Quantum Integrability and Algebraic Geometry

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References

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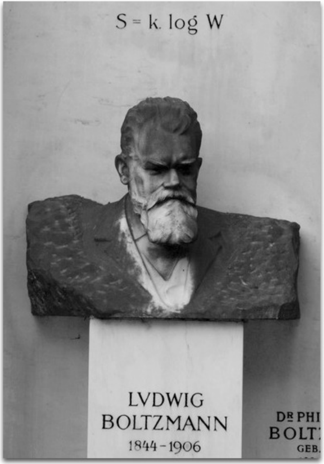
Classical Lattice Models of Statistical Thermodynamics



- Is the framework of Statistical Thermodynamics able to handle Phase Transitions ?

Brief History of Magnetic Phase Transition

- Pierre Curie, Paul Langevin and Pierre Weiss (1895-1910):

Boltzmann's framework for non-interacting micro-magnets.

- Wilhelm Lenz and Ernest Ising (1920):

Basic elements are dipoles which turn over among two positions,

$$Z = \sum_{\sigma_1=\pm} \sum_{\sigma_2=\pm} \cdots \sum_{\sigma_N=\pm} \exp[-\beta E(\sigma_1, \ldots, \sigma_N)], \quad \beta = \frac{1}{k_B T}$$

- Werner Heisenberg and Paul Dirac (1928-1929):

$$E = \sum_{i,j} J_{ij}^x \sigma_i^x \sigma_j^x + J_{ij}^y \sigma_i^y \sigma_j^y + J_{ij}^z \sigma_i^z \sigma_j^z$$

- Rudolf Pierls (1936):

2D Ising model display spontaneous magnetization for low temperatures prompting new interest.

Brief History of Magnetic Phase Transition

- Hendrick Kramers and Gregory Wannier (1941):

In 2D derived the Curie temperature and a combinatorial sums translated into a linear algebra problem,

$$Z = Tr[T^N]$$

- Lars Onsager (1944):

2D Lenz-Ising model is exactly solvable since $\Lambda_0 > \Lambda_1 > \dots$ was found,

$$Z = \Lambda_0^N \left[1 + (\frac{\Lambda_0}{\Lambda_1})^N + \dots \right]$$

The free-energy has no power law singularity as hypothesized,

$$f_s \sim |T - T_c|^2 \log(|T - T_c|)$$

- Leo Kadanoff and Kenneth Wilson (1963-1975):

At criticality we have scale invariance and the Curie critical point is universal.

Hamiltonian Limit

Thermal fluctuations (D+1) classical system $\sim$ quantum effects D-spatial field theory,

- Path Integral with space-time lattice,

$$K(x_a, t_a, x_b, t_b) = \langle x_a| \exp\left[-iH(t_b - t_a)\right] |x_b\rangle$$

$$= \int dx_{N-1} \ldots dx_1 \langle x_b| T |x_{N-1}\rangle \langle x_{N-1}| T |x_{N-2}\rangle \ldots \langle x_1| T |x_a\rangle$$

with $t_b - t_a = N\tau$ and $T = \exp(-i\tau H)$

- Amplitude with periodic boundary,

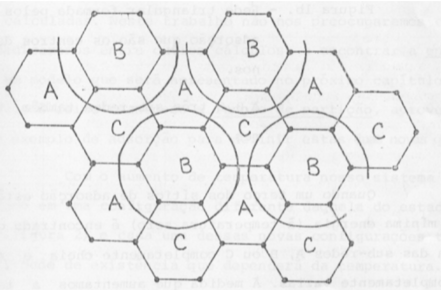
$$Z = \int K(x_0, N\tau, x_0, 0) dx_0 = Tr[T^N]$$

Free energy density	Vaccum energy density
Correlation function	Propagator
Correlation lenght	Inverse mass gap

Modeling Adsorption in Surfaces

- Greg Dash and Michael Bretz (1971):

Thin films of gases adsorbed on regular crystal surfaces: graphite has a hexagonal lattice.



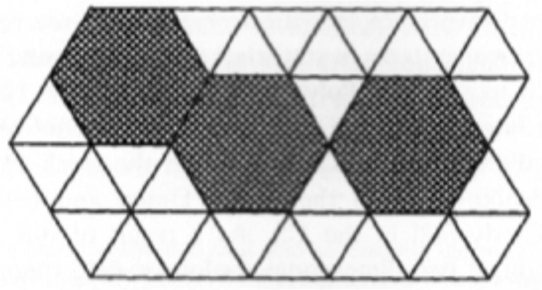
The gas atoms slightly larger than a basic hexagon and two adjacent hexagons cannot both be occupied.

$$C \sim |T - T_c|^{-\alpha}, \qquad \alpha \cong 0.36$$

Tiling the Triangular Lattice

- Rodney Baxter (1980):

Consider a triangular lattice and place hexagonal tiles without overlapping

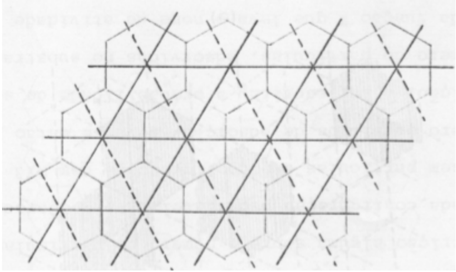


Let g(m,N) be the number of ways of placing m hexagons on N sites,

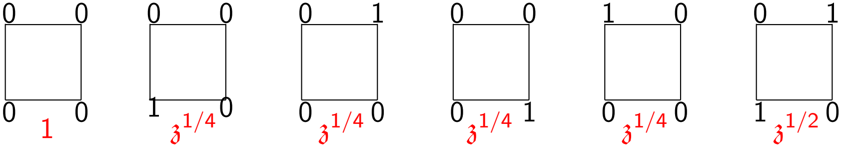
$$f(\mathfrak{z}) = \lim_{N\to\infty} \log\left[Z_N\right]/N, \qquad Z_N = \sum_{m=0}^{N/3} \mathfrak{z}^m g(m,N)$$

Interaction Around Face Model

Spin $\sigma = 0, 1$ and instead sites of adsorption one use face sites

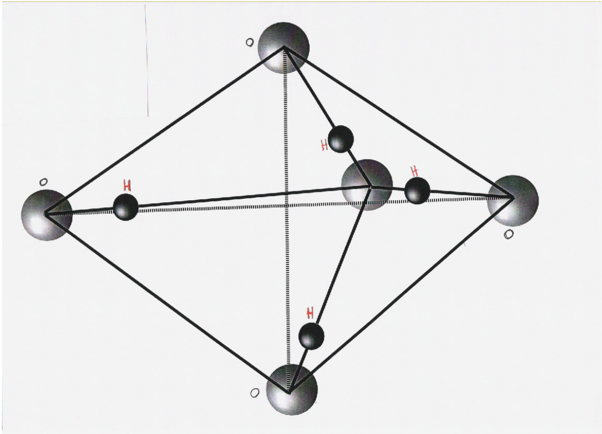


$\sigma_i \sigma_j = 0$ for all next-neighbors face variables



The Ice Model

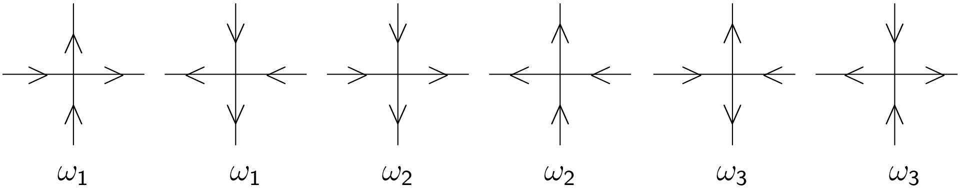
- Introduced by Pauling in 1935 to explain the experimental fact that certain phase of Ice has a residual entropy.



The lattice sites are occupied by Oxygens **O** having four nearest neighbors Hydrogens **H** atoms:O-O >>> O-H.

The Vertex Representation

- To make water molecule $\mathbf{H_2O}$ two Hydrogens are close to the central Oxygen and the other two are farther away.



- The statistical configuration sits on the edges and can be represented by an arrow whose tip points forwards the side where the Oxygen **O** is sited.

- The residual entropy can be computed,

$$S = k_B \log [\Lambda_0]$$

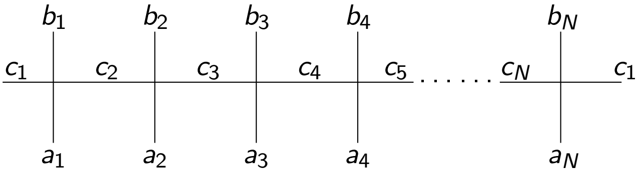
Brief History of Integrability

- Hans Bethe plane wave function (1931):

$$H = -\sum_{i,j} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \Delta \sigma_i^z \sigma_j^z$$

- Elliott Lieb for $\Delta = 1/2$ (1967):

$$\mathrm{T}_{a_1 \ldots a_N}^{b_1 \ldots b_N} =$$



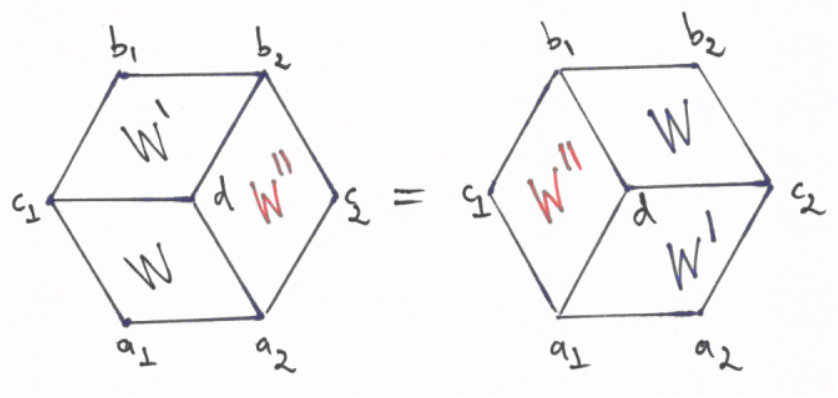
- Barry MacCoy and F. Wu (1968):

$$[T, H] = 0,$$

provided the weights sit in the quadric,

$$\omega_1^2 + \omega_2^2 - \omega_3^2 - 2\Delta\omega_1\omega_2 = 0.$$

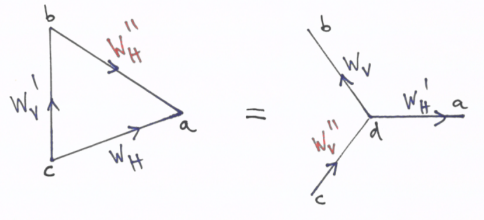
Face Models Yang-Baxter



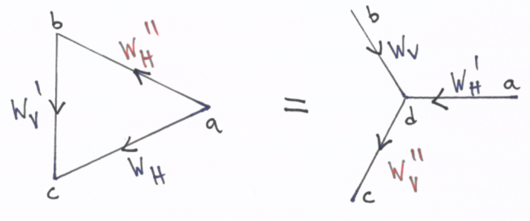
$$\sum_{d} W(a_1, a_2, d, c_1) W'(c_1, d, b_2, b_1) W''(d, a_2, c_2, b_2)$$

$$= \sum_{d} W''(c_1, a_1, d, b_1) W'(a_1, a_2, c_2, d) W(d, c_2, b_2, b_1)$$

Onsager Star-Triangle Relation

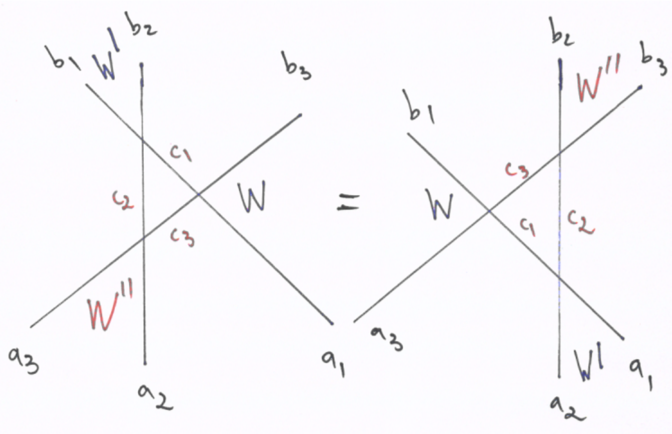


$$W_h(c,a)\,W_v'(c,b)\,W_h''(b,a) = \sum_d W_v''(c,d)\,W_h'(d,a)\,W_v(d,b)$$



$$W_h''(a,b)\,W_v'(b,c)\,W_h(a,c) = \sum_d W_v(b,d)\,W_h'(a,d)\,W_v''(d,c)$$

Vertex Models Yang-Baxter



$$\sum_{c_1,c_2,c_3} W''^{\,c_2,c_3}_{\,a_2,a_3} W^{c_1,b_3}_{a_1,c_3} W'^{\,b_1,b_2}_{\,c_1,c_2} = \sum_{c_1,c_2,c_3} W'^{\,c_1,c_2}_{\,a_1,a_2} W^{b_1,c_3}_{c_1,a_3} W''^{\,b_2,b_3}_{\,c_2,c_3}$$

- McGuire (1964), C.N. Yang (1968)