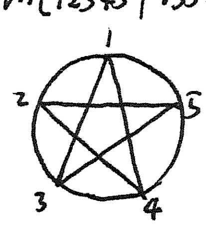
  - graphical rule:
    * use α ordering to define a planar ordering
    * use β to define a loop linking the α-ordered points. This loop is in general folded.
    * translate each polygon from the folded loop to a set of planar cubic graphs, and glue.

  e.g. $m[12345|12354]$



  * example of vanishing integral

    $m[12345|13524]$



- $m[\alpha|\beta]$ describes the double-partial amplitudes in Φ^3 flavored in $U(N)\times U(N')$, with interaction

$$f_{abc} f_{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'}$$

"double-partial": after color decomposition twice.

More specifically, $m[\alpha|\beta]$ is the summation of cubic diagrams that can be embedded into both α and β planar orderings.

- Obviously, the graphical rule above can also be inversely applied

$\Rightarrow$ for any cubic diagram, $\exists\, m[\alpha|\beta]$ that reproduces it. (though in general $m[\alpha|\beta]$ is non-unique)

$\Rightarrow$ Any amplitude of massless particles can be represented using the CHY formalism

$$A = \sum_{\text{diagrams}} \frac{N}{D} = \sum_{\substack{\text{cubic}\\ \text{diagrams}}} \frac{N'}{D'} = \sum_{\substack{\text{cubic}\\ \text{diagrams}}} N' \int d\mu_n\, m[\alpha'|\beta']$$

$$= \int d\mu_n \underbrace{\sum_{\substack{\text{cubic}\\ \text{diagrams}}} N' m[\alpha'|\beta']}_{I_n} .$$

though naively the resulting I_n can be highly complicated and non-intuitive.

- Q: what is then the advantage of this formalism?

— Magic: gluon scattering (any number, dimension, helicity config) receives a compact closed formula ("closed" in the sense that one formula applies to all amplitudes in a given theory)

$$A_n^{YM}[\alpha] = \int d\mu_n \, I_n^{YM}[\alpha] \quad , \quad I_n^{YM}[\alpha] = C[\alpha] \, \mathrm{Pf}' \Psi_n$$

* $C[\alpha]$ is already introduced before

* Ψ_n is a $2n \times 2n$ anti-symmetric matrix consisting of four blocks

$$\Psi_n = \left(\begin{array}{c|c} A_n & -C_n^T \\ \hline C_n & B_n \end{array} \right)$$

$$(A_n)_{ab} = \frac{k_a \cdot k_b}{z_a - z_b} , \quad (B_n)_{ab} = \frac{\epsilon_a \cdot \epsilon_b}{z_a - z_b} \quad \text{(diagonals are zero)}$$

$$(C_n)_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot k_b}{z_a - z_b} & a \neq b \\ -\displaystyle\sum_{\substack{c=1 \\ \neq a}}^{n} \dfrac{\epsilon_a \cdot k_c}{z_a - z_c} & a = b \end{cases}$$

$! = 0, 1$

There are two null eigenvectors to Ψ_n : $(z_1^!, z_2^!, \cdots, z_n^! ; 0, \cdots, 0)$.
Hence, to define a nontrivial invariant, we choose an arbitrary pair of labels (i, j) and delete the corresponding rows and column:

$$\mathrm{Pf}' \Psi_n \equiv \frac{(-1)^{i+j}}{z_i - z_j} \, \mathrm{Pf} (\Psi_n)_{i,j}^{i,j}$$

↑ from the first block of labels.

— Gravity amplitudes:

$$I_n^{GR} = (\mathrm{Pf}' \Psi_n)^2 \equiv \det' \Psi_n$$

Even thought perturbative expansion of Einstein-Hilbert action yields infinite number of vertices!

– Four dimensions.

Why Parke-Taylor formula (MHV) only relies on one solution?

In terms of spinor-helicity formalism, our one-form

$$\omega_{\alpha\dot\alpha} = \sum_{a=1}^{n} \frac{k_{a,\alpha\dot\alpha}}{z - z_a} dz \equiv \frac{P_{\alpha\dot\alpha}(z)\, dz}{\prod_a (z - z_a)}$$

$P_{\alpha\dot\alpha}(z)$ has degree $n-2$.

$$Q \equiv \omega\cdot\omega \equiv 0 \implies P_{1\dot1}(z) P_{2\dot2}(z) \equiv P_{1\dot2}(z) P_{2\dot1}(z)$$

This indicates that $P_{\alpha\dot\alpha}(z)$ has to factorize

$$P_{\alpha\dot\alpha}(z) = \underset{\substack{\uparrow \\ \text{degree } d}}{\lambda_\alpha(z)}\ \underset{\substack{\uparrow \\ \text{degree } n-d-2}}{\tilde\lambda_{\dot\alpha}(z)}$$

d can take the value

$$\underset{\text{"MHV"}}{1},\quad \underset{\text{"NMHV"}}{2},\quad \cdots,\quad \underset{\text{"}\overline{\text{MHV}}\text{"}}{n-3}$$

← call these "solution sectors"

* Each solution fall into one of the solution sector, in total $(n-3)!$ solutions from all sectors.
* When fixing the helicity sector for the polarization $\{\epsilon_a\}$, $\mathrm{Pf}'\Psi_n$ is non-vanishing only in the corresponding solution sector!
* The "MHV" and "$\overline{\text{MHV}}$" sector only contain one solution respectively. These are always rational in $\{k\}$.
* We immediately see A_n^{GR} in MHV sector also depends only on the same "MHV" solution as A_n^{YM}.

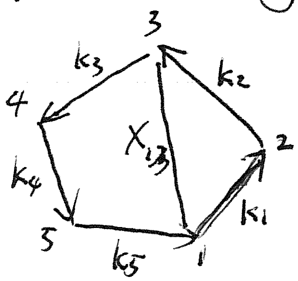
— Let us get back to Φ^3 theory and understand the ~~kinematics~~ connection between the kinematics and the space of punctured spheres better.

— A core ingredient here is to ~~under~~ treat amplitude as a form.

instead of $A \xrightarrow{s\to 0} A_L \frac{1}{s} A_R$ + subleading

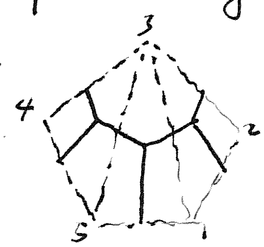
we have $\mathrm{Res}_{s=0} A = A_L A_R$.

— planar scattering form:



- e.g. $X_{1,3} = (k_1+k_2)^2$
- in total $\frac{n(n-3)}{2}$ X non-vanishing and they form a basis for the kinematics
- for every tree that can be embedded into this plane, every propagator $\Leftrightarrow$ some X

e.g.



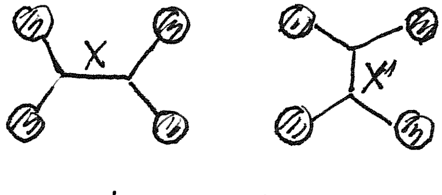
$\Rightarrow \frac{1}{X_{1,3}X_{3,5}}$

- promote each to a dlog form: $\frac{dX_{1,3}}{X_{1,3}} \wedge \frac{dX_{3,5}}{X_{3,5}}$

planar scattering form is to sum up these planar diagrams

$$\Omega_n[123\cdots n] = \sum_{\text{diagrams}} \pm \bigwedge^{n-3} d\log X$$

to determine the relative signs, require Ω_n to be projective i.e., invariant under $X_{i,j} \longrightarrow \Lambda(X) X_{i,j}$



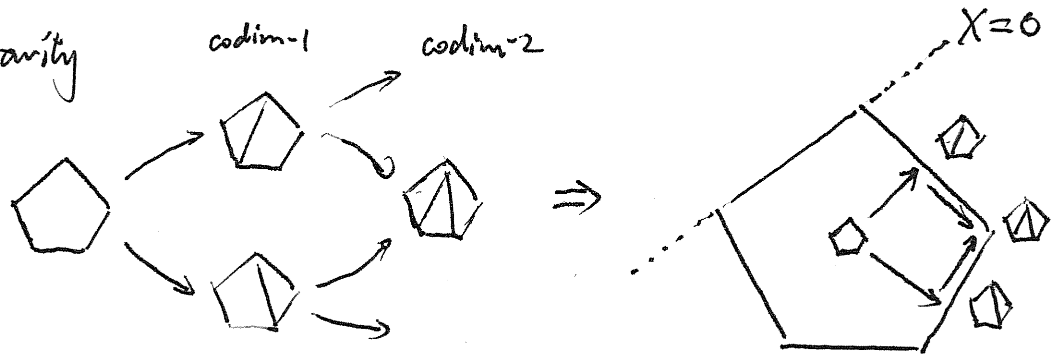
differ by a sign.

hence, at 4-points

$$\frac{ds}{s} - \frac{dt}{t} \equiv d\log\frac{s}{t}$$

This is not yet an amplitude (need to extract it from some top form)

— Unitarity



Can we identify such an associahedron in the kinematics?

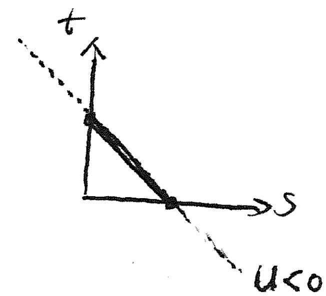
Yes : $*$ $X_{i,j} \geq 0$ (any i,j)

$*$ $c_{ij} = X_{i,j} + X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1} \equiv -(k_i+k_j)^2 > 0$ fixed.

(Name this $K[\alpha]$)

e.g., 4-point : $s \geq 0$, $t \geq 0$

$u<0$ fixed



in this case $ds + dt = 0$

$$\Omega_4[1234]\Big|_{K[1234]} = \left(\frac{1}{s}+\frac{1}{t}\right) ds$$

Alternatively, if we choose a different ordering for restriction

$s \geq 0$, $u \geq 0$

$t < 0$ fixed $\Rightarrow dt = 0$

$$\Omega_4[1234]\Big|_{K[1243]} = \frac{ds}{s}$$

In general : $\Omega_n[\alpha]\Big|_{K[\beta]} \Rightarrow m[\alpha|\beta]$.

— Worldsheet

another ~~assoh~~ associahedron

$$\mathcal{M}^+_{0,n}[\alpha] := \{z_1 < z_2 < \cdots < z_n\}/SL(2,\mathbb{R})$$

or equivalently

$$(1 \leq i < j-1 < n) \quad u_{ij} = \frac{(z_i - z_{j-1})(z_{i-1} - z_j)}{(z_i - z_j)(z_{i-1} - z_{j-1})}, \quad u_{i,j} = 1 - \prod_{\substack{i \; l \\ k \; j}} u_{k,l}, \quad u_{ij} \geq 0$$

(for every line crossing $\overline{ij}$)

all boundary at $u_{i,j} = 0$.

Canonical form of $\mathcal{M}^+_{0,n}[12\cdots n]$ is

$$\omega_n[12\cdots n] = \frac{1}{\mathrm{vol}\,SL(2)} \prod_{a=1}^{n} \frac{dz_a}{z_a + z_{a+1}} .$$

— scattering equations:

For some ordering β, and $\{X\}$ associated to β

i.e., $[\beta] = [12\cdots n]$, the equations can be written into

$$X_{a,b} = \sum_{\substack{1\le i<a\\ a<j<b}} z_{a,j}\frac{c_{i,j}}{z_{i,j}} + \sum_{\substack{a\le i<b\\ b\le j<n}} z_{i,b-1}\frac{c_{i,j}}{z_{i,j}} + \sum_{\substack{1\le i<a\\ b\le j<n}} z_{a,b-1}\frac{c_{i,j}}{z_{i,j}}$$

* $X_{a,b}$ consistent with $c_{i,j}$

* $X_{a,b} > 0$ if $\{z_a\} \in \mathcal{M}^+_{0,n}[\beta]$

* ~~boundari~~ boundaries of $\mathcal{M}^+_{0,n}[\beta] \longrightarrow$ boundaries of $K[\beta]$

Diffeomorphism: given X, precisely one solution in $\mathcal{M}[\beta]$
in $K[\beta]$

— the scattering form for ordering α is

the pushforward ~~to~~ of $\omega_n[\alpha]$, by the map from scattering eqns.

– Lecture 3.

– Kawai-Lewellen-Tye relations

– closed string contains both left and right movers.

– In computing tree amplitudes, contour deformation leads to

$$A_{\text{closed}} = \sum_{\alpha,\beta \in S_{n-3}} A_{\text{open}}[1,\alpha,n-1,n]\, K_{\text{string}}[\alpha|\beta]\, A_{\text{open}}[1,\beta,n,n-1]$$

$$\Big\downarrow \text{ low energy limit}$$

$$A^{GR} = \sum_{\alpha,\beta \in S_{n-3}} A^{YM}[1,\alpha,n-1,n]\, K[\alpha|\beta]\, A^{YM}[1,\beta,n,n-1]$$

(The way of fixing labels is just one choice out of many, but we'll keep to this)

$$K[\alpha|\beta] = \prod_{c=2}^{n-2}\left(s_{1,\alpha(c)} + \sum_{d=2}^{c-1} \theta^{\beta}_{\alpha(d),\alpha(c)}\, s_{\alpha(d),\alpha(c)}\right)$$

1 if $\alpha(d)$ comes before $\alpha(c)$ in β ordering otherwise 0.

e.g., 4pt

$$\begin{array}{c|c} & 2 \\ \hline 2 & s_{12} \end{array}$$

5pt

$$\begin{array}{c|cc} & 23 & 32 \\ \hline 23 & s_{12}(s_{13}+s_{23}) & s_{12}s_{13} \\ 32 & s_{12}s_{13} & s_{13}(s_{12}+s_{23}) \end{array} \xrightarrow{\text{inverse}} \frac{1}{s_{45}}\begin{pmatrix} -\frac{1}{s_{12}}-\frac{1}{s_{23}} & \frac{1}{s_{23}} \\ \frac{1}{s_{23}} & -\frac{1}{s_{13}}-\frac{1}{s_{23}} \end{pmatrix}$$

– KLT orthogonality

let
$$V^i_\alpha = \frac{C[1,\alpha,n-1,n]}{\sqrt{J}}\Big|_{\text{soln. } i}$$

$$U^j_\beta = \frac{C[1,\beta,n,n-1]}{\sqrt{J}}\Big|_{\text{soln. } j}$$

(recall: J is the Jacobian from solving the scattering eqns)

"KLT orthogonality": $\sum_{\alpha,\beta} V^i_\alpha K[\alpha|\beta] U^j_\beta = \delta^{ij}$.

I will not provide proof here, but discuss the consequence.

* as matrices: $VKU = \underset{\text{labeled by solutions}}{\mathbb{1}} \Rightarrow UVKUU^{-1} = \underset{\text{labeled by orderings}}{\mathbb{1}}$

$$\Rightarrow UV = \sum_{\text{soln } i} \frac{C[1,\alpha,n-1,n]\, C[1,\beta,n,n-1]}{J}\Big|_{\text{soln } i} = m[1,\alpha,n-1,n|1,\beta,n,n-1] = K[\alpha|\beta]^{-1}$$

* $\sum_{\alpha\beta} A_{YM}[1,\alpha,n-1,n]\, K[\alpha|\beta]\, A_{YM}[1,\beta,n,n-1]$

$$= \sum_{\alpha,\beta} \sum_{i,j} \left(\frac{Pf'\Psi}{\sqrt{J}} \Big|_{soln\, i} \right) V^i_\alpha\, K[\alpha|\beta]\, U^j_\beta \left(\frac{Pf'\Psi}{\sqrt{J}} \Big|_{soln\, j} \right)$$

$$= \sum_i \frac{(Pf'\Psi)^2}{J} \Big|_{soln.\, i} = A_{GR}$$

Hence we re-derived KLT relations under the CHY formalism

* In fact, the above indicates that

if we ~~sta~~ start with ~~to~~ partial amplitude in theory X and Y (both contains color/flavor) such that

$$\cancel{A_X[\alpha] = \int d\mu\, I_X I}$$

$$A_X[\alpha] = \int d\mu\, C[\alpha]\, I_X \ , \quad A_Y[\alpha] = \int d\mu\, C[\alpha]\, I_Y$$

then there exist another theory XY such that

$$A_{XY} = \int d\mu\, I_X I_Y = \sum_{\alpha,\beta} A_X[1,\alpha,n-1,n]\, K[\alpha|\beta]\, A_Y[1,\beta,n,n-1] .$$

As long as X and Y are consistent at tree level, then ~~then~~ the tree level consistency with unitarity of A_{XY} is ensured.

- Let us check what other theories we can easily describe using CHY.

- Compactification

$$K^M_a = (k^\mu_a \,|\, 0, \dots, 0)$$

$$\mathcal{E}^M_a = \begin{cases} (\epsilon^\mu_a \,|\, 0, \dots, 0) & \text{external} \\ (0, \dots, 0 \,|\, e^I_a) & \text{internal} \end{cases}$$

choose an orthonormal basis such that $e_a \cdot e_b = \delta_{I_a, I_b}$

Then Ψ_n reduces to

$$\begin{array}{r} \\ \\ \text{external} \to \\ \text{internal} \to \end{array} \left(\begin{array}{c|c:c} \frac{k\cdot k}{z-z} & \frac{k\cdot \epsilon}{z-z} & 0 \\ \hline \frac{\epsilon\cdot k}{z-z} & \frac{\epsilon\cdot\epsilon}{z-z} & 0 \\ \hdashline 0 & 0 & \frac{\delta^{I_a I_b}}{z_a - z_b} \end{array} \right)$$

$$Pf'\Psi_n \longrightarrow Pf(X_n)_S\ Pf'(\Psi_n)_{:\hat{S}}$$

$$(X_n)_{ab} = \frac{\delta^{I_a I_b}}{z_a - z_b}$$

$\hat{S}$: set of labels to be deleted. in correspondence to particles in the internal e.g. photon minimally coupled to gravity

$$I = \mathrm{Pf}\,X_{\gamma}\ \mathrm{Pf}(\Psi_n)_{:\hat{\gamma}}\ \mathrm{Pf}'\Psi_n$$

$\uparrow$ photon labels.

– extreme case: when all particles are internal

$$\mathrm{Pf}'\Psi_n \longrightarrow \mathrm{Pf}'A_n\ \mathrm{Pf}\,X_n \qquad \text{recall: } (A_n)_{ab} = \frac{k_a\cdot k_b}{z_a - z_b}$$

under $SL(2,\mathbb{C})$: $\quad \mathrm{Pf}'A_n \longmapsto \mathrm{Pf}'A_n \prod_{a=1}^{n} (\gamma z_a+\delta)^{1}$

Hence we are able to construct a bunch of other consistent CHY integrands, e.g.:

* $(\mathrm{Pf}'A_n)^2\ \mathrm{Pf}'\Psi_n \implies$ Born-Infeld (BI)
* $(\mathrm{Pf}'A_n)^3\ \mathrm{Pf}\,X_n \implies$ Dirac-Born-Infeld (DBI)
* $C[\alpha]\,(\mathrm{Pf}'A_n)^2 \implies U(N)$ non-linear sigma model (NLSM)
* $(\mathrm{Pf}'A_n)^4 \implies$ a special Galilean scalar with enhanced symmetries (sGal)

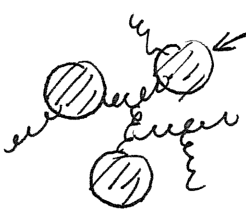
Recall KLT relations: $GR = YM \otimes_{KLT} YM$.

$BI = NLSM \otimes_{KLT} YM$.

$sGal = NLSM \otimes_{KLT} NLSM$.

– Another operation: "squeeze"

Is it also simple to describe amplitudes where YM couples to GR?



gluon trace

use $C[\alpha]$ with smaller cycle for each gluon trace

Not quite correct:

additional ϵ for gravitons

$$I_n = C[\alpha_1]\,C[\alpha_2]\cdots C[\alpha_r]\ \mathrm{Pf}'\Psi_n\ \boxed{?}$$

SL(2) weight of gravitons

We need to figure out $\boxed{?}$.

It turns out it descends from $\mathrm{Pf}'\Psi_m$ by "squeezing".

We call it $\mathrm{Pf}'\Pi$.

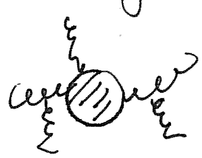
* for each set of ~~labl~~ labels in a gluon trace, delete the corresponding rows and insert a new row which is their summation.
  This is performed in both the first and second label blocks.
* Do the same ~~opera~~ operation to the columns.
* replace $\epsilon_a \mapsto z_a k_a$ for each gluon label.
  to get rid of the extra polarization.

This results in a new matrix Π, w/ graviton labels & trace labels.

$$
\begin{array}{c|c|c}
 & & \\
\hline
\text{trace } i & \sum_a \frac{k_a \cdot k_b}{z_a - z_b} & \sum_a \frac{k_a \cdot \epsilon_b}{z_a - z_b} \\
\hline
\text{trace } i & \sum_a \frac{\epsilon_a \cdot k_b}{z_a - z_b} & \sum_a \frac{\epsilon_a \cdot \epsilon_b}{z_a - z_b}
\end{array}
\quad /.\{\epsilon_a \to z_a k_a\}
$$

Then $\mathrm{Pf}'\Pi \equiv \mathrm{Pf}\,\Pi_{\hat{i}:\hat{j}} \equiv \frac{(-1)^a}{z_a} \mathrm{Pf}\,\Pi_{\hat{a}:\hat{j}} \equiv \frac{(-1)^a}{z_a} \mathrm{Pf}\,\Pi_{\hat{a}\hat{i}:} \equiv \frac{(-1)^{a+b}}{z_a - z_b} \mathrm{Pf}\,\Pi_{\hat{a}\hat{b}:}$

- e.g. 1. ~~only~~ one trace of gluons (denote the set of gravitons as h)



$$I = C[\text{gluons}]\ \underbrace{\mathrm{Pf}\,\Pi_{\hat{i}:\hat{i}}}_{\cancel{= \mathrm{Pf}(\Psi_n)_{\bar{h}:h}} \;\to\; = \mathrm{Pf}(\Psi_n)_{h:h}}\ \mathrm{Pf}'\Psi_n$$

- e.g. 2. two traces of gluons. no external gravitons.
  In this case Π is a 4x4 matrix

$$I = C[\text{trace } 1]\ C[\text{trace } 2]\ \underbrace{\sum_{\substack{c,d \in \text{trace } 1 \\ c \neq d}} \frac{z_c\, k_c \cdot k_d}{z_c - z_d}}_{\frac{1}{2} s}$$