Topological Theories and Quantum Computing

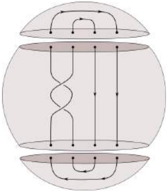
Dmitry Melnikov
International Institute of Physics – UFRN

ICTP-SAIFR – Sao Paulo
August, 2018

Plan of the Talk

Work in collaboration with A. Mironov, S. Mironov, A. Morozov and A. Morozov

Project: Modern applications of knot theory [MMMMM'17]



- Knots and Quantum Computing
  - QC basics
    - Quantum computing (qubits, entanglement)
    - Topological Field Theories and Topological invariants
    - Quantum entanglement in TQFTs
    - Entanglement of closed and open curves

Quantum computing

Quantum computer

1. Initial state – vector in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots$, *e.g.*

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle\right)$$

2. (A sequence of) unitary transformations (quantum gates) U

$$|\Psi\rangle \;\to\; U|\Psi\rangle$$

3. Measurement (collapse of the wavefunction on a given state)

$$\langle\Psi_0|U|\Psi\rangle\,, \qquad e.g.\ |\Psi_0\rangle = |\uparrow\rangle$$

(probabilistic output)

a generic linear combination $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$ is called *qubit*

Quantum computing

Universal quantum gates

- Why unitary? – Unitary operations are invertible; in an ideal computer the energy is only consumed in erasing the data (Landauer's principle)
- a minimal set of unitary operations generating a dense subset of all unitaries – *universal* gates

Example

$$\mathtt{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{i\pi/4} \end{pmatrix}$$

$$\mathtt{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathtt{CNOT} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Quantum computing

Quantum states: pure vs mixed

We consider quantum systems that consist of two or more subsystems: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$

- The state is called *pure*, if there is a wavefunction, *e.g.* EPR-state

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \right)$$

  One can introduce a density matrix as $\rho_{AB} = |\Psi_{AB}\rangle \otimes \langle\Psi_{AB}|$

- Conversely, not every matrix ρ_{AB} satisfying Tr $\rho_{AB} = 1$ is separable. Such d.m. are said to describe a *mixed* state

Quantum computing

Quantum state: pure vs mixed vs entangled

- A pure state is *entangled*, if the wavefunction (or density matrix) is *not* separable.

$$|\Psi_{AB}\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle$$

- if the state is mixed, it is entangled unless

$$\rho_{AB} = \rho_A \otimes \rho_B$$

Examples:

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \right) \qquad \text{EPRs is entangled}$$

$\alpha|\uparrow\rangle \otimes |\uparrow\rangle + \beta|\uparrow\rangle \otimes |\downarrow\rangle + \gamma|\downarrow\rangle \otimes |\uparrow\rangle + \delta|\downarrow\rangle \otimes |\downarrow\rangle$ is entangled unless $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 0$

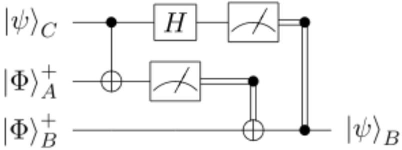
Quantum computing

Quantum computing and entanglement

- a unitary operator U is called *entangling*, if there exists a non-entangled state $|\Psi\rangle$, such that $U|\Psi\rangle$ is entangled
- *Brylinskis' theorem.* The set of quantum gates is universal if and only if it is entangling.

Example

$$\mathtt{CNOT} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



is entangling

Quantum computing

Entanglement measures

- problem: quantify entanglement

(Von Neumann) entanglement entropy

$$S = -\mathrm{Tr}_A(\rho_A \log \rho_A)$$

where $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ is the reduced density matrix

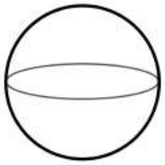
Example

$$\text{EPR:} \qquad \rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad S = \log 2$$

Topological Quantum Field Theories

Definition [Witten, Atiyah]

- Functor Z between the category of topological spaces and the category of linear spaces:
  1. With a d-dimensional Σ associates a vector space $V = Z(\Sigma)$
  2. With a $d+1$ dimensional M, $\Sigma = \partial M$ associates a vector $v = Z(M) \in V$
  3. $\forall\ \Sigma_1, \Sigma_2$ and M, $\partial M = \Sigma_1 \cup \Sigma_2$, associates a linear map $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$



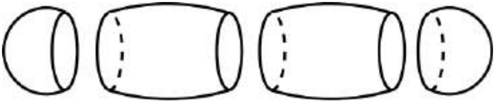
Hilbert space

vector/operator

Topological Quantum Field Theories

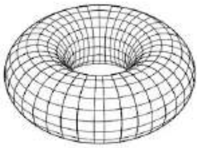
TQFT functor [Atiyah]

- $Z(\Sigma^\dagger) = Z^\dagger(\Sigma)$, where $\Sigma^\dagger$ stands for the reversed Σ orientation
- $Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$ for a disjoint union
- For a composition $M_2 \circ M_1$ of cobordisms $M_1 : \Sigma_1 \to \Sigma_2^\dagger$ and $M_2 : \Sigma_2 \to \Sigma_3$, $Z(M_2 \circ M_1)$ is a composition of linear maps $Z(M_2) \circ Z(M_1) : Z(\Sigma_1) \to Z(\Sigma_3)$
- $Z(\phi) = \mathbb{C}$, where ϕ is an empty manifold
- For a unit interval I, such that $\Sigma \times I$ is an identity cobordism, $Z(\Sigma \times I)$ is an identity map $Z(\Sigma) \to Z(\Sigma)$

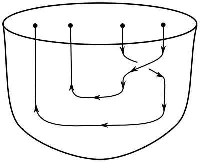


Topological Quantum Field Theories

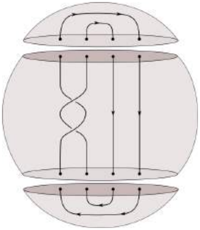
Visualization



Hilbert space



state



matrix element

- If Σ has punctures – they are extended inside states M
- Representation of the braid group
- Topological invariants

Topological Quantum Field Theories

Explicit definition [Witten'89]

$$S_{\text{CS}}[\mathcal{M}_3] \;=\; \frac{k}{4\pi}\int_{\mathcal{M}_3} \text{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

HOMFLY-PT polynomials

$$Z(\mathcal{M}_3, K; G, R, k) \;=\; \int \mathcal{D}A \;\; \text{Tr}_R\, \text{P}\exp\left(i\int_K A\right) \text{e}^{iS_{\text{CS}}[\mathcal{M}_3]}$$

Topological Quantum Field Theories

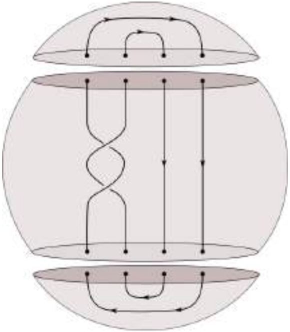
Examples of Hilbert spaces

$SU(N)$ Chern-Simons

- $\Sigma = S^2$: $\dim \mathcal{H} = 1$, $S_E = 0$:
- $\Sigma = T^2$: $\dim \mathcal{H} \ = \ k + 1, k \in \mathbb{Z}$
- $\Sigma = S^2 \backslash \{P\}$: $\dim \mathcal{H}$ depends on number of points and their "charges".

Topological Quantum Field Theories

Topological quantum computation [Kitaev *et al*'02]



- Initial state of is represented by a 3D manifold with a 2D boundary.
- Unitary representations of braid group realize quantum gates
- Measurement corresponds to evaluation of matrix elements of the braids
- Topological invariants compute quantum amplitudes

Quantum Entanglement and TQFT

Quantum states

- Given a manifold $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \ldots$ states are cobordisms of Σ

Two classes of states:

"separable"

"entangled"

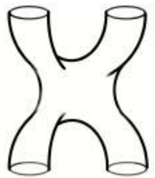
Quantum Entanglement and TQFT

Quantum operators

- Linear maps are also cobordisms

Typical operators

"density matrix"
(mixed state)



"entangler"

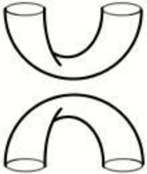
Quantum Entanglement in TQFT

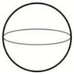
Entanglement entropy

[Dong,Fradkin,Leigh,Nowling'08]

- Replica trick: compute $\mathrm{Tr}\ \rho_A^n$

$$S = -\frac{d}{dn}\,\mathrm{Tr}\ \rho_A^n\Big|_{n=1}$$



$\rho_1^A = \Big[$  $\Big]^{-1} \times$ 

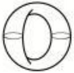
$\rho_2^A = \Big[$  $\Big]^{-1} \times$ 

Quantum Entanglement and TQFT

Entanglement entropy

In the examples above

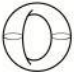
$$\mathrm{Tr}\,\left(\rho_1^A\right)^n = 1\,, \qquad \mathrm{Tr}\,\left(\rho_2^A\right)^n = \Bigg[$$



$$\Bigg]^{1-n}$$

Consequently,

$$S_{\mathrm{E}}(\rho_1) = 0\,, \qquad S_{\mathrm{E}}(\rho_2) = \log \Bigg[$$



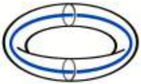
$$\Bigg]$$

In the trivial case the donut is $\mathrm{Tr}_{\mathcal{H}}\,\mathbf{1} = \dim \mathcal{H} = Z(\Sigma \times S^1)$

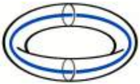
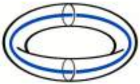
Closed Curves

Torus Hilbert space

Basis vectors

$$|R\rangle = \Bigg|$$  $$\Bigg\rangle$$

Scalar product

$$\langle R_i | R_j \rangle = \Bigg\langle$$  $$\Bigg|$$  $$\Bigg\rangle = \mathrm{tr}$$ 

$$= Z(S^2 \times S^1; R_i, R_j) = \delta_{ij}$$

Closed Curves

Operators

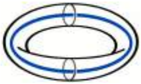
$SL(2,\mathbb{Z})$ diffeomorphisms ($SL(2,\mathbb{Z}) = \{S, T | S^2 = 1, (ST)^3 = 1\}$)

$$|m,n;R\rangle = \sum_i W^{(m,n)}_{R,R_i}|R_i\rangle = \Bigg|$$

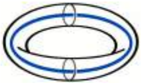
$$\Bigg\rangle \qquad W^{(m,n)}_{ij} \in SL(2,Z)$$

Twisted scalar product

$$\langle R_i|S|R_j\rangle = \Bigg\langle$$



$$\Bigg| \; S \; \Bigg|$$



$$\Bigg\rangle =$$

$$= Z(S^3;\text{Hopf link})$$

Closed Curves

Entanglement entropy [Balasubramanian *et al*'16]

- Cut a tubular neighbourhood of a link in $S^3 \Rightarrow \Sigma = T^2 \otimes T^2 \cdots$
- Define a state associated with the link

$$|\mathcal{L}\rangle = \sum_{R_1,\ldots,R_l} H_{R_1,\ldots,R_l} |R_1\rangle \otimes \ldots \otimes |R_l\rangle$$

- Compute the full and reduced density matrices

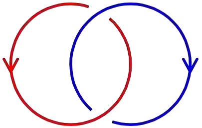
$$\rho = \frac{|\mathcal{L}\rangle\langle\mathcal{L}|}{\langle\mathcal{L}|\mathcal{L}\rangle}, \qquad \bar{\rho} = \frac{\sum_{\vec{b}} H_{a'\vec{b}} H_{\vec{b}a''}}{\sum_{\vec{a}} H_{\vec{a}} H_{\vec{a}}} \cdot |a'\rangle\langle a''|$$

- Compute the entanglement entropy associated with the link

Closed Curves

Examples ($SU(2)$ case) [Balasubramanian *et al*'16]

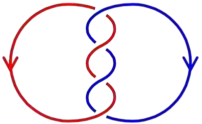
Hopf link:



$$H_{R_1,R_2} = \frac{S_{R_1R_2}}{S_{00}},$$

$$S^2 = I \quad \Rightarrow \quad S_E = \log\dim\mathcal{H} = \log(k+1) \text{ – max entanglement}$$

$(2m, 2)$ family:



$$\begin{aligned} H_{R_1R_2} &= \langle 0|ST^{2m}|R_1, R_2\rangle \\ &= \sum_R \langle 0|ST^{2m}|R\rangle\langle R|R_1, R_2\rangle \end{aligned}$$

In general, $S_E(2m, 2) \leq S_E(\text{Hopf link})$

Open Curves

Invariants in $S^2 \times S^1$

Master formula

$$S_{\mathrm{E}}(\rho_2) = \log \Big[$$

$$\Big]$$

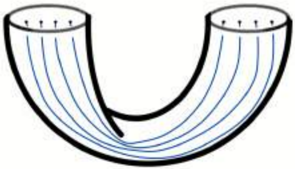
Two Wilson lines

$$S_{\mathrm{E}} = \log \dim \mathcal{H} = 0$$

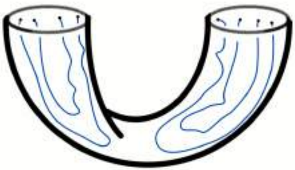
$$S_{\mathrm{E}} = 0$$

Open Curves

Four Wilson lines



$$S_{\mathrm{E}} = \log \dim \mathcal{H}$$



$$S_{\mathrm{E}} = 0$$

- Again, the entropy of unlinked lines is maximal

Conclusions

In this talk we reviewed

- Basics of a topological quantum computer: code spaces, operations, entanglement
- Entanglement entropy in TQFTs. Entropy of open and closed curves

Observations

- Links and tangles can be endowed with a physical characteristics such as entropy
- Counter-intuitive relation between quantum and topological entanglement: The entropy is maximal on "simple" configurations