Basic Notions in Algebraic Geometry

Marcio J. Martins

Universidade Federal de São Carlos
Departamento de Física

July 2018

Ring and Fields

- Commutative Ring R:

Consists of a set R and two binary operations $*$ and $+$ with the following conditions for $a, b, c \in R$:

(i) Associative: $(a + b) + c = a + (b + c)$ and $(a * b) * c = a * (b * c)$,

(ii) Commutative: $a + b = b + a$ and $a * b = b * a$,

(iii) Distributive: $a * (b + c) = a * b + a * c$,

(iv) Identities: there are $0, 1 \in R$ such that $a + 0 = a * 1 = a$,

(v) Additive Inverse: given $a \in R$ there is $b \in R$ with $a + b = 0$,

- Field K:

(vi) Multiplicative Inverse: given $a \in K \quad a \neq 0$ there is $c \in K$ with $a * c = 1$.

Projective Space $\mathbb{P}^n(K)$

- Definition:

The n-dimensional projective space over a field K is a set of equivalence classes of $K^{n+1}/\{0, 0, \ldots, 0\}$ under the equivalence relation,

$$(x_0, x_1, \ldots, x_n) \sim (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$$

for any $\lambda \in K/\{0\}$.

- A way of looking at projective space:

Let $U_0 = \{(x_0, x_1, \ldots, x_n) \in \mathbb{P}^n(K) | x_0 \neq 0\}$ then

$$(x_0, x_1, \ldots, x_n) = (1, x_1, \ldots, x_n) \in \mathbb{A}^n(K)$$

The set $\mathbb{P}^n(K)/U_0$ is clearly a copy of $\mathbb{P}^{n-1}(K)$ and therefore,

$$\mathbb{P}^n(K) = \mathbb{A}^n(K) \cup \mathbb{P}^{n-1}(K)$$

The idea of Ideal

- A subset $I \subset K[x_1, \ldots, x_n]$ is an ideal if it satisfies:

(i) $0 \in I$

(ii) If $f, g \in I$ then $f + g \in I$,

(iii) If $f \in I$ and $h \in K[x_1, \ldots, x_n]$ then $f * h \in I$.

- Hilbert Basis Theorem,

Every ideal $I \in K[x_1, \ldots, x_n]$ has a finite generating set.

- Hilbert Nullstellensatz,

A system of polynomial equations $f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0$ fail to have a common solution in $\mathbb{C}^n$ if only if

$$1 \in < f_1(x_1, \ldots, x_n) \ldots f_m(x_1, \ldots, x_n) >$$

Change of Basis

- Let $I \subset K[x_1, x_2, x_3, y_1, y_2, y_3]$

$$\begin{aligned} x_1 + x_2 - y_1 &= 0, \\ x_1 - x_2 - y_2 &= 0, \\ x_1 + 2x_2 - y_3 &= 0. \end{aligned}$$

$$\begin{aligned} &x_1 + x_2 - y_1 = 0, \\ &-2x_2 + y1 - y_2 = 0, \\ &x_2 + y_1 - y_3 = 0. \end{aligned}$$

$$\begin{aligned} &x_1 + x_2 - y_1 = 0, \\ &-2x_2 + y1 - y_2 = 0, \\ &3y_1 - y_2 - 2y_3 = 0. \end{aligned}$$

Groebner Basis

1. The ideal generated by the leading terms of polynomials in $I =< f_1, ..., f_n >$ equals the ideal generated by the leading terms in the *$G - basis$*.

2. The leading term of any polynomial in I is divisible by the leading term of some polynomial in G.

3. The multivariable division by G of any polynomial in the ideal I gives zero as remainder.

For $I =< x^2 - y, x^3 - x >$ the G-basis is not generated by the leading terms!.

Buchberger's Algorithm

$$I =< f1 = x^2+y^2+z^2-1, f2 = x^2+z^2-y, f_3 = x-z >\subset Q[x,y,z]$$

- Mathematica

GroebnerBasis[{f1,f2,f3},{x,y,z}]

$G =< 4z^4 + 2z^2 - 1, y - 2z^2, x - z >$

- Maple

with(Groebner); G:=Basis([g1,g2,g3],plex(x,y,z));

$G =< 4z^4 + 2z^2 - 1, y - 2z^2, x - z >$

- Singular

ring R=0, (x,y,z),dp; (global reverse ordering)

ideal I= $x^2 + y^2 + z^2 - 1, x^2 + z^2 - y, x - z$;

ideal G=std(I);

$G =< y^2 + y - 1, y - 2z^2, x - z >$

For an empty set you obtain $G =< 1 >$

Implicitization Problem

Consider the spatial curve C defined by

$$x = t^4, y = t^3, z = t^2$$

what are the equations for C in $K[x, y, z]$?.

- Compute the G-basis of
  $I =< x - t^4, y - t^3, t^2 - z >\subset K[t, x, y, z]$

$G =< x - z^2, y^2 - z^3, z - t^2, ty - z^2, tz - y >$

and observe that the first two polynomials depend only on x, y, z.

- Singular

ring R=0, (x,y,z,t),dp;

ideal I= $x - t^4, y - t^3, z - t^2$;

ideal C=eliminate(I,t);

$C =< x - z^2, y^2 - xz >$

Mapping

- Birational Equivalence: $S \backslash (z = 0) \cong \mathbb{C}^2 \backslash (uv = 0)$

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\sigma} & \mathrm{S} = z^3 - xy \subset \mathbb{C}^2 \\ (u, v) & \longmapsto & (u^2 v, uv^2, uv) \end{array}$$

$$\begin{array}{ccc} \mathrm{S} = z^3 - xy \subset \mathbb{C}^3 & \xrightarrow{\sigma^{-1}} & \mathbb{C}^2 \\ (x, y, z) & \longmapsto & (x/z, y/z) \end{array}$$

- Rational Parametrization:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\sigma} & \mathrm{S} = x^3 + xyz + x + y^3 + yz^2 \subset \mathbb{C}^2 \\ (u, v) & \longmapsto & (f_1/f_4, f_2/f_4, f_3/f_4) \end{array}$$

$$f_1 = u^3(u^2 - 2v), \quad f_2 = 2uv - u^3,$$
$$f_3 = u(v^2 + u^6 - 1), \quad f_4 = -1 - v^2 + u^2 v + u^6$$

$(u, v) = (1, -1)$ or $(-1, 1/3)$ leads to same point at S.

Dimension

Consider the spatial curve C defined by

$$x = t^3, y = t^4, z = t^5$$

The ideal $I(C)$ is generated by three polynomials !

$$I = < y^2 - xz, x^2y - z^2, x^3 - yz >$$

- Dimension with Singular

ring R=0, (x,y,z,),dp;

ideal $I = < y^2 - xz, x^2y - z^2, x^3 - yz >$;

dim(groebner(I));

1

Algebraic variety may not be given as complete intersection.

Differential Forms

- Let $V =< f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n) >$ define the set,

$$\Omega^1(V) = \frac{< dx_1, \ldots, dx_n >_{K[V]}}{< df_1, \ldots, df_n >}$$

and the space of differential forms of degree $q = dim[V]$ is,

$$\Omega^q(V) = \overset{q}{\Lambda} \, \Omega^1(V)$$

- The plurigenera of V are non-negative integers,

$$P_l(V) = dim[\overset{l}{\otimes} \, \Omega^q(V)]$$

- For example in the case of a curve C we have,

$$P_1(C) = g, \quad P_l(C) = (2l - 1)(g - 1), \quad l > 1.$$

- Kodaira Dimension $P_l(V) \sim l^{k(V)}$.

Complete Intersection

- An n-dimensional algebraic variety $V \in \mathbb{P}^{n+m}$ is a complete intersection,

$$V =< f_1(x_0, \ldots, x_{n+m}), \ldots, f_m(x_0, \ldots, x_{n+m}) >$$

- For such complete intersection,

$$k(V) = \begin{cases} -\infty, & deg(f_1) + \cdots + def(f_m) - (n+m+1) < 0. \\ 0, & deg(f_1) + \cdots + def(f_m) - (n+m+1) = 0. \\ n, & deg(f_1) + \cdots + def(f_m) - (n+m+1) > 0. \end{cases}$$