Practical QCD at colliders

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5th Lecture

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Today

Today I want to cover briefly two big areas:

- jets
- Monte Carlos

Both are ubiquitous at the LHC!
Where do jets enter?

Essentially everywhere at colliders!

Jets are an essential tool for a variety of studies:

- top reconstruction
- mass measurements
- most Higgs and NP searches
- general tool to attribute structure to an event
- instrumental for QCD studies, e.g. inclusive-jet measurements

⇒ important input for PDF determinations
Jets

Jets provide a way of projecting away the multiparticle dynamics of an event ⇒ leave a simple quasi-partonic picture of the hard scattering.

The projection is fundamentally ambiguous ⇒ jet physics is a rich subject.

Ambiguities:
1) Which particles should belong to a same jet?
2) How does recombine the particle momenta to give the jet-momentum?
Jet developments

- 1975: Sterman Weinberg
- 1980: UA1+2 cones
- 1985: Jade, seq. rec.
- 1990: Snowmass (cone)
- 1995: $k_t$
- 2000: Cambridge Aachen
- 2005: Tev Run II wkshp (midpoint cone)

- fast-$k_t$, SISCones, anti-$k_t$, jet-areas, jet-flavour, non-perturbative effects, quality measures, jet-substructure ...
Two broad classes of jet algorithms

Today many extensions of the original Sterman-Weinberg jets. Modern jet-algorithms divided into two broad classes

Jet algorithms

Cone type
(UA1, JetCLU, Midpoint, SIS Cone..)

Sequential
(kt-type, Jade, Cambridge/Aachen...)

top down approach: cluster particles according to distance in coordinate-space
Idea: put cones along dominant direction of energy flow

bottom up approach: cluster particles according to distance in momentum-space
Idea: undo branchings occurred in the PT evolution
Jet requirements

Snowmass accord

Toward a Standardization of Jet Definitions

Several important properties that should be met by a jet definition are

[3]:

1. Simple to implement in an experimental analysis;
2. Simple to implement in the theoretical calculation;
3. Defined at any order of perturbation theory;
4. Yields finite cross section at any order of perturbation theory;
5. Yields a cross section that is relatively insensitive to hadronization.
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Other desirable properties:
- flexibility
- few parameters
- fast algorithms
- transparency
- ...
Inclusive $k_t$/Durham-algorithm

Catani et. al ’92-’93; Ellis&Soper ’93

Inclusive algorithm:

1. For any pair of final state particles i,j define the distance

$$d_{ij} = \frac{\Delta y_{ij}^2 + \Delta \phi_{ij}^2}{R^2} \min\{k_{ti}^2, k_{tj}^2\}$$
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2. For each particle $i$ define a distance with respect to the beam

$$d_{iB} = k_{ti}^2$$
**Inclusive \( k_t/Durham \)-algorithm**

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3. Find the smallest distance. If it is a \( d_{ij} \) recombine \( i \) and \( j \) into a new particle (\( \Rightarrow \) recombination scheme); if it is \( d_{iB} \) declare \( i \) to be a jet and remove it from the list of particles

NB: if \( \Delta R_{ij} \equiv \Delta y_{ij}^2 + \Delta \phi_{ij}^2 < R \) then partons (ij) are always recombined, so \( R \) sets the minimal interjet angle
Inclusive \( k_t/Durham \)-algorithm

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4. repeat the procedure until no particles are left
Exclusive $k_t$/Durham-algorithm

Inclusive algorithm gives a variable number of jets per event, according to the specific event topology
Exclusive $k_t$/Durham-algorithm

**Inclusive algorithm** gives a variable number of jets per event, according to the specific event topology.

**Exclusive version:** run the inclusive algorithm but stop when either
- all $d_{ij}, d_{iB} > d_{\text{cut}}$ or
- when reaching the desired number of jets $n$
\( k_t \) / Durham-algorithm in \( e^+e^- \)

\( k_t \) originally designed in \( e^+e^- \), most widely used algorithm in \( e^+e^- \) (LEP)

\[ y_{ij} = 2 \min\{E_i^2, E_j^2\} \left(1 - \cos \theta_{ij}^2\right) \]

- can classify events using \( y_{23}, y_{34}, y_{45}, y_{56} \) ...
- resolution parameter related to minimum transverse momentum between jets
\( k_t / \text{Durham-algorithm in } e^+e^- \)

\( k_t \) originally designed in \( e^+e^- \), most widely used algorithm in \( e^+e^- \) (LEP)

\[
y_{ij} = 2 \min\{ E^2_i, E^2_j \} (1 - \cos \theta^2_{ij})
\]

- can classify events using \( y_{23}, y_{34}, y_{45}, y_{56} \) ...
- resolution parameter related to minimum transverse momentum between jets

**Satisfies fundamental requirements:**

1. **Collinear safe:** collinear particles recombine early on
2. **Infrared safe:** soft particles do not influence the clustering sequence

\[ \Rightarrow \text{collinear + infrared safety important: it means that cross-sections can be computed at higher order in } p\text{QCD (no divergences)!} \]
The CA and the anti-\( k_t \) algorithm

The Cambridge/Aachen: sequential algorithm like \( k_t \), but uses only angular properties to define the distance parameters

\[
d_{ij} = \frac{\Delta R_{ij}^2}{R^2} \quad d_{iB} = 1 \quad \Delta R_{ij}^2 = (\phi_i - \phi_j)^2 + (y_i - y_j)^2
\]

*Dotshitzer et. al ’97; Wobisch & Wengler ’99*
The CA and the anti-\(k_t\) algorithm

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The anti-\(kt\) algorithm: designed not to recombine soft particles together

\[
d_{ij} = \min\{1/k_{ti}^2, 1/k_{tj}^2\} \frac{\Delta R_{ij}^2}{R^2} \quad d_{iB} = 1/k_{ti}^2
\]

*Cacciari, Salam, Soyez ’08*
Recombination schemes in e^+e^-

Given two massless momenta $p_i$ and $p_j$ how does one recombine them to build $p_{ij}$? Several choices are possible.

**Most common ones:**

1. **E-scheme**  
   
   $p_{ij} = p_i + p_j$

2. **E₀-scheme**  
   
   $\vec{p}_{ij} = \vec{p}_i + \vec{p}_j$  
   
   $E_{ij} = |\vec{p}_{ij}|$

3. **P₀-scheme**  
   
   $E_{ij} = E_i + E_j$  
   
   $\vec{p}_{ij} = \frac{E_{ij}}{|\vec{p}_i + \vec{p}_j|}(\vec{p}_i + \vec{p}_j)$

**E₀/P₀-schemes give massless jets**, along with the idea that the hard parton underlying the jet is massless.

**E-scheme give massive jets.** Most used in recent analysis.
Recombination schemes in hh

Most common schemes:

- E-scheme (as in e+e-)
- $p_t$, $p_t^2$, $E_t$, $E_t^2$ schemes
  - first preprocessing, i.e. make particles massless, rescaling the 3-momentum in the $E_t$, $E_t^2$ schemes or the energy in the $p_t$, $p_t^2$ schemes
  - then define
    \[
    p_{t,ij} = p_{t,i} + p_{t,j}
    \]
    \[
    \phi_{ij} = (w_i \phi_i + w_j \phi_j) / (w_i + w_j)
    \]
    \[
    y_{ij} = (w_i y_i + w_j y_j) / (w_i + w_j)
    \]

where the weights $w_i$ are $p_{ti}$ for the $p_t$, $E_t$ schemes and $p_{ti}^2$ for the $p_t^2$ and $E_t^2$ schemes

NB: a jet-algorithm is fully specified only once all parameters and the recombination scheme is specified too
Cone algorithms

1. A particle $i$ at rapidity and azimuthal angle $(y_i, \Phi_i) \subset$ cone $C$ iff

$$\sqrt{(y_i - y_C)^2 + (\phi_i - \phi_C)^2} \leq R_{cone}$$
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2. Define

$$\bar{y}_C \equiv \frac{\sum_{i \in C} y_i \cdot p_{T,i}}{\sum_{i \in C} p_{T,i}} \quad \bar{\phi}_C \equiv \frac{\sum_{i \in C} \phi_i \cdot p_{T,i}}{\sum_{i \in C} p_{T,i}}$$
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3. If weighted and geometrical averages coincide $(y_C, \phi_C) = (\bar{y}_C, \bar{\phi}_C)$ a stable cone ($\Rightarrow$ jet) is found, otherwise set $(y_C, \phi_C) = (\bar{y}_C, \bar{\phi}_C)$ & iterate
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4. Stable cones can overlap. Run a split-merge on overlapping jets: merge jets if they share more than an energy fraction $f$, else split them and assign the shared particles to the cone whose axis they are closer to.

Remark: too small $f$ ($<0.5$) creates huge jets, not recommended
Cone algorithms

- The question is where does one start looking for stable cone?
- The direction of these trial cones are called seeds
- Ideally, place seeds everywhere, so as not to miss any stable cone
- Practically, this is unfeasible. Speed of recombination grows fast with the number of seeds. So place only some seeds, e.g. at the \((y, \Phi)\)-location of particles.
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Seeds make cone algorithms infrared unsafe
Jets: infrared unsafety of cones

3 hard \Rightarrow 2 \text{ stable cones} \quad 3 \text{ hard } + 1 \text{ soft } \Rightarrow 3 \text{ stable cones}

Soft emission changes the hard jets \Rightarrow \text{algorithm is IR unsafe}

Midpoint algorithm: take as seed position of emissions \text{and midpoint between two emissions} (postpones the infrared safety problem)
Seedless cones

Solution:
use a seedless algorithm, i.e. consider all possible combinations of particles as candidate cones, so find all stable cones \([\Rightarrow\) jets\]

Blazey ’00
Seedless cones

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The problem:
clustering time growth as \( N2^N \). So for an event with 100 particles need \( 10^{17} \) ys to cluster the event \( \Rightarrow \) prohibitive beyond PT \( (N=4,5) \)
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Better solution:
\textit{SISCone} recasts the problem as a computational geometry problem, the identification of all distinct circular enclosures for points in 2D and finds a solution to that \(\Rightarrow N^2 \ln N\) time IR safe algorithm

\textit{Salam, Soyez ’07}
**IR safety test & time comparisons**

IR safety test: take a random hard event, add very soft emissions, count the number of times the hard jets change due to soft emissions.

<table>
<thead>
<tr>
<th>Method</th>
<th>Fraction (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>JetClu</td>
<td>50.1%</td>
</tr>
<tr>
<td>SearchCone</td>
<td>48.2%</td>
</tr>
<tr>
<td>MidPoint</td>
<td>16.4%</td>
</tr>
<tr>
<td>Midpoint-3</td>
<td>15.6%</td>
</tr>
<tr>
<td>PxCone</td>
<td>9.3%</td>
</tr>
<tr>
<td>Seedless [SM-p_{T}]</td>
<td>1.6%</td>
</tr>
<tr>
<td>Seedless [SM-MIP]</td>
<td>0.17%</td>
</tr>
<tr>
<td>0 (none in 4x10^9)</td>
<td>Seedless (SISCone)</td>
</tr>
</tbody>
</table>

Fraction of hard events failing IR safety test

![Graph showing run time comparisons](image)
Physical impact of infrared unsafety

Up to 40% difference in mass spectrum

IR-unsafety is an issue at the LHC

<table>
<thead>
<tr>
<th>Observable</th>
<th>1st miss cones at</th>
<th>Last meaningful order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inclusive jet cross section</td>
<td>NNLO</td>
<td>NLO</td>
</tr>
<tr>
<td>3 jet cross section</td>
<td>NLO</td>
<td>LO (NLO in NLOJet)</td>
</tr>
<tr>
<td>(W/Z/H + 2) jet cross sect.</td>
<td>NLO</td>
<td>LO (NLO in MCFM)</td>
</tr>
<tr>
<td>jet masses in 3 jets</td>
<td>LO</td>
<td>none (LO in NLOJet)</td>
</tr>
</tbody>
</table>

If you don’t want theoretical efforts to be wasted!
Jet area

Given an infrared safe, fast jet-algorithm, can define the jet area $A$ as follows: fill the event with an infinite number of infinitely soft emissions uniformly distributed in $\eta$-$\phi$ and make $A$ proportional to the # of emissions clustered in the jet.

NB: cone, not circular!

NB: new anti-kt
What jet areas are good for

jet-area ≡ catching area of the jet when adding soft emissions

⇒ use the jet area to formulate a simple area based subtraction of pile-up events

1. cluster particle with an IR safe jet algorithm
2. from all jets (most are pile-up ones) in the event define the median

\[
\rho = \frac{p_{t,j}}{A_j}
\]

3. the median gives the typical \( p_t/A_j \) for a given event
4. use the median to subtract off dynamically the soft part of the soft events

\[
p_{j,\text{sub}} = p_j - A_j \rho
\]

Pileup = generic p-p interaction (hard, soft, single-diffractive...) overlapping with hard scattering
Sample 2 TeV mass reconstruction

\[ p_{ij}/A_j \text{ [GeV]} \]

\[ k_t \text{ algorithm, } R=0.5 \]
Sample 2 TeV mass reconstruction

Cacciari et al. ’07
Quality measures of jets

Suppose you are searching for a heavy state $({H \rightarrow gg, Z' \rightarrow qq, \ldots})$

The object is reconstructed through its decay products
$
\Rightarrow \text{Which jet algorithm (JA) is best? Does the choice of R matter?}
$

**Define:** $Q^w_{f}(JA, R) \equiv \text{width of the smallest mass window that contains a fraction } f \text{ of the generated massive objects}$

- good algo $\Leftrightarrow$ small $Q^w_{f}(JA, R)$
- ratios of $Q^w_{f}(JA,R)$: mapped to ratios of effective luminosity (with same $S/\sqrt{B}$)

$$
\mathcal{L}_2 = \rho_\mathcal{L} \mathcal{L}_1 \\
\rho_\mathcal{L} = \frac{Q^f_{f}(JA_2, R_2)}{Q^f_{f}(JA_1, R_1)}
$$

![Graph showing W reconstruction and anti-k_t (R=0.4) and anti-k_t (R=0.7) at 10.75 GeV and 15.0 GeV]
Quality measures: sample results

NB: Here “fake Higgs” = narrow resonance decaying to gluons

- At 100GeV: use a Tevatron standard algo ($k_t$, $R=0.7$) instead of best choice (SISCone, $R=0.6$) ⇒ lose $\rho_L = 0.8$ in effective luminosity
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- At 2 TeV: use $M_{Z'}=100$GeV best choice (or $k_t$) instead SISCone, $R=1.1$ ⇒ lose $\rho_L = 0.6$ in effective luminosity
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- At 2 TeV: use $M_{Z'}=100\text{GeV}$ best choice (or $k_t$) instead SISCone, R=1.1 ⇒ lose $\rho_L = 0.6$ in effective luminosity

A good choice of jet-algorithm does matter!
Bad choice of algo ⇔ lost in discrimination power!
Jet substructure: $Z/W + H \rightarrow bb$

$\Rightarrow$ Light Higgs hard: Higgs mainly produced in association with $Z/W$, decay $H \rightarrow bb$ is dominant, but overwhelmed by QCD backgrounds
Recall why searching for $pp \rightarrow WH(bb)$ is hard:

$$\sigma(pp \rightarrow WH(bb)) \sim \text{few pb} \quad \sigma(pp \rightarrow Wbb) \sim \text{few pb}$$

$$\sigma(pp \rightarrow tt) \sim 800\text{pb} \quad \sigma(pp \rightarrow Wjj) \sim \text{few } 10^4\text{pb} \quad \sigma(pp \rightarrow bb) \sim 400\text{pb}$$

$\Rightarrow$ signal extraction very difficult

**Conclusion [ATLAS TDR]:**

The extraction of a signal from $H \rightarrow bb$ decays in the WH channel will be very difficult at the LHC even under the most optimistic assumptions [...]
Z/W+ H (→bb) rescued?

But ingenious suggestions open up to window of opportunity

Central idea: require high-p_T W and Higgs boson in the event

- leads to back-to-back events where two b-quarks are contained within the same jet
- high p_T reduces the signal but reduces the background much more
- improve acceptance and kinematic resolution
Then use a jet-algorithm geared to exploit the specific pattern of $H \rightarrow bb$ vs $g \rightarrow gg, q \rightarrow gg$

- QCD partons prefer soft emissions (hard $\rightarrow$ hard + soft)
- Higgs decay prefers symmetric splitting
- try to beat down contamination from underlying event
- try to capture most of the perturbative QCD radiation

1. **cluster** the event with e.g. CA algo and large-ish $R$
2. undo last recomb: **large mass drop** + symmetric + $b$ tags
3. **filter** away the UE: take only the 3 hardest sub-jets
Z/W+ H (→bb) rescued?

Mass of the three hardest sub-jets:

- with common & channel specific cuts:
  \( p_{tV}, p_{tH} > 200 \text{GeV} \), ...
- real/fake b-tag rate: 0.7/0.01
- NB: very neat peak for WZ (Z → bb)
  Important for calibration

\textit{Butterworth, Davison, Rubin, Salam ’08}

Suggested to have 5.9\(\sigma\) at 30 fb\(^{-1}\). This and other works opened a new field of jet-substructure… (would be a whole new lecture)
Recap on jets

Two major jet classes: sequential ($k_t$, CA, ...) and cones (UA1, midpoint, ...)

Jet algo is fully specified by: clustering + recombination + split merge or removal procedure + all parameters

Standard cones based on seeds are IR unsafe

SISCone is a infrared safe cone algorithm (no seeds)

anti-kt a sequential algorithm used in most analyses now

Using IR unsafe algorithms you can not use perturbative QCD calculations

IR safe algorithm: sophisticated studies e.g. jet-area for pile-up subtraction

Not all algorithms fare the same for BSM searches: quality measures

Very active novel field of jet substructure [example of $ZH(bb)$ with]
Parton shower & Monte Carlo methods

today one can compute infrared-safe quantities at NLO, NNLO and very few ones at $N^3$LO. Progress is steady but somehow limited.

Fixed-order calculations involve few particles in the final state. This is quite different from “realistic” LHC events with hundreds of particles in the detectors.

we have also seen that sometimes large logs spoil the convergence of perturbative calculations, i.e. NLO (NNLO…) becomes unreliable.

now we adopt a different approach: we seek for an approximate result such that enhanced terms are taken into account to all orders.

this will lead to a ‘parton shower’ picture, which can be implemented in computer simulations, usually called Monte Carlo programs or event generators.
Parton shower & Monte Carlo methods
Parton branching: the time-like case

Assume: $p^2_b, p^2_c \ll p^2_a \equiv t$ (scale of the branching)

$$p_a = (E_a, 0, 0, p_{az})$$
$$p_b = (E_b, 0, E_b \sin \theta_b, E_b \cos \theta_b)$$
$$p_c = (E_c, 0, -E_c \sin \theta_c, E_c \cos \theta_c)$$

Time-like branching: $t > 0$

Kinematics:

$$z = \frac{E_b}{E_a} = 1 - \frac{E_c}{E_a}$$

$$t = (p_b + p_c)^2 = 2E_bE_c(1 - \cos \theta) \sim z(1 - z)E_a^2\theta^2$$

$$E_b \sin \theta_b = E_c \sin \theta_c \Rightarrow z\theta_b \sim (1 - z)\theta_c$$

$$\theta = \theta_b + \theta_c = \frac{\theta_b}{1 - z} = \frac{\theta_c}{z}$$
Parton branching: gluon case

Three-gluon vertex:

$$V_{ggg} = ig_s f_{ABC} \epsilon_a^\mu \epsilon_b^\nu \epsilon_c^\rho (g_{\mu \nu} (p_a - p_b)_\rho + g_{\nu \rho} (p_b - p_c)_\mu + g_{\rho \mu} (p_c - p_a)_\nu)$$
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Use: \( \epsilon_i \cdot p_i = 0 \) and \( p_a + p_b + p_c = 0 \)

\[ V_{ggg} = -2ig_s f_{ABC} [(\epsilon_a \cdot \epsilon_b) (\epsilon_c \cdot p_b) - (\epsilon_b \cdot \epsilon_c) (\epsilon_a \cdot p_b) - (\epsilon_c \cdot \epsilon_a) (\epsilon_b \cdot p_c)] \]
Parton branching: gluon case

Three-gluon vertex:

\[ V_{ggg} = i g_s f_{ABC} \epsilon^\mu_a \epsilon^\nu_b \epsilon^\rho_c (g_{\mu\nu}(p_a - p_b)_{\rho} + g_{\nu\rho}(p_b - p_c)_{\mu} + g_{\rho\mu}(p_c - p_a)_{\nu}) \]

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\[ V_{ggg} = -2i g_s f_{ABC} [(\epsilon_a \cdot \epsilon_b)(\epsilon_c \cdot p_b) - (\epsilon_b \cdot \epsilon_c)(\epsilon_a \cdot p_b) - (\epsilon_c \cdot \epsilon_a)(\epsilon_b \cdot p_c)] \]

Branching: in a plane. Natural to split polarization vectors in \( \epsilon_i^{\text{in}} \) and \( \epsilon_i^{\text{out}} \)

Properties:

\[ \epsilon^{\text{in}}_i \cdot \epsilon^{\text{in}}_j = \epsilon^{\text{out}}_i \cdot \epsilon^{\text{out}}_j = -1 \quad \epsilon^{\text{in}}_i \cdot \epsilon^{\text{out}}_j = \epsilon^{\text{out}}_i \cdot p_j = 0 \]

Explicitly:

\[ \epsilon^{\text{in}}_a = (0, 0, 1, 0) \]
\[ \epsilon^{\text{in}}_b = (0, 0, \cos \theta_b, -\sin \theta_b) \]
\[ \epsilon^{\text{in}}_c = (0, 0, \cos \theta_c, \sin \theta_c) \]

\[ \epsilon^{\text{in}}_a \cdot p_b = -E_b \theta_b = -z(1 - z)E_a \theta \]
\[ \epsilon^{\text{in}}_b \cdot p_c = E_c \theta = (1 - z)E_a \theta \]
\[ \epsilon^{\text{in}}_c \cdot p_b = -E_b \theta = -zE_a \theta \]
Parton branching: the gluon case

Squared matrix element for n+1 partons becomes:

$$|\mathcal{M}_{n+1}|^2 = \frac{4g_s^2}{t} C_A F(z; \epsilon_a, \epsilon_b, \epsilon_c) |\mathcal{M}_n|^2$$

NB: one “t” cancels completely

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>$F(z; \epsilon_a, \epsilon_b, \epsilon_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>in</td>
<td>in</td>
<td>in</td>
<td>$(1-z)/z + z/(1-z) + z(1-z)$</td>
</tr>
<tr>
<td>in</td>
<td>out</td>
<td>out</td>
<td>$z(1-z)$</td>
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<tr>
<td>out</td>
<td>in</td>
<td>out</td>
<td>$(1-z)/z$</td>
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<tr>
<td>out</td>
<td>out</td>
<td>in</td>
<td>$z/(1-z)$</td>
</tr>
</tbody>
</table>

Averaging over incoming and summing over outgoing pol. we get

$$C_A \langle F \rangle = \hat{P}_{gg} = C_a \left[ \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right]$$
The gluon case: remarks

Soft singularities \((z \to 0,1)\) are associated to soft gluon in the plane of the branching.

Correlation between plane of branching and polarization of incoming gluon: take polarization of gluon at an angle \(\phi\) to the plane then

\[
F_\phi = \sum_{b,c} |\cos \phi \mathcal{M}(\epsilon^\text{in}_a, \epsilon_c, \epsilon_c) + \sin \phi \mathcal{M}(\epsilon^\text{out}_a, \epsilon_c, \epsilon_c)|^2
\]

\[
= \frac{1 - z}{z} + \frac{z}{1 - z} + z(1 - z) + z(1 - z) \cos 2\phi
\]

Correction favors polarization of branching gluon in the branching plane, but is weak (no soft enhancements).
Gluon splitting to quarks

Similarly start from 3-particle vertex:

\[ V_{qg\bar{q}} = -i g_s t^A_{bc} \bar{u}(p_b) \gamma_\mu \epsilon^\mu_a v(p_c) \]

Fix a representation of the Dirac algebra (called Dirac rep.):

\[ \gamma^0 = \begin{pmatrix} 1_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -1_{2\times2} \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0_{2\times2} & \sigma_i \\ -\sigma_i & 0_{2\times2} \end{pmatrix} \]

To first order in the small angles the spinors are

\[
\frac{u_+(p_b)}{\sqrt{E_b}} = \begin{pmatrix} 1 \\ \theta_b/2 \\ 1 \\ \theta_b/2 \end{pmatrix}, \quad \frac{u_-(p_b)}{\sqrt{E_b}} = \begin{pmatrix} \theta_b/2 \\ -1 \\ \theta_b/2 \\ -1 \end{pmatrix}, \quad \frac{v_+(p_c)}{\sqrt{E_c}} = i \begin{pmatrix} -\theta_c/2 \\ -1 \\ \theta_c/2 \\ 1 \end{pmatrix}, \quad \frac{v_-(p_c)}{\sqrt{E_c}} = i \begin{pmatrix} -1 \\ \theta_c/2 \\ -1 \\ \theta_c/2 \end{pmatrix}
\]
Gluon splitting to quarks

Explicitly we find e.g.

\[-i g_s \bar{u}_+(p_b) \gamma_\mu \epsilon^{a.\mu}_a \nu_-(p_c) = \sqrt{E_b E_c} (\theta_b - \theta_c) = \sqrt{z(1-z)(1-2z)} E_a \theta \]

Similarly to before define

\[ |M_{n+1}|^2 = \frac{4g_s^2}{t} T_R F(z; \epsilon_a, \lambda_b, \lambda_c) |M_n|^2 \]

Averaged splitting function:

\[ T_R \langle F \rangle \equiv \hat{P}_{qg}(z) = T_R \left[ z^2 + (1-z)^2 \right] \]

Angular correlation:

\[ F_\phi = z^2 + (1-z)^2 - 2z(1-z) \cos 2\phi \] (more important)
Last case: quark emitting gluon

Similarly to the two previous cases one obtains

\[ |M_{n+1}|^2 = \frac{4g_s^2}{t} C_F F(z; \lambda_a, \lambda_b, \epsilon_c) |M_n|^2 \]

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<thead>
<tr>
<th></th>
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<th>F(z; \lambda_a, \lambda_b, \epsilon_c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>in</td>
<td>±</td>
<td>±</td>
<td></td>
<td>(1+z)^2/(1-z)</td>
</tr>
<tr>
<td>out</td>
<td>±</td>
<td>±</td>
<td></td>
<td>1-z</td>
</tr>
</tbody>
</table>

NB: helicity of the quark does not change during the branching

Averaged splitting function:

\[ C_F \langle F' \rangle \equiv \hat{P}_{qq}(z) = C_F \frac{1 + z^2}{1 - z} \]

Angular correlation:

\[ F_\phi = \frac{1 + z^2}{1 - z} + \frac{2z}{1 - z} \cos 2\phi \]
Phase space

n-particle phase space (without branching): \(d\Phi_n = d\Phi_{n-1} \frac{d^3 p_a}{(2\pi)^3 2E_a}\)

(n+1)-particle phase space (with branching): \(d\Phi_{n+1} = d\Phi_{n-1} \frac{d^3 p_b}{(2\pi)^3 2E_b} \frac{d^3 p_c}{(2\pi)^3 2E_c}\)

At fixed \(p_b\): \(d^3 p_a = d^3 p_c \Rightarrow d\Phi_{n+1} = d\Phi_n \frac{d^3 p_b}{(2\pi)^3 2E_b} \frac{E_a}{E_c}\)

\[
\begin{align*}
    d^3 p_b &= p_b^2 dp_b \sin \theta d\theta d\phi \\
    &\sim E_b^2 dE_b \theta d\theta d\phi \\
    &= E_a^3 z^2 dz \frac{dt}{2z(1-z)E_a^2} d\phi
\end{align*}
\]

N-particle cross-section: \(d\sigma_n = F |\mathcal{M}_n|^2 d\Phi_n\) with \(|\mathcal{M}_{n+1}|^2 = \frac{4g_s^2}{t} CF |\mathcal{M}_n|^2\)

\[
d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz d\phi \frac{\alpha_s}{2\pi} CF
\]
Azimuthal averaged result

Averaging over azimuthal angles:

\[ \int \frac{d\phi}{2\pi} C F = \hat{P}_{ba}(z) \]

The evolution equation becomes:

\[ d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_s}{2\pi} \hat{P}_{ba}(z) \]
Space-like branching

What are the modifications needed if an incoming parton splits?

The kinematics changes: \( p_a^2, p_c^2 \ll |p_b^2| \equiv t \)

Space-like branching: \( t < 0 \)

Small angle approximation: \( t = E_a E_c \theta_c^2 \) (verify)

\((n+1)\) particle phase space becomes: \( d\Phi_{n+1} = d\Phi_n \frac{1}{4(2\pi)^3} dt \frac{dz}{z} d\phi \)

The additional “\(z\)” is compensated by the different flux-factor, we find

Space-like or time-like branching: \( d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_s}{2\pi} \hat{P}_{ba}(z) \)
Perturbative evolution

In exact analogy with what done for parton densities inside hadrons we want to write an evolution equation for the probability to have partons at the momentum scale $Q^2$ with momentum fraction $z$ during PT branching.

Start from DGLAP equation

$$Q^2 \frac{\partial f(x, Q^2)}{\partial Q^2} = \int_0^1 dz \frac{\alpha_s}{2\pi} \hat{P}(z) \left( \frac{1}{z} f \left( \frac{x}{z}, Q^2 \right) - f(x, Q^2) \right)$$

Introduce a cut-off to regulate divergences

$$Q^2 \frac{\partial f(x, Q^2)}{\partial Q^2} = \int_0^{1-\epsilon} dz \frac{\alpha_s}{2\pi} \hat{P}(z) f \left( \frac{x}{z}, Q^2 \right) - f(x, Q^2) \int_0^{1-\epsilon} dz \frac{\alpha_s}{2\pi} \hat{P}(z)$$

Introduce a Sudakov form factor

$$\Delta(Q^2) = \exp \left\{ - \int_{Q_0}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^{1-\epsilon} dz \frac{\alpha_s}{2\pi} \hat{P}(z) \right\}$$
Perturbative evolution

The DGLAP equation becomes

$$Q^2 \frac{\partial}{\partial Q^2} \left( \frac{f(x, Q^2)}{\Delta(Q^2)} \right) = \frac{1}{\Delta(Q^2)} \int_0^{1-\epsilon} \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z) f \left( \frac{x}{z}, Q^2 \right)$$

Integrating the above equation one gets

$$f(x, Q^2) = f(x, Q_0^2) \frac{\Delta(Q^2)}{\Delta(Q_0^2)} + \int_{Q_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \frac{\Delta(Q^2)}{\Delta(k_\perp^2)} \int_0^{1-\epsilon} \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z) f \left( \frac{x}{z}, k_\perp^2 \right)$$

This equation has a \textbf{probabilistic interpretation}

- \textbf{First term:} probability of evolving from $Q_0^2$ to $Q^2$ without emissions (ratio of Sudakovs $\Delta(Q^2)/\Delta(Q_0^2)$)
- \textbf{Second term:} emission at scale $k_\perp^2$ and evolution from $k_\perp^2$ to $Q^2$ without further emissions
Multiple branchings

Multiple branching can now be described using the above probabilistic equation.

Denote by $t$ the evolution variable (e.g. $t = Q^2$)
Start from one parton at scale $t_1$ and momentum fraction $x_1$

The question is how to generate the values of $t_2, x_2$ and $\varphi_2$
Multiple branchings

1. $t_2$ generated with the correct probability by solving the equation ($r = \text{random number in } [0,1]$)

$$\frac{\Delta(t_1)}{\Delta(t_2)} = r$$

If $t_2$ smaller than cut-off evolution stops (no further branching)
Multiple branchings

1. \( t_2 \) generated with the correct probability by solving the equation 
   \( \Delta(t_1)/\Delta(t_2) = r \) 
   ( \( r = \) random number in \([0,1]\) )

   If \( t_2 \) smaller than cut-off evolution stops (no further branching)

2. Else, generate momentum fraction \( z = x_2/x_1 \) with
   \[ \text{Prob.} \sim \frac{\alpha_s}{2\pi} P(z) \]

   \[ \int_{\epsilon}^{x_2/x_1} dz \frac{\alpha_s}{2\pi} P(z) = r' \int_{\epsilon}^{1-\epsilon} dz \frac{\alpha_s}{2\pi} P(z) \]

   \( \epsilon: \) IR cut-off for resolvable branching
Multiple branchings

1. $t_2$ generated with the correct probability by solving the equation
   ( $r = \text{random number in } [0,1]$ )
   \[
   \Delta(t_1)/\Delta(t_2) = r
   \]
   If $t_2$ smaller than cut-off evolution stops (no further branching)

2. Else, generate momentum fraction $z = x_2/x_1$ with $\text{Prob.} \sim \frac{\alpha_s}{2\pi} P(z)$
   \[
   \int_{\epsilon}^{x_2/x_1} dz \frac{\alpha_s}{2\pi} P(z) = r' \int_{\epsilon}^{1-\epsilon} dz \frac{\alpha_s}{2\pi} P(z)
   \]
   $\epsilon$: IR cut-off for resolvable branching

3. Azimuthal angles: generated uniformly in $(0, 2\pi)$ (or taking into account polarization correlations)
Space-like vs time-like evolution

**Time-like:** $t$ evolves from a hard-scale downwards to an IR cut-off

$$Q > t_1 > t_2 > \cdots > Q_0$$

**Space-like:** $t$ increases in the evolution up to the hard scale $Q^2$

$$Q_0 < t_1 < t_2 < \cdots, Q$$

Each outgoing parton becomes a source of the new branching until the “no-branching” step is met *(cut-off essential in parton shower)*

$\Rightarrow$ a parton cascade develops, when all branchings are done partons are converted into hadrons via a hadronization model
Backward evolution

In space-like cases it is more convenient to start from the momentum fraction of the outgoing parton $x_n$ and generate $x_{n-1}, \ldots, x_0$ by backward evolution.

Essentially, the evolution proceeds as before but with a modified form factor which take the local parton density into account.

We will not discuss backward evolution, despite its wide-spread use.
Angular ordering

*In the branching formalism discussed now we considered collinear enhancements to all orders in PT. But there are also soft enhancements.*

When a soft gluon is radiated from a \((p_ip_j)\) dipole one gets a universal eikonal factor

\[
\omega_{ij} = \frac{p_ip_j}{p_ip_j} = \frac{1 - \nu_i \nu_j \cos \theta_{ij}}{\omega_k^2 (1 - \nu_i \cos \theta_{ik})(1 - \nu_j \cos \theta_{jk})}
\]

Massless emitting lines \(\nu_i = \nu_j = 1\), then

\[
\omega_{ij} = \omega_{ij}^{[i]} + \omega_{ij}^{[j]}
\]

\[
\omega_{ij}^{[i]} = \frac{1}{2} \left( \omega_{ij} + \frac{1}{1 - \cos \theta_{ik}} - \frac{1}{1 - \cos \theta_{jk}} \right)
\]

Angular ordering

\[
\int_0^{2\pi} \frac{d\phi}{2\pi} \omega_{ij}^{[i]} = \begin{cases} 
\frac{1}{\omega_k^2 (1-\cos \theta_{ik})} & \text{if } \theta_{ik} < \theta_{ij} \\
0 & \text{if } \theta_{ik} > \theta_{ij}
\end{cases}
\]

*Proof: see e.g. QCD and collider physics, Ellis, Stirling, Webber*
Angular ordering & coherence

A. O. is a manifestation of coherence of radiation in gauge theories

In QED
suppression of soft bremsstrahlung from an e+e- pair (Chudakov effect)
At large angles the e^+e^- pair is seen coherently as a system without total charge ⇒ radiation is suppressed
Angular ordering & coherence

Coherent $a \rightarrow b + c$ branching: replace the ordering variable $t = p_a^2$ with

$$\zeta = \frac{p_b p_c}{E_b E_c} \sim 1 - \cos \theta_{bc}$$

and require $\zeta' < \zeta$ at successive branchings

The basic formula for coherent branching

$$d\sigma_{n+1} = d\sigma_n \frac{d\zeta}{\zeta} dz \frac{\alpha_s}{2\pi} \hat{P}_{ba}(z)$$

NB: need collinear cut-off. Simplest choice: $\zeta_0 = \frac{t_0}{E^2}$
AO: time like vs space-like case

NB: angles decrease when moving away from the hard vertex, i.e. in the space-like case angles increase during the evolution.
Accuracy issue

Formally, Monte Carlos are Leading Logs showers
✦ because they don’t include any higher order corrections to the $1 \rightarrow 2$ splitting
✦ because they don’t have any $1 \rightarrow 3$ splittings
✦ ....
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However, they fare better than analytic Leading Log calculations
• because they have energy conservation (NLO effect) implemented
• because they have coherence
• because they have optimized choices for the coupling
• because they provide an exclusive description of the final state
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However, they fare better than analytic Leading Log calculations
• because they have energy conservation (NLO effect) implemented
• because they have coherence
• because they have optimized choices for the coupling
• because they provide an exclusive description of the final state

So, despite not guaranteeing NLL accuracy, they fare usually better than Leading Log analytic calculations

The real issue is that it is very difficult to estimate the uncertainty
Warning

The above discussion is a simplification

› many details/subtleties not discussed enough, some not at all
› various MC differ in the choice of the ordering variable and in many details, but the basic idea remains the same
› purpose was to give an overall idea of how Monte Carlos and what they can/can’t do
Recap on Monte Carlos

- **parton evolution as branching process** from higher to lower $x$
- parton shower based on **Sudakov form factor** (Prob. of evolving without branching) with corresponding **evolution equation**
- branching described by picking randomly 3 numbers $(t, x, \varphi)$ with the right prob. distributions
- virtuality ordered shower: **collinear approximation**
- angular ordering needed to describe also soft effects
- parton shower supplemented by hadronization + U.E. (various models $\Rightarrow$ MC tuning) $\Rightarrow$ **full event generator**
- by construction PS fail to describe multiple hard radiation
- Lots of work on **merging/matching parton shower and fixed order calculations** (POWHEG, MC@NLO, NNLOPS …)