Notation $\eta_{\mu\nu} = \text{diag}\,(+---)$

$(\partial_\mu \phi)^2 \equiv \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \equiv \partial_\mu \phi \partial^\mu \phi$

Lecture 0

Review, review, review.

(i) Spontaneous symmetry breaking (SSB)

Action integral: $S[\phi] = \int d^d x \, [\underbrace{\partial_\mu \phi^* \partial^\mu \phi - V(|\phi|)}_{\mathcal{L} \text{ "lagrangian density"}}]$, with $\phi = \phi(x)$, a complex scalar field

Note: Complex scalar

$\phi(x) = \frac{A(x) + iB(x)}{\sqrt{2}}$

$\mathcal{L} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 - V$

$S[\phi]$ is a "functional" $\{\phi(x)\} \mapsto \mathbb{R}$

Symmetry: $S[e^{i\alpha}\phi(x)] = S[\phi(x)]$

Group structure: General statement: the set of physical transformations of a system forms a group

- group product = composition of transformations (one after another)
- identity, inverse obvious
- closure $A \to B \to C$ is same as $A \to C$

Symmetry transformations form a (sub) group

Exercise: prove this ↑

Above, $\phi(x) \to e^{i\alpha}\phi(x)$ the set of $e^{i\alpha}$ form a group, $U(1)$

$U(N) = \{ N \times N \text{ matrices } g \mid g^\dagger g = \mathbb{1} \}$ ie "unitary" matrices

(Notation $A^\dagger = (A^*)^T$, $*$ = complex conjugate T=transpose $(A^T)_{ij} = A_{ji}$)

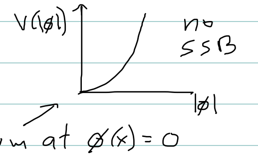
Energy $E[\phi(x)] = \int d^{d-1}x \, [\, |\Pi|^2 + |\vec{\nabla}\phi|^2 + V(|\phi|)]$ $\left(\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* \right)$

↑ space, not time

SSB occurs when there are more than one minima of the energy functional AND minima are related by a symmetry transformation

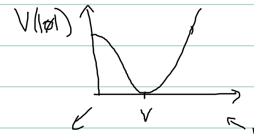
In our example if $\phi_0(x)$ is a minimum so is $e^{i\alpha}\phi_0(x)$. These are distinct provided $\phi_0 \neq 0$.

Example

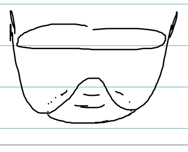


minimum at $\phi(x) = 0$

vs



minimum at $|\phi(x)| = v$



"mexican hat" or "bottom of wine bottle" potential

- Exercise: $|\phi(x)| = v$ is not sufficient to characterize minimum. Why? (Hint: compute $E[e^{i\alpha(x)} v]$).

- Exercise: If $\chi(x)$ is a real scalar field, invent an action integral with SSB, and explain which symmetry(ies) is (are) broken.

In SSB the equations of motion $\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \right)$ are symmetric. But the spectrum is not.

Language: "spectrum": collection of states (often 1 particle states)

Example: let's complicate the model; add a second complex scalar χ.

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \partial_\mu \chi^* \partial^\mu \chi - V(\phi, \chi)$$

where the potential $V(\phi,\chi)$ is a function of ϕ and χ and their complex conjugates invariant under $\phi(x) \to e^{i\alpha}\phi(x)$ and $\chi(x) \to e^{i\alpha}\chi(x)$.

Take $V(\phi,\chi) = V_1(|\phi|) + m^2 \chi^* \chi + \frac{1}{2}\lambda\left[(\phi^*\chi)^2 + (\chi^*\phi)^2\right]$

with $V_1(|\phi|)$ has a minimum at $|\phi(x)| = v$.

To find the spectrum, we quantize small fluctuations about the minimum at $\phi_0(x) = v$, $\chi_0(x) = 0$

$\phi(x) = v + \delta\phi(x)$ $\chi(x) = 0 + \delta\chi(x)$ (so we may as well just use $\chi(x)$).

Focusing on terms quadratic in χ

$$V = \cdots + m^2 \chi^* \chi + \frac{1}{2}\lambda v^2 (\chi^2 + \chi^{*2})$$

Write $\chi = \frac{A + iB}{\sqrt{2}} \Rightarrow V = \frac{1}{2}m^2(A^2+B^2) + \frac{1}{2}\lambda v^2 (A^2 - B^2) \Rightarrow m_A^2 = m^2 + \lambda v^2$ $m_B^2 = m^2 - \lambda v^2$

Compare to non-SSB case (minimum of ϕ at $\phi = 0$). Then

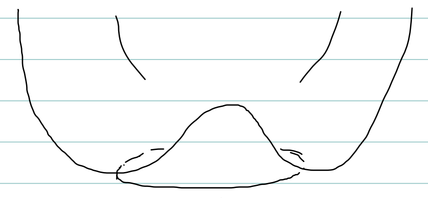
$$V = \cdots + m^2 \chi^* \chi = \frac{1}{2}m^2(A^2+B^2) \Rightarrow m_A^2 = m_B^2 = m^2$$

- Exercise: What if $m^2 < \lambda v^2$?

Goldstone's theorem, Nambu-Goldstone bosons.

For every continuous symmetry that is spontaneously broken there is a massless scalar in the spectrum

Intuition:



$\longrightarrow$ flat direction ⇒ no curvature ⇒ no mass

Let's see this in two ways:

(i) Explicitly. Take $V(|\phi|) = \frac{1}{4}\lambda(|\phi|^2 - v^2)^2 = \frac{1}{4}\lambda|\phi|^4 - \frac{1}{2}\lambda v^2|\phi|^2 + \text{constant}$

Clearly the minimum is at $|\phi| = v$. Looks like negative $(\text{mass})^2$, $-\frac{1}{2}\lambda v^2$, but this is just local maximum of V. Pick a phase for minimum, say, $\phi_0(x) = v$, and to quantize fluctuations about this write $\phi(x) = \phi_0(x) + \varphi(x) = v + \varphi(x)$

$$\text{Then } V(\varphi(x)) = \frac{1}{4}\lambda\left(|v+\varphi|^2 - v^2\right)^2 = \frac{1}{4}\lambda\left(2v\,\text{Re}\,\varphi + |\varphi|^2\right)^2 = \underset{\text{masses}}{\lambda v^2(\text{Re}\,\varphi)^2} + \underset{\text{interactions}}{\mathcal{O}(\varphi^3)}$$

So if $\varphi = \frac{A + iB}{\sqrt{2}}$ $\quad \mathcal{L} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\lambda v^2 A^2 + \text{interactions}$

⇒ two real scalars, 1 massless and 1 massive of $\text{mass}^2 = \lambda v^2$.

- Exercise: Repeat if no SSB: $V(|\phi|) = \mu^2|\phi|^2 + \frac{1}{4}\lambda|\phi|^4$. Show there are 2 degenerate states.

(ii) Indirect: $V(e^{i\alpha}\phi, e^{-i\alpha}\phi^*) = V(\phi, \phi^*)$ ⇒ (α infinitesimal) $\phi\frac{\partial V}{\partial \phi} - \phi^*\frac{\partial V}{\partial \phi^*} = 0$

Trivial at $\phi = \phi_0$, $\left.\frac{\partial V}{\partial \phi}\right|_{\phi_0} = 0$. But, 2nd derivative

$$\frac{\partial}{\partial \phi^*}: \quad \phi\frac{\partial^2 V}{\partial\phi\partial\phi^*} - \frac{\partial V}{\partial \phi^*} - \phi^*\frac{\partial^2 V}{\partial\phi^*\partial\phi^*} = 0 \Rightarrow i\phi_0\left.\frac{\partial^2 V}{\partial\phi\partial\phi^*}\right|_0 - i\phi_0^*\left.\frac{\partial^2 V}{\partial\phi^*\partial\phi^*}\right|_0 = 0 \qquad \text{(and it's c.c.)}$$

$$\text{Now } V(\phi,\phi^*) = \underset{\text{irrelevant const}}{V(\phi_0,\phi_0)} + \underbrace{(\varphi-\phi_0)\left.\frac{\partial V}{\partial\phi}\right|_0 + (\phi^* - \phi_0^*)\left.\frac{\partial V}{\partial \phi^*}\right|_0}_{\text{zero}} + \underbrace{\frac{1}{2}(\phi-\phi_0)^2\left.\frac{\partial^2 V}{\partial\phi^2}\right|_0 + (\phi-\phi_0)(\phi-\phi_0)^*\left.\frac{\partial^2 V}{\partial\phi\partial\phi^*}\right|_0 + \frac{1}{2}(\phi^*-\phi_0^*)\left.\frac{\partial^2 V}{\partial\phi^{*2}}\right|_0}_{\text{mass terms}} + \cdots$$

mass in terms of $\varphi = \phi - \phi_0$: $\frac{1}{2}\begin{pmatrix}\varphi\\ \varphi^*\end{pmatrix}^\dagger \mathcal{M}^2 \begin{pmatrix}\varphi\\ \varphi^*\end{pmatrix}$ with $\mathcal{M}^2 \equiv \begin{pmatrix} \left.\frac{\partial^2 V}{\partial\phi\partial\phi^*}\right|_0 & \left.\frac{\partial^2 V}{\partial\phi^{*2}}\right|_0 \\ \left.\frac{\partial^2 V}{\partial\phi^2}\right|_0 & \left.\frac{\partial^2 V}{\partial\phi\partial\phi^*}\right|_0 \end{pmatrix}$ $\quad$ note $(\mathcal{M}^{2\dagger} = \mathcal{M}^2)$.

Symmetry gave us, above, $\mathcal{M}^2\begin{pmatrix} i\phi_0 \\ -i\phi_0^* \end{pmatrix} = 0$ ⇒ a zero eigenvalue of $\mathcal{M}^2$

Expand $\begin{pmatrix}\varphi\\ \varphi^*\end{pmatrix}$ in an orthonormal basis $\frac{1}{N}\begin{pmatrix} i\phi_0 \\ -i\phi_0^* \end{pmatrix}, \frac{1}{N}\begin{pmatrix} \phi_0 \\ \phi_0^* \end{pmatrix}$ $\quad N^2 = \begin{pmatrix} i\phi_0 \\ -i\phi_0^* \end{pmatrix}^\dagger \begin{pmatrix} i\phi_0 \\ -i\phi_0^* \end{pmatrix} = 2|\phi_0|^2$, $\begin{pmatrix}\varphi\\ \varphi^*\end{pmatrix} = \frac{1}{N}\begin{pmatrix} \phi_0 \\ \phi_0^* \end{pmatrix}A + \frac{1}{N}\begin{pmatrix} i\phi_0 \\ -i\phi_0^* \end{pmatrix}B$

Then there is no B^2 term ⇒ $B = \frac{1}{N}\begin{pmatrix} i\phi_0 \\ -i\phi_0^* \end{pmatrix}^\dagger \begin{pmatrix}\varphi\\ \varphi^*\end{pmatrix} = -i\frac{\phi_0^*\varphi - \phi_0\varphi^*}{\sqrt{2}|\phi_0|}$ is massless $\left(\phi_0 = v \Rightarrow B = \sqrt{2}\,\text{Im}\,\varphi\right)$.

Caveats:

(i) Symmetry is global, but not local (local symmetry not really a symmetry but a redundancy).

(ii) Symmetry is bosonic. Fermionic symmetries (with anticommuting infinitesimal parameter), appear in supersymmetry. If spontaneously broken they give massless spinors (rather than scalars).

Gauge theories and the Higgs mechanism.

Local symmetry: $\phi(x) \to e^{i\alpha(x)}\phi(x)$

$\partial_\mu \phi^* \partial^\mu \phi$ not symmetric: $\partial_\mu\phi \to e^{i\alpha}(\partial_\mu \phi + i\partial_\mu\alpha \cdot \phi)$

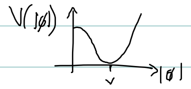
Covariant derivative: $D_\mu = \partial_\mu + ieA_\mu$, $D_\mu\phi \to e^{i\alpha} D_\mu \phi$ $\qquad D_\mu\phi^* D^\mu\phi$ invariant

Need $A_\mu \to A'_\mu = ?$ $\Rightarrow$ $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi \to e^{i\alpha}(\partial_\mu\phi + i\partial_\mu\alpha\phi + ieA'_\mu\phi)$ $\Rightarrow$ $A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha$

So $S = \int d^d_x \mathcal{L}$ with $\mathcal{L} = D_\mu\phi^* D^\mu\phi - V(|\phi|)$ is locally $\overline{U(1)\text{-invariant}}$

"Has U(1) gauge symmetry".

Towards the BEH mechanism: now take $V(|\phi|)$ as before



the energy of $\phi(x) = e^{i\alpha(x)}v$

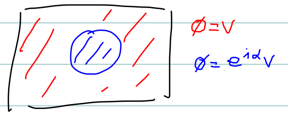
- Exercise: Find the energy functional.

$V(|\phi|)$ is independent of $\alpha(x)$.

The term $|\vec{D}\phi|^2 = v^2|\vec{\nabla}\alpha + e\vec{A}|^2$ is minimized by choosing $\vec{A} = -\frac{1}{e}\vec{\nabla}\alpha$.

The (complicated) Π, A_0 term, when minimized over Π at $\phi = e^{i\alpha(x)}v$ is $\alpha(x)$ independent.

Global symmetry → ∞ energy barrier between different minima v and $ve^{i\alpha}$ in this sense



While all minima are equivalent once we choose one, the system stays there, modulo small, localized fluctuations → hence $\phi(x) = v + \delta\phi$ makes sense.

Local symmetry: $\overset{\leftarrow\text{"zero"}}{0}$-energy barrier, an infinitude of minima of the form $\phi(x) = e^{i\alpha(x)}v$ but nothing depends on $\alpha(x)$.

- Puzzle: Nambu-Goldstones
  - Yes for Global
  - No for Local

For S or E to be independent of $\alpha(x)$ in $\phi = ve^{i\alpha(x)}$ we need to choose $A_\mu(x)$. This should be automatic, due to dynamics, ie, need to let A_μ evolve → give it a kinetic term.

But we know how to do this from Maxwell's theory

Review. Let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

then $A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha \Rightarrow F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu}$

Then $S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu\phi^* D^\mu \phi - V(|\phi|) \right]$ "scalar QED"

is invariant (under $\phi \to e^{i\alpha(x)}\phi$ $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha$)

(Useful: $[D_\mu, D_\nu] = ie F_{\mu\nu}$)

- Exercise: Find the equations of motion (EOM) for A_0 and $\vec{A}$. What's peculiar about the one for A_0?

A mass term in $\mathcal{L}$, $\mathcal{L} = \cdots + m^2 A_\mu A^\mu$ does not respect gauge invariance.

It would seem gauge invariance implies massles vector fields.
The only way we know to consistently quantize vector fields is in gauge theories (caveat: renormalizable)
It would seem quantum field theory (QFT) cannot describe W's & Z's.

BEH mechanism (used to be "Higgs" mechanism)

- Exercise for historians: what are the 6 names associated with the "Higgs" mechanism?

Now take $V(|\phi|)$ as before (with "SSB", but not really)

Use $\phi \to v$ in $D_\mu\phi^* D^\mu\phi$ get $D_\mu v\, D^\mu v = (-ieA_\mu v)(ieA^\mu v) = e^2v^2 A_\mu A^\mu$

This is a mass for A_μ ?!

More carefully: without loss of generality write $\phi(x) = e^{i\eta(x)}(v + \rho(x))$

Then make a gauge transformation $\phi(x) \to e^{-i\eta(x)}\phi(x)$, $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \eta$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + e^2v^2 A_\mu A^\mu + \partial_\mu\rho\,\partial^\mu\rho - V(v+\rho) + e^2(\rho^2 + 2v\rho)A_\mu A^\mu$$

Note that $$V(v+\rho) = \underset{\substack{\uparrow \\ \text{irrelevant}}}{V(v)} + \rho\, \overset{\to 0}{\cancel{V'(v)}} + \frac{1}{2}\rho^2 \underset{\substack{\uparrow \\ \text{mass term for } \rho^2}}{V''(v)} + \cdots$$

BEH mechanism: A_μ became massive, NGB dissapeared

$$A_\mu \begin{cases} \text{massless : 2 polarizations} \to \text{2 degrees of freedom (DOF)} & : +1 \text{ NGB } = 3\text{DOF} \\ \text{massive : 3 polarizations} \to 3 \text{ DOF} & \end{cases}$$

We say that the NGB was "eaten" by the vector field.
Note also ρ is a massive real scalar. The particle associated with quantizing the fluctuations is called a "higgs" particle generally, and more specifically in the Standard Model, as we will see.

Quick review of Lie-Algebras and representations.

$U(N)$ = group of $N\times N$ unitary matrices, $U^\dagger U = 1$

$SU(N) \subset U(N)$ with $\det U = 1$ $\quad$ ($\det UU^\dagger = 1 \Rightarrow |\det U| = 1$ $\Rightarrow \det U = e^{i\gamma}$).

$O(N)$ = group of $N\times N$ real orthogonal matrices $R^T R = 1$

$SO(N) \subset O(N)$ with $\det R = 1$.

If $g \in G$ is infinitesimally close to $1 \in G$, $\quad g = 1 + i\underset{\nearrow}{\epsilon^a} \underset{\nwarrow}{T^a}$ $N\times N$ matrices

infinitesimal parameters

- $\det g = 1 \rightarrow \mathrm{Tr}\, T^a = 0$
- $g^\dagger g = 1 \rightarrow T^{a\dagger} = T^a$
- Finite distance from $\mathbb{1}$: $\quad g = e^{i\epsilon^a T^a}$ $\uparrow$ no longer infinitesimal
- Closure $g_1 g_2 = g_3 \Leftrightarrow [T^a, T^b] = T^a T^b - T^b T^a$ is itself in the span of T^c:

$$[T^a, T^b] = i f^{abc} T^c \qquad \text{Lie algebra}$$

$\uparrow$ "structure constants"

- Can always take $\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$; then f^{abc} real, completely antisymmetric

(mathematicians take $\mathrm{Tr}\, T^a T^b = \delta^{ab}$

but we are stuck with spin history $\vec{s} = \frac{1}{2}\vec{\sigma} \leftarrow$ Pauli matrices

Lie Algebra of $SU(2)$: $\left[\frac{1}{2}\sigma^a, \frac{1}{2}\sigma^b\right] = i\epsilon^{abc}\frac{1}{2}\sigma^c$ and $\mathrm{Tr}\,\frac{\sigma^a}{2}\frac{\sigma^b}{2} = \frac{1}{2}\delta^{ab}$)

- n-dimensional representation: linear map $T^a \rightarrow D(T^a)$ into $n\times n$ matrices &

$[D(T^a), D(T^b)] = i f^{abc} D(T^c)$. Just write $t^a = D(T^a)$, or even T^a (get it from context).

Example: $t^3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $t^2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $t^1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ gives $[t^a, t^b] = i\epsilon^{abc} t^c$

$\Rightarrow$ 3-dim rep of $SU(2)$.

- Irreducible representations: if S exists such that $SD(T^a)S^{-1}$ are all

block diagonal $SD(T^a)S^{-1} = \begin{pmatrix} \boxtimes & 0 & 0 \\ 0 & \boxtimes & 0 \\ 0 & 0 & \boxtimes \end{pmatrix}$ (for all T^a), then D is "reducible"; else

it is irreducible. The 3-dim rep of $SU(2)$ above is irreducible. It corresponds to spin-1

The irreducible representations of $SU(2)$ correspond to spin $= 0, 1/2, 1, 3/2, \ldots$

and have dimension $2s+1$: $\qquad$ dim $= 1, 2, 3, 4, \ldots$

☆ Exercise: Find explicitly a 4-dimensional irreducible respresentation of $SU(2)$ (ie, give the three 4×4 matrices).

Follow up: find explicitly the 1-dimensional representation (called "trivial")

* Reducible reps can be decomposed into irreducible ones: $D = D_1 \oplus D_2 \oplus \cdots$

ie $$D = \begin{pmatrix} D_1 & 0 & 0 & \cdots \\ 0 & D_2 & 0 & \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \end{pmatrix}$$

☆ Exercise: Find all inequivalent 4-dim representations of (the Lie-Algebra of) $SU(2)$

For $SU(N)$ or $SO(N)$ the N-dim irreducible rep is the "defining" of "fundamental" representation

☆ Exercise: Suppose the Lie algebra is spanned by n matrices, T^a $a=1,\dots,n$

Then $(t^a)^{bc} = \kappa f^{abc}$ are a representation, n-dimensional, for some κ.

Show this and determine κ.

This is the "Adjoint" representation.

Example: 3-dim rep for $SU(2)$ we saw above.

☆ Exercise: show for $SU(N)$, $\dim(\text{Adj}) = N^2-1$. Can you find $\dim(\text{Adj})$ for $SO(N)$?

Leave out tensor products $D \otimes D' = D_1 \oplus D_2 \oplus \cdots$

<u>SU(3)</u>: $$T^{1,2,3} = \frac{1}{2}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\sigma^{1,2,3}} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & 0 \end{array}\right) \quad T^4 = \frac{1}{2}\begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}, \quad T^5 = \frac{1}{2}\begin{pmatrix} 0&0&i\\0&0&0\\-i&0&0 \end{pmatrix}$$

$$T^6 = \frac{1}{2}\begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix} \quad T^7 = \frac{1}{2}\begin{pmatrix} 0&0&0\\0&0&i\\0&-i&0 \end{pmatrix} \quad T^8 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix}$$

The defining (3-dim) rep acts on "triplets" (ie, 3-component complex vectors). The adjoint (8-dim) acts on "octets" (ie, 8-component real vectors).

☆ Exercise: why real octets but complex triplets? (this is equally true for N^2-1 vs N reps of $SU(N)$).

Convenient way of representing octets (or vectors in Adjoint rep):

octets = 8-dim real vector space $\longleftrightarrow$ space spanned by T^a $a=1,\dots,8$ with real coefficients.

$$\vec{c} = \begin{pmatrix} c^1 \\ c^2 \\ \vdots \\ c^8 \end{pmatrix} \longleftrightarrow T = c^a T^a, \quad c^a = 2\,\mathrm{Tr}\, T^a T$$

$T \to T' = U T U^\dagger \Rightarrow \vec{c} \to D(U)\vec{c}$ $\quad D = \text{adjoint}$

To see this, do infinitesimally: $$(1 + i\epsilon^c T^c)\, c^a T^a (1 - i\epsilon^d T^d) = c^a T^a + c^a i \epsilon^b [T^b, T^a]$$
$$= c^a T^a - c^a f^{bac} \epsilon^b T^c$$
$$= (c^c + i\epsilon^b D(T^b)^{ca} c^a) T^c$$

Non-abelian (Yang-Mills) gauge theory.

(i) From $U(1)$ to $U(N)$ (or other G) global symmetry

$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_N(x) \end{pmatrix}$ $\quad \phi(x) \to U\phi(x) \quad U \in G = SU(N)$ (or possibly $U(N) = SU(N) \times U(1)$).

$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi)$ is invariant

(ii) Extend to local symmetry $\phi(x) \to U(x)\phi(x)$

Need $D_\mu = \partial_\mu \mathbb{1} + ig A_\mu$ ($\mathbb{1}$ is an $N \times N$ identity $\to$ from now on understood
g is a number, the "coupling constant")
with A_μ an $N \times N$ matrix valued vector field.

($D_\mu \phi \to U[\partial_\mu \phi + (U^{-1}\partial_\mu U)\phi + ig\, U^{-1} A'_\mu U \phi]$ so $igU^{-1}A'_\mu U = igA_\mu - (U^{-1}\partial_\mu U)$

$\Rightarrow A'_\mu = UA_\mu U^{-1} - \frac{1}{ig}(\partial_\mu U)U^{-1} = U(A_\mu + \frac{1}{ig}\partial_\mu)U^{-1}$

$D_\mu \phi(x) \to U(x) D_\mu \phi(x)$ if $A_\mu(x) \to A'_\mu(x) = U(x) A_\mu(x) U^{-1}(x) + \frac{1}{ig} U(x) \partial_\mu U(x)^{-1}$

The term $UA_\mu U^{-1}$ is adjoint. (The non-homogeneous term makes it a gauge field.) There can also be a singlet in A_μ. The singlet piece A_μ^{singlet} has $UA_\mu^{\text{singlet}}U^{-1} = A_\mu^{\text{singlet}} \Rightarrow A_\mu^{\text{singlet}} = \mathbb{1} \times A_\mu^0$ where A_μ^0 is not a matrix. Since $U(N) = SU(N) \times U(1)$ this piece is the gauge field for $U(1)$ (can have its own coupling g_0). leaves us with $SU(N)$. For this $A_\mu(x) = T^a A_\mu^a(x)$

$\underset{\text{N×N matrices, Lie algebra}}{\uparrow}$ $\quad \uparrow$ fields

Note we can write, compactly: $A'_\mu(x) = \frac{1}{ig} U(x) D_\mu U^{-1}(x)$

Dynamics. As before

$$F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

☆ Exercise: Show, under $A_\mu \to A'_\mu = U(A_\mu + \frac{1}{ig}\partial_\mu)U^{-1}$, that $F_{\mu\nu} \to UF_{\mu\nu}U^{-1}$

☆ Exercise: Show $\operatorname{Tr} F_{\mu\nu} = 0$.

Quadratic invariants: $(\operatorname{Tr} F_{\mu\nu})(\operatorname{Tr} F^{\mu\nu}) = 0$, $\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}$ ✓

so $\mathcal{L} = -\frac{1}{2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger D_\mu \phi - V(\phi^\dagger\phi)$

☆ Exercise: Write $-\frac{1}{2}\operatorname{Tr} F^{\mu\nu}F_{\mu\nu}$ in terms of $A_\mu^a(x)$ (where $A_\mu(x) = T^a A_\mu^a(x)$).

(iii) Vacuum : ϕ at the minimum, henceforth $\langle\phi\rangle = \langle 0|\phi(x)|0\rangle$ or "vacuum expectation value" or VEV

$$\langle\phi\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \quad \text{find } U \ni \quad U\langle\phi\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} \qquad v = \sqrt{|v_1|^2 + \cdots + |v_N|^2}$$

(irrelevant)

So take $\langle\phi\rangle$ of this simpler form (physics does not depend on choice).

$$\text{If } \quad U = \left(\begin{array}{c|c} U' & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0\,0\cdots 0 & 1 \end{array}\right) \quad \text{with } U' \in SU(N-1) \quad \Rightarrow \quad U\langle\phi\rangle = \langle\phi\rangle$$

Symmetry breaking leaves $SU(N-1)$ still a symmetry : "$SU(N) \to SU(N-1)$"

\# Generators: $SU(N)$: N^2-1

$SU(N-1)$: $(N-1)^2-1$

$\Rightarrow$ # broken generators: $2N-1$ = # NGBs $\langle\phi\rangle^\dagger T^a \phi - \phi^\dagger T^a \langle\phi\rangle$

If Global sym, these are massless bosons. If gauged $\to$ eaten "would-be NGBs"

*) Exercise : Display explicitly the $2N-1$ broken symmetry generators

(iv) Spectrum - vectors

$$(D_\mu\phi)^\dagger D^\mu\phi = (ig A_\mu\langle\phi\rangle)^\dagger (ig A^\mu\langle\phi\rangle) + \cdots$$

$$= g^2 \langle\phi\rangle^\dagger A_\mu^\dagger A^\mu \langle\phi\rangle$$

$$= \frac{1}{2} g^2 \langle\phi^\dagger\rangle \{T^a, T^b\} \langle\phi\rangle A_\mu^a A^{\mu b} = \frac{1}{2}(m^2)^{ab} A_\mu^a A_\nu^b \eta^{\mu\nu}$$

$(m^2)^{ab}$ is a $(2N-1)\times(2N-1)$ mass^2 matrix.

v) Gauges.

- Unitary gauge $\phi(x) = e^{i\alpha^a(x)T^a}\left(\langle\phi\rangle + \chi(x)\right)$

$\uparrow$ only broken ones

This is a change of variables if same number of components : $\dim\phi = 2N$ (real), $\dim\alpha = 2N-1$ $\Rightarrow$ $\dim\chi = 1$: the "radial" field

$$\langle\phi\rangle + \chi(x) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{v+\rho(x)}{\sqrt{2}} \end{pmatrix}$$

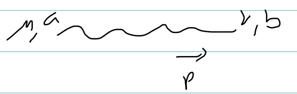
(I put in a $\frac{1}{\sqrt{2}}$ for convenience $\to$ kin. energy of ρ, real, $\to \frac{1}{2}$).

Then gauge transform $\phi \to e^{-i\alpha T}\phi = \langle\phi\rangle + \chi$

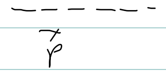
$$D_\mu\phi^\dagger D_\mu\phi = \frac{1}{2}(\partial_\mu\rho)^2 + g^2\frac{(v+\rho)^2}{2}(T^aT^b)_{NN} A_\mu^a A_\mu^b$$

This includes the mass term $\frac{1}{2}(m^2)^{ab} A_\mu^a A_\mu^b$

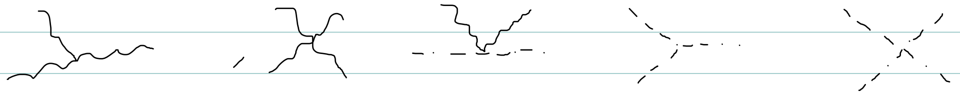
~~A~~Exercise: for SU(2) show all masses eigenvalues in $\mathcal{M}^2$ are equal, say M^2. Then compute all Feynman rules in unitary gauge:



$= -i\,\dfrac{\eta_{\mu\nu} - p_\mu p_\nu/M^2}{p^2 - M^2 + i\epsilon}$ $\qquad\rho$:



$\dfrac{i}{p^2 - M_\rho^2}$



- 't Hooft gauge

The propagator above (unitary gauge) is physically intuitive: 3 polarizations of massive vector:

$$\sum_{\lambda=+,-,0} \epsilon^{\lambda}_{\mu}\epsilon^{\lambda *}_{\nu} = -\eta_{\mu\nu} + p_\mu p_\nu/M^2$$

but cause trouble in loops: at large p,  $\sim \dfrac{i p_\mu p_\nu}{p^2} \sim \mathcal{O}(1)$ as opposed to the usual $1/p^2$. Very divergent. Renormalization becomes difficult.

Instead, use a renormalizable gauge: include a (Lorentz invariant) gauge fixing term.

Go back to QED (U(1) gauge theory)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

$$= -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial_\mu A_\nu - \frac{1}{2\xi}(\partial_\mu A_\mu)^2$$

$$= \frac{1}{2}\left[A_\nu \partial^2 A_\nu - (1-\tfrac{1}{\xi})A_\nu \partial_\nu\partial_\mu A_\mu\right]$$

$$= \frac{1}{2}A_\nu\left[\eta_{\mu\nu}\partial^2 - (1-\tfrac{1}{\xi})\partial_\mu\partial_\nu\right]A_\mu$$

- Need gauge fixing: if $1/\xi = 0$ then no inverse of $\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu$ (because $(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\partial^\nu = 0$).

- Invert by using projector operators: $P^2 = P$ so if $X = aP + b(1-P)$ then $X^{-1} = a^{-1}P + b^{-1}(1-P)$

Use $P = \eta_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2$, $(1-P) = \partial_\mu\partial_\nu/\partial^2$ $\quad a = \partial^2 \quad b = \frac{1}{\xi}\partial^2$

$$\Delta = (\eta_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2)\cdot(\partial^2)^{-1} + (\partial_\mu\partial_\nu/\partial^2)(\tfrac{1}{\xi}\partial^2)^{-1} = \frac{\eta_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu/\partial^2}{\partial^2} \rightarrow -i\,\frac{\eta_{\mu\nu} - (1-\xi)p_\mu p_\nu/p^2}{p^2 + i\epsilon}$$

$\xi = 1$ "Feynman gauge" $\quad \xi = 0$ "Landau gauge"

Now for non-abelian case:

If no SSB: take G.F. $= -\frac{1}{\xi}\mathrm{Tr}(\partial_\mu A^\mu)^2 = -\frac{1}{2\xi}(\partial_\mu A^{a\mu})^2$

If SSB (and the reason for reviewing this)

Use $\phi(x) = \langle\phi\rangle + \varphi(x)$. Then

$D_\mu\phi^\dagger D_\mu\phi = D_\mu\varphi^\dagger D_\mu\varphi + D_\mu\langle\phi\rangle^\dagger D_\mu\langle\phi\rangle$ + interesting

Interesting: $D_\mu\langle\phi\rangle^\dagger D_\mu\varphi + D_\mu\varphi^\dagger D_\mu\langle\phi\rangle$

$$= -ig\langle\phi\rangle^\dagger A_\mu\partial_\mu\varphi + ig\,\partial_\mu\varphi^\dagger A_\mu\langle\phi\rangle + \cdots$$

$$= ig\,\partial_\mu A^a_\mu\left[\langle\phi\rangle^\dagger T^a\varphi - \varphi^\dagger T^a\langle\phi\rangle\right]$$

- Mixing of ∂A_μ with φ:  agh!
- φ is really only the would-be NGB's

't Hooft: use GF: $-\frac{1}{2\xi}\left(\partial_\mu A^a_\mu + ig\xi(\langle\phi\rangle^\dagger T^a\varphi - \varphi^\dagger T^a\langle\phi\rangle)\right)^2$

Expanding the square, first term is $-\frac{1}{2\xi}(\partial A)^2$ as before, cross term cancels  (by design) and new term is $\frac{1}{2}\xi g^2(\langle\phi\rangle^\dagger T^a\varphi - \varphi^\dagger T^a\langle\phi\rangle)^2$ a mass term for the would-be NGBs.

Summing all graphs for a particular physical quantity gives a ξ-independent (gauge choice independent) result, but each individual Feynman graph may depend on ξ.

★Exercise: Feynman rules

Additional fields

We can add to this spinors: $\overline{\Psi} i \not{D} \psi$

with $D_\mu = \partial_\mu + ig A_\mu^a t^a$ where t^a is in some rep (often fundamental)

Spectrum is not symmetric:

Intuitively, if $a_{\vec{k},i}$ $i=1,\dots,N$ is a collection of creation operators, $a^{\dagger}_{\vec{k},i}|0\rangle = |\vec{k},i\rangle$ are 1-particle states; symmetry $a^{\dagger}_{\vec{k},i} \to U a^{\dagger}_{\vec{k},i} U^{\dagger} = g_{ij}\, a^{\dagger}_{\vec{k},j}$ ($g \in U(N)$, for example) (and $UHU^{\dagger} = H$, of course). Note $g \to U(g)$ an operator on the Hilbert space.

If $U|0\rangle = |0\rangle$ (invariant vacuum) then

$U|\vec{k},i\rangle = g_{ij}|\vec{k},j\rangle$ and if $H|\vec{k},i\rangle = m_i|\vec{k},i\rangle \Rightarrow g_{ij}\, H|\vec{k},j\rangle = m_i\, g_{ij}|\vec{k},j\rangle \Rightarrow m_j = m_i$

(or $H|\vec{k}\rangle = m|\vec{k}\rangle$ with m a matrix and $|\vec{k}\rangle$ a vector, then $g^{\dagger} m g = m$

Schur's lemma then says $m \propto \mathbb{1}$).

(Not quite: center of group, but $m^{\dagger} = m$ takes care of this possibility).

ASIDE 2: the standard convention is

$U|i\rangle = |j\rangle\langle j|U|i\rangle = |j\rangle g_{ji} = g^{\dagger}_{ij}|j\rangle = g^{-1}_{ij}|j\rangle$

More ASIDES (#3):

$V(z, z^*) = V(x, y)$

$V_0 + \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \dots$ vs $V_0 + \frac{\partial V}{\partial z}(dx + i\,dy) + \frac{\partial V}{\partial z^*}(dx - i\,dy) + \dots$

so finding an extremum requires $\frac{\partial V}{\partial z} + \frac{\partial V}{\partial z^*} = 0$ and $\frac{\partial V}{\partial z} - \frac{\partial V}{\partial z^*} = 0 \Leftrightarrow \frac{\partial V}{\partial z} = \frac{\partial V}{\partial z^*} = 0$

$$(t^1)_{ij} = -i\epsilon_{1ij} = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad t^2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad t^3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[t^1, t^2] = -\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i t^3 \quad \checkmark$$