ICTP mini-course: ~~Q...~~ and

# Introduction to QC and its classical simulability

## Lecture I : Physical states and the (convenient) illusion of Hilbert space

Refs: • D. Poulin, A. Qarry, R. D. Somma and F. Verstraete PRL 11. Arxiv: 1102.1360

• M. Kliesch, T. Barthel, C. Gogolin, M. Kastoryano and J. Eisert PRL 11.

• Nielsen & Chuang Book. Sec 4.5.4

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## Hilbert space is a big place!

• Deterministic N-bit state

$$\boxed{\bar{s} = (s_1, s_2, \ldots s_N)}$$

N binary parameters

$\Rightarrow$ $2^N$ possible different states

• "Deterministic" N-qubit state

$$\boxed{|\psi\rangle = \sum_{\bar{s}} \psi_{\bar{s}} \, |\bar{s}\rangle}$$

comp basis

$2^N$ complex parameters

$\Rightarrow \left(1/\epsilon\right)^{2^N}$ possible different states

constant accuracy in $\mathbb{C}$

Example: # atoms in the observable univers estimated (Google)

to be $10^{78} - 10^{82} \lesssim 2^{272}$

$\Rightarrow$ A 272-qubit state contain more complex parameters to specify than atoms in the known, observable universe!

→ A bit too much...

Makes one wonder:

- Is QM's description of many body system the correct one?
- Can QM's " " " " be falsified at all anyway?

Or, perhaps, not all states allowed by QM are physical..

Definition: Physical states of N particles ⟺ generated by a local, time-varying Hamiltonian (of arbitrary long-range interactions) in Poly(N) time

starting from an easy reference state ferromagnetic, vacuum, etc $|0\rangle$

local interaction term of X particles

$$\boxed{H(t) = \sum_{X \subset \{1,\dots,N\}} H_X(t)}$$

local Hamiltonians

with $\|H_X\| \leq E \;\; \forall X$ (bounded norm)

$H_X = 0$ if $|X| > K$, for K constant

Examples: classical spins

$\Rightarrow$ All states $\bar{s}$ can be generated in constant time by the trivial Hamiltonian starting from $\bar{s} = \bar{0}$ (all down)

constant magnetic field

$$H = \sum_i B_x \cdot \mu_x$$

dipole moment in the x direction

(corresponds to a constant-depth (depth = 1) classical circuit.)

~~Example 2: Quantum spins - qubits~~

• Physical quantum states

Schrödinger's equation $\Rightarrow$ $\boxed{U(0,t) = \mathcal{T}\left[e^{-i\int_0^t H(s)\,ds}\right]}$

time ordering operator

$$\Rightarrow U(0,T) = U((n-1)\Delta t, n\Delta t) \cdots U(\Delta t, 2\Delta t)\, U(0, 0+\Delta t)$$

$T = n\Delta t$

$\Rightarrow$ For slowly varying Hamiltonians

$$H(t) \approx H(j\Delta t) \quad \forall\, t \in [(j-1)\Delta t, j\Delta t]$$

$$\Rightarrow \boxed{U(0,T) \approx \mathcal{T}\left[\prod_{j=0}^{n-1} e^{-i H(j\Delta t)\Delta t}\right]}$$

$$= e^{-i H((n-1)\Delta t)\Delta t} \cdots\cdots e^{-i H(\Delta t)\Delta t} \; e^{-i H(0)\Delta t}$$

$n$ scales linearly with the evolution time $T$ : $n = \frac{T}{\Delta t}$

valid if $\boxed{\Delta t \ll \left\| \frac{\partial H}{\partial t} \right\|^{-1}}$ (slowly varying condition)

$\forall \; t \in [0, T]$

$n \approx \text{Poly}(N)$

$\Delta t \approx \text{cte}$

$T \approx \text{Poly}(N)$

Now open $e^{-i H(j\Delta t)\Delta t}$ :

Trotter formula (the simplest decomposition)

$$\boxed{e^{\Delta t (A+B)} = e^{\Delta t A} \, e^{\Delta t B} + O(\Delta t^2)}$$

$$\Rightarrow \boxed{e^{\frac{\Delta t}{m}(A+B)} = \left(e^{\Delta t A/m} \, e^{\Delta t B/m}\right)^m + m\,O\left(\left(\Delta t/m\right)^2\right)}$$

(for $\Delta t$ not small enough)

and there are higher-order expansions e.g.: Lie-Trotter-Suzuki

follows from the Baker-Campbell-Hausdorff formula:

~~$A+B$: Baker-Campbell-Hausdorff formula for $[A,B]\neq 0$~~

~~$e^{A+B} = e^{A} e^{B}$~~

<u>Exercise</u> : Show that

$$\boxed{e^{-i H(\Delta t j)\Delta t} = \prod_{x} e^{-i H_x(\Delta t j)\Delta t} + O\left(\#NC(X)\,\Delta t^2\right)}$$

number of non-commuting tersm X

Hint: - BCH: $e^{A+B} = e^{A} e^{B}$ for $[A,B]=0$

- $\| e^{-i H_x(\Delta t j)\Delta t} \| = 1$ (unitarity)

$$\Rightarrow \quad U(0,T) = \mathcal{T}\left[\prod_{j=0}^{n-1}\prod_{x} e^{-i H_x(\Delta t j)\Delta t}\right] + O\left(n\,\#NC(X)\,\Delta t^2\right)$$

$\Rightarrow$ Take $n\,\#NC(X)\,\Delta t^2 = T\,\#NC(X)\,\Delta t = \varepsilon$ (constant error)

$$\Rightarrow \quad \boxed{U(0,T) = \mathcal{T}\left[\prod_{j=0}^{n-1}\prod_{x} e^{-i H_x(\Delta t j)\Delta t}\right] + O(\varepsilon)}$$

k-particle gate

$\frac{1}{\Delta t} = \frac{T\,\#NC(X)}{\varepsilon}$

# k-particle gates : $n \times \#X = \frac{T}{\Delta t}\,\#X =$

$$= \frac{T^2\,\#NC(X)\,\#X}{\varepsilon}$$

Now, from the Solovay-Kitaev theorem

"Any K-qubit gate can be ε-approximated

using $\underbrace{O\left[K^2\, 4^K \log^{c}\left(K^2 4^K/\varepsilon\right)\right]}_{F(K)}$ gates" (constant in N)

$c \approx 2$

⇒ # of ~~universal gates~~ 1- and 2-qubit gates:

$$\Rightarrow \quad \frac{T^2 F(K)}{\varepsilon} \, \# NC(X) \, \#(X) \leq \frac{T^2 F(K)}{\varepsilon} \, \# X^2$$

For physical states $T = O(\mathrm{Poly}(N))$ (evol time not exp greater than the space a system occupies) and $\# X = O(N)$ (interaction terms no t exp with N)

$$\Rightarrow \boxed{U(0,T) \text{ can be } \varepsilon\text{-approximated by a UQC using } \mathrm{Poly}(N, \varepsilon) \text{ gates}}$$

And for quickly-varying Hamiltonians substitute

$H_X(\Delta t\, j)$ by its time average $\displaystyle \frac{1}{\Delta t} \int_{(j)\Delta t}^{(j+1)\Delta t} H_X(s)\, ds$

and the same conclusion follows

$$\Rightarrow \boxed{\text{Physical states} = \text{efficiently } \overset{\varepsilon\text{-simulatable}}{\cancel{\text{preparable}}} \text{ states on a UQC}}$$

## • Counting the number of physical states

• Count the number of different Poly(N)-sized circuits and consider an $\varepsilon$-ball around each output!

$\Rightarrow$ Take $\text{Poly}(N) = N^{\alpha}$ — degree of the polynomial

$M$ = number of gates in the universal set of gates

$$\Rightarrow \boxed{N_{circuits} = (M N^2)^{N^{\alpha}}} \quad (\text{exp in } N)$$

$M$ possible

$N^2$: # pairs of qubits on which to apply each gate

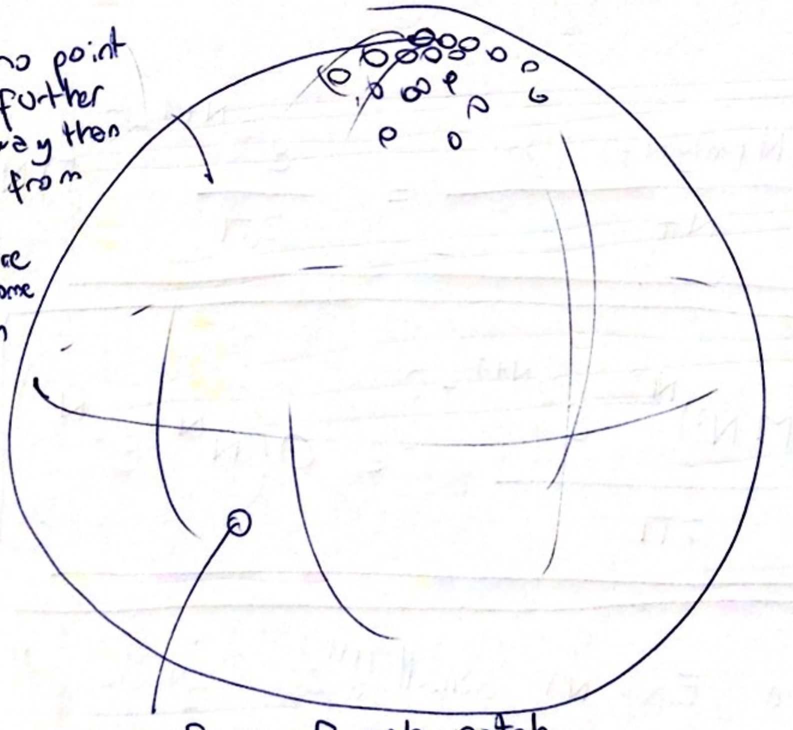
That is: # of patches of radius $\varepsilon$ that can be reached

Number of total states: $\varepsilon$-net

$$|\psi\rangle = \sum_{z} \psi_z |z\rangle$$

$$\Rightarrow \sum_{z} |\psi_z|^2 = 1$$

no point is further away than $\varepsilon$ from the centre of some patch



surface of each patch $\approx$ Volume of a $2^{N+1}-2$ sphere of $r = \varepsilon$

• Pure states of n qubits live on the surface of a $2^{N+1}-1$ dimensional Euclidean sphere of $r=1$

$$S_D(r) = \frac{2\pi^{\frac{D+1}{2}}}{\Gamma\left(\frac{D+1}{2}\right)} r^D \qquad V_D(r) = \frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}+1\right)} r^D$$

surface of a D-sphere — Volume of a D-sphere

$\Rightarrow$ Total number of states : $N_T = \dfrac{S_{2^{N+1}-1}(1)}{V_{2^{N+1}-2}(\varepsilon)} =$

$$= \frac{\dfrac{2\pi^{\cancel{2^N}}}{\cancel{\Gamma(2^N)}} \times 1^{2^{N+1}-1}}{\dfrac{\cancel{\pi^{2^N-1}}}{\cancel{\Gamma(2^N)}} \varepsilon^{2^{N+1}-2}} = \frac{2\pi}{\varepsilon^{2^{N+1}-2}}$$

$$\Rightarrow \boxed{N_T = 2\pi \left(\frac{1}{\varepsilon}\right)^{2^{N+1}-2} = O\left[\left(\frac{1}{\varepsilon}\right)^{2^N}\right]}$$ (doubly exp in N)!

$$\frac{\text{Surface of phys states}}{\text{Surface of } \cancel{\text{other}} \text{ total states}} = \frac{N_{\text{poly}(N,\varepsilon) \text{ circuits}}}{N_T} = \frac{\varepsilon^{2^{N+1}-2}}{2\pi} (\Pi N^2)^{N^\alpha}$$

$$\Rightarrow \boxed{\frac{N_{phys}}{N_{total}} = \frac{(M N^2)^{N^\alpha} \varepsilon^{2^{N+1}-2}}{2\pi} = O\left(N^N \varepsilon^{2^N}\right)}$$

More than Exp(N) small !!!! $2^{N\log(N)} \varepsilon^{2^N}$

$\Rightarrow$ ~~Area~~

$$\boxed{\text{Actual area of Hilbert space occupied by physical states} = O\left(N^N \varepsilon^{2N}\right)}$$

- All other states have never existed and will never exist: they are an illusion!
- Is $\mathbb{H}$ actually a good description of physical systems for large N?
- Computational complexity as a new fundamental physical axiom to complement quantum mechanics??

  ↳ Since physical states admit a Q circuit decomposition of Poly(N) gates
  $\Rightarrow$ the are described by Poly(N) parameters instead of Exp(N) parameters

$\Rightarrow$ Exp. complexity of $\mathbb{H}$ actually not required

- Quantum info giving input to foundations of QM!