SAIFR - ICTP Summer School 2018 Early Universe Laura Covi

Exercise Sheet 2: Inflation

Exercise 1

Consider an inflationary phase driven by a scalar field with a simple monomial potential:

$$V(\phi) = \lambda\, \phi^{\alpha}\ . \tag{1}$$

a) Compute the classical dynamic of the scalar field in the slow-roll approximation, i.e. assuming $\ddot{\phi} \ll 3H\dot{\phi}$, with H constant, using as time-variable the e-folding number defined as

$$N(t) = \int_t^{t_f} dt'\; H(t')\ , \tag{2}$$

where t_f is the time at the end of inflation.

b) Determine the value of N needed to explain the homogeneity and isotropy of the CMB radiation.

c) Compute the power spectrum of the primordial fluctuations from the inflaton potential at horizon exit, i.e. using

$$P_{\mathcal{R}}(k) = \frac{1}{12\pi^2 M_P^6} \frac{V^3}{(V')^2}|_{k=aH}\ . \tag{3}$$

Derive then the power spectrum to obtain the spectral index

$$n(k) - 1 = \frac{\partial \ln P_{\mathcal{R}}(k)}{\partial \ln(k)} \tag{4}$$

and compare the result with the slow-roll approximation.

d) Use the slow-roll approximation to obtain the curves $r = f(n-1)$ obtained in the plane r vs $n-1$ by varying N, for $\alpha = 2; 4; 6$.

Exercise 2

The equations of motion for the Fourier modes of a scalar field fluctuation $\delta\phi_k$ in a de-Sitter background can be written as

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(\frac{k^2}{a^2} - m^2\right)\delta\phi_k \ . \tag{5}$$

a) Rewrite the equation in conformal time $\eta = -\frac{1}{aH}$ obtained from $dt = a(\eta)d\eta$. Note that in de-Sitter H is exactly constant.

b) Rescale the scalar modes as $u_k = a\ \delta\phi_k$ and obtain the equation for u_k.

c) Show that for $k, m \sim 0$ one solution of the equation of motion is simply $u_0 = a$. Which is the second solution ?

d) Check that for a massless field, the full solution is given by a combination of Hankel Functions:

$$\begin{aligned} H^1_{3/2}(k\eta) &= -\sqrt{\frac{2}{\pi k\eta}}\left(1 + i\frac{1}{k\eta}\right)e^{ik\eta} \\ H^2_{3/2}(k\eta) &= H^1_{3/2}(k\eta)^* = -\sqrt{\frac{2}{\pi k\eta}}\left(1 - i\frac{1}{k\eta}\right)e^{-ik\eta} \end{aligned}$$

and uses the Minkowski limit $k \gg H$ to fix the appropriate mode expansion and normalization.