Topological Theories and Quantum Computing

Dmitry Melnikov
International Institute of Physics – UFRN

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Plan of the Talk

Work in collaboration with A. Mironov, S. Mironov, A. Morozov and A. Morozov

Project: Modern applications of knot theory

Knots and Quantum Computing

- QC basics
  - Quantum computing (qubits, entanglement)
  - Topological Field Theories and Topological invariants
  - Quantum entanglement in TQFTs
  - Entanglement of closed and open curves
Quantum computing

Quantum computer

1. Initial state – vector in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots$, e.g.

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$$

2. (A sequence of ) unitary transformations (quantum gates) $U$

$$|\Psi\rangle \rightarrow U|\Psi\rangle$$

3. Measurement (collapse of the wavefunction on a given state)

$$\langle \Psi_0 | U | \Psi \rangle , \quad e.g. \ |\Psi_0\rangle = |\uparrow\rangle$$

(probabilistic output)

_a generic linear combination $\alpha |\uparrow\rangle + \beta |\downarrow\rangle$ is called qubit_
Quantum computing

Universal quantum gates

- Why unitary? – Unitary operations are invertible; in an ideal computer the energy is only consumed in erasing the data (Landauer’s principle)
- A minimal set of unitary operations generating a dense subset of all unitaries – *universal* gates

Example

\[
T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}
\]

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

\[
\text{CNOT} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]
Quantum states: pure vs mixed

We consider quantum systems that consist of two or more subsystems: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$

- The state is called \textit{pure}, if there is a wavefunction, \textit{e.g.} EPR-state

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$$

One can introduce a density matrix as $\rho_{AB} = |\Psi_{AB}\rangle \otimes \langle \Psi_{AB}|$

- Conversely, not every matrix $\rho_{AB}$ satisfying $\text{Tr} \rho_{AB} = 1$ is separable. Such d.m. are said to describe a \textit{mixed} state
Quantum computing

Quantum state: pure vs mixed vs entangled

- A pure state is *entangled*, if the wavefunction (or density matrix) is *not* separable.

$$|\Psi_{AB}\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle$$

- If the state is mixed, it is entangled unless

$$\rho_{AB} = \rho_A \otimes \rho_B$$

Examples:

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle) \quad \text{EPRs is entangled}$$

$$\alpha |\uparrow\rangle \otimes |\uparrow\rangle + \beta |\uparrow\rangle \otimes |\downarrow\rangle + \gamma |\downarrow\rangle \otimes |\uparrow\rangle + \delta |\downarrow\rangle \otimes |\downarrow\rangle \text{ is entangled unless } \det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 0$$
Quantum computing

Quantum computing and entanglement

- A unitary operator $U$ is called *entangling*, if there exists a non-entangled state $|\Psi\rangle$, such that $U|\Psi\rangle$ is entangled.

- Brylinski’s theorem. The set of quantum gates is universal if and only if it is entangling.

Example

$$
\text{CNOT} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

is entangling.
Quantum computing

Entanglement measures

- problem: quantify entanglement

(Von Neumann) entanglement entropy

\[ S = - \text{Tr}_A (\rho_A \log \rho_A) \]

where \( \rho_A = \text{Tr}_B (\rho_{AB}) \) is the reduced density matrix

Example

EPR: \[ \rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad S = \log 2 \]
Topological Quantum Field Theories

Definition

- Functor $Z$ between the category of topological spaces and the category of linear spaces:
  1. With a $d$-dimensional $\Sigma$ associates a vector space $V = Z(\Sigma)$
  2. With a $d + 1$ dimensional $M$, $\Sigma = \partial M$ associates a vector $v = Z(M) \in V$
  3. $\forall \Sigma_1, \Sigma_2$ and $M$, $\partial M = \Sigma_1 \cup \Sigma_2$, associates a linear map $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$

Hilbert space \hspace{3cm} vector/operator
Topological Quantum Field Theories

TQFT functor

- $Z(\Sigma^\dagger) = Z^\dagger(\Sigma)$, where $\Sigma^\dagger$ stands for the reversed $\Sigma$ orientation
- $Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$ for a disjoint union
- For a composition $M_2 \circ M_1$ of cobordisms $M_1 : \Sigma_1 \rightarrow \Sigma_2^\dagger$ and $M_2 : \Sigma_2 \rightarrow \Sigma_3$, $Z(M_2 \circ M_1)$ is a composition of linear maps $Z(M_2) \circ Z(M_1) : Z(\Sigma_1) \rightarrow Z(\Sigma_3)$
- $Z(\phi) = \mathbb{C}$, where $\phi$ is an empty manifold
- For a unit interval $I$, such that $\Sigma \times I$ is an identity cobordism, $Z(\Sigma \times I)$ is an identity map $Z(\Sigma) \rightarrow Z(\Sigma)$

\[ \begin{tikzpicture}
\end{tikzpicture} \]
Topological Quantum Field Theories

Visualization

- If $\Sigma$ has punctures – they are extended inside states $M$
- Representation of the braid group
- Topological invariants
Topological Quantum Field Theories

Explicit definition

\[ S_{CS}[\mathcal{M}_3] = \frac{k}{4\pi} \int_{\mathcal{M}_3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \]

HOMFLY-PT polynomials

\[ Z(\mathcal{M}_3, K; G, R, k) = \int \mathcal{D}A \ \text{Tr}_R \text{P exp} \left( i \int_K A \right) \text{e}^{iS_{CS}[\mathcal{M}_3]} \]
Topological Quantum Field Theories

Examples of Hilbert spaces

$SU(N)$ Chern-Simons

- $\Sigma = S^2$: $\dim \mathcal{H} = 1, S_E = 0$
- $\Sigma = T^2$: $\dim \mathcal{H} = k + 1, k \in \mathbb{Z}$
- $\Sigma = S^2 \backslash \{P\}$: $\dim \mathcal{H}$ depends on number of points and their "charges".
Topological quantum computation

- Initial state of is represented by a 3D manifold with a 2D boundary.
- Unitary representations of braid group realize quantum gates.
- Measurement corresponds to evaluation of matrix elements of the braids.
- Topological invariants compute quantum amplitudes.
Quantum Entanglement and TQFT

Quantum states

- Given a manifold $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \ldots$ states are cobordisms of $\Sigma$

Two classes of states:

"separable"

"entangled"
Quantum Entanglement and TQFT

Quantum operators

- Linear maps are also cobordisms

Typical operators

"density matrix"
(mixed state)

"entangler"
Quantum Entanglement in TQFT

Entanglement entropy

- Replica trick: compute $\text{Tr} \rho_A^n$

\[ S = -\frac{d}{dn} \text{Tr} \rho_A^n \bigg|_{n=1} \]

\[ \rho_1^A = \left[ \begin{array}{c} \text{\textcircled{}} \\ \text{\textcircled{}} \end{array} \right]^{-1} \times \left[ \begin{array}{c} \text{\textcircled{}} \\ \text{\textcircled{}} \end{array} \right] \]

\[ \rho_2^A = \left[ \begin{array}{c} \text{\textcircled{}} \end{array} \right]^{-1} \times \left[ \begin{array}{c} \text{\textcircled{}} \end{array} \right] \]

[Dong,Fradkin,Leigh,Nowling’08]
Quantum Entanglement and TQFT

Entanglement entropy

In the examples above

\[ \text{Tr} \left( \rho_1^A \right)^n = 1, \quad \text{Tr} \left( \rho_2^A \right)^n = \begin{bmatrix} \hline 1- \end{bmatrix} \]

Consequently,

\[ S_E(\rho_1) = 0, \quad S_E(\rho_2) = \log \begin{bmatrix} \hline \end{bmatrix} \]

In the trivial case the donut is \( \text{Tr} \mathcal{H} \mathbf{1} = \dim \mathcal{H} = Z(\Sigma \times S^1) \)
Closed Curves

Torus Hilbert space

Basis vectors

| \( R \rangle = | \text{torus} \rangle 

Scalar product

\[ \langle R_i | R_j \rangle = \langle \text{torus} | \text{torus} \rangle = \text{tr} \]

\[ = Z(S^2 \times S^1; R_i, R_j) = \delta_{ij} \]
Closed Curves

Operators

$\text{SL}(2, \mathbb{Z})$ diffeomorphisms ($\text{SL}(2, \mathbb{Z}) = \{ S, T | S^2 = 1, (ST)^3 = 1 \}$)

$|m, n; R\rangle = \sum_i W^{(m,n)}_{R,R_i} |R_i\rangle = \left| \begin{array}{c} \text{image} \end{array} \right\rangle \quad W^{(m,n)}_{ij} \in \text{SL}(2, \mathbb{Z})$

Twisted scalar product

$\langle R_i | S | R_j \rangle = \left\langle \begin{array}{c} \text{image} \end{array} \right| S \left| \begin{array}{c} \text{image} \end{array} \right\rangle = \mathbb{Z}(S^3; \text{Hopf link})$
Closed Curves

Entanglement entropy

[Balasubramanian et al' 16]

- Cut a tubular neighbourhood of a link in $S^3 \Rightarrow \Sigma = T^2 \otimes T^2 \ldots$
- Define a state associated with the link

$$|\mathcal{L}\rangle = \sum_{R_1, \ldots, R_l} H_{R_1, \ldots, R_l} |R_1\rangle \otimes \ldots \otimes |R_l\rangle$$

- Compute the full and reduced density matrices

$$\rho = \frac{|\mathcal{L}\rangle \langle \mathcal{L}|}{\langle \mathcal{L}|\mathcal{L}\rangle}, \quad \bar{\rho} = \frac{\sum_{\tilde{b}} H_{a'\tilde{b}} H_{\tilde{b}a''}}{\sum_{\tilde{a}} H_{\tilde{a}} H_{\tilde{a}}} \cdot |a'\rangle \langle a''|$$

- Compute the entanglement entropy associated with the link
Closed Curves

Examples ($SU(2)$ case)

Hopf link:

\[ H_{R_1,R_2} = \frac{S_{R_1 R_2}}{S_{00}} , \]

\[ S^2 = I \quad \Rightarrow \quad S_E = \log \dim \mathcal{H} = \log(k + 1) - \text{max entanglement} \]

$(2m, 2)$ family:

\[ H_{R_1 R_2} = \langle 0 | ST^{2m} | R_1, R_2 \rangle \]
\[ = \sum_R \langle 0 | ST^{2m} | R \rangle \langle R | R_1, R_2 \rangle \]

In general, $S_E(2m, 2) \leq S_E(\text{Hopf link})$
Open Curves

Invariants in $S^2 \times S^1$

Master formula

$$S_E(\rho_2) = \log \left[ \begin{array}{c} \text{Diagram} \end{array} \right]$$

Two Wilson lines

$$S_E = \log \dim \mathcal{H} = 0$$  \quad  $$S_E = 0$$
Open Curves

Four Wilson lines

\[ S_E = \log \dim \mathcal{H} \]

\[ S_E = 0 \]

- Again, the entropy of unlinked lines is maximal
Conclusions

In this talk we reviewed

- Basics of a topological quantum computer: code spaces, operations, entanglement
- Entanglement entropy in TQFTs. Entropy of open and closed curves

Observations

- Links and tangles can be endowed with a physical characteristics such as entropy
- Counter-intuitive relation between quantum and topological entanglement: The entropy is maximal on "simple" configurations