Exercise 1

Consider a stable particle decoupling from the thermal bath and compute its final abundance $Y_X = n_X/s$ by solving approximately the Boltzmann equation:

$$\frac{dY_X}{dx} = -\frac{s(x)\langle \sigma v \rangle_x}{xH(x)}(Y_X^2 - Y_{X, eq}^2),$$

where $x = \frac{m_X}{T}$, $s = s(m_X)x^{-3}$ is the entropy density and $H(T) = H(m_X)x^{-2}$ the Hubble parameter during radiation domination, i.e. $s(m_X) = \frac{2\pi^2}{45} g_s m_X^3$, $H(m_X) = \frac{\pi}{3} \left( \frac{g_\rho}{10} \right)^{1/2} \frac{m_P^2}{m_X}$ with $g_s \sim g_\rho$ counting the relativistic degrees of freedom. (This equation can be obtained from the original Boltzmann equation in the slides by doing the redefinitions above).

a) Estimate the temperature $x_f$, when the particle density starts deviating from equilibrium by the approximation

$$n_{X, eq}(x_f)\langle \sigma v \rangle_{x_f} = H(x_f),$$

taking the temperature expansion

$$\langle \sigma v \rangle_x \approx a + bx^{-1} + \ldots ,$$

and the non-relativistic Maxwell-Boltzmann distribution for the equilibrium number density, i.e. $n_{X, eq}(x) = \frac{g}{\pi^2} m_X^3 x^{-3} e^{-x}$.

b) Give an estimate of $x_f$ for $a = \frac{\alpha^2}{m_X}, b = 0$ with $\alpha \approx 0.01$ and for the maximal annihilation cross-section allowed by the unitarity bound, i.e. $a = \frac{16\pi}{m_X}$.

c) Solve the Boltzmann equation after $x_f$ by neglecting the term $Y_{X, eq}^2$ in the eq. (1) for $x > x_f$.

d) Match the solution to $Y_{X, eq}(x)$ at $x_f$ and discuss the dependence of the final result from $Y_{X, eq}(x_f)$ and $x_f$. 
Exercise 2

Assume that the (non-relativistic) Dark Matter velocity distribution is given by a Maxwell-Boltzmann distribution with constant velocity dispersion, i.e.

\[ f(E) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} e^{-E/\sigma^2} \]

where \( E = \frac{1}{2} m \vec{v}^2 + \Phi(r) \), \( \rho_0 \) is a normalization constant, \( \Phi(r) \) the local gravitational potential and \( m, \vec{v} \) the Dark Matter mass and velocity.

a) Determine then \( \rho(\Phi(r)) \) and insert this into the Poisson equation

\[ \Delta \Phi(r) = 4\pi G_N \rho(r) . \]

to find a differential equation for the DM density \( \rho(r) \):

b) Show that the corresponding form for \( \rho(r) \) is

\[ \rho(r) = \frac{\sigma^2}{2\pi m G_N} \frac{1}{r^2} . \]

c) To avoid the singularity at \( r = 0 \), where in any case the baryon density is not negligible, modify this solution to

\[ \rho(r) = \frac{\rho_0}{1 + \left(\frac{r}{r_0}\right)^2} \]

where \( r_0 = \frac{\sigma}{\rho_0} \frac{1}{\sqrt{2\pi m G_N}} \) is the typical radius where \( \rho_b \sim \rho \). How does then the gravitational potential grow in the centre?