Exercise 1

Consider an inflationary phase driven by a scalar field with a simple monomial potential:

$$V(\phi) = \lambda \phi^\alpha.$$  \hspace{1cm} (1)

a) Compute the classical dynamic of the scalar field in the slow-roll approximation, i.e. assuming $\dot{\phi} \ll 3H\phi$, with $H$ constant, using as time-variable the e-folding number defined as

$$N(t) = \int_t^{t_f} dt' H(t'),$$  \hspace{1cm} (2)

where $t_f$ is the time at the end of inflation.

b) Determine the value of $N$ needed to explain the homogeneity and isotropy of the CMB radiation.

c) Compute the power spectrum of the primordial fluctuations from the inflaton potential at horizon exit, i.e. using

$$P_R(k) = \frac{1}{12\pi^2 M_P^4} \left( \frac{V^3}{(V')}^2 \right) |_{k = aH}.$$  \hspace{1cm} (3)

Derive then the power spectrum to obtain the spectral index

$$n(k) - 1 = \frac{\partial \ln P_R(k)}{\partial \ln(k)}$$  \hspace{1cm} (4)

and compare the result with the slow-roll approximation.

d) Use the slow-roll approximation to obtain the curves $r = f(n - 1)$ obtained in the plane $r$ vs $n - 1$ by varying $N$, for $\alpha = 2; 4; 6$. 
Exercise 2

The equations of motion for the Fourier modes of a scalar field fluctuation $\delta \phi_k$ in a de-Sitter background can be written as

$$\delta \ddot{\phi}_k + 3H\dot{\delta \phi}_k + \left(\frac{k^2}{a^2} - m^2\right)\delta \phi_k .$$

(5)

a) Rewrite the equation in conformal time $\eta = -\frac{1}{aH}$ obtained from $dt = a(\eta)d\eta$. Note that in de-Sitter $H$ is exactly constant.

b) Rescale the scalar modes as $u_k = a \delta \phi_k$ and obtain the equation for $u_k$.

c) Show that for $k, m \sim 0$ one solution of the equation of motion is simply $u_0 = a$. Which is the second solution?

d) Check that for a massless field, the full solution is given by a combination of Hankel Functions:

$$H_{3/2}^1(k\eta) = -\sqrt{\frac{2}{\pi k\eta}} \left(1 + i\frac{1}{k\eta}\right) e^{ik\eta}$$

$$H_{3/2}^2(k\eta) = H_{3/2}^1(k\eta)^* = -\sqrt{\frac{2}{\pi k\eta}} \left(1 - i\frac{1}{k\eta}\right) e^{-ik\eta}$$

and uses the Minkowski limit $k \gg H$ to fix the appropriate mode expansion and normalization.