Hands-on Exercise: Code a simple ODE solver

- Write a code than can solve systems of ODEs, using the forward Euler, RK2 and RK4 methods.
  Use it on simple ODEs first, check convergence and quantify the numerical error:

$$y'(t) = t^n \qquad y' = \lambda y \qquad y'(t) = sin(t)$$

- How large can you make the time step? What happens when the time step is too large?
- Does the solution converge to the exact solution?
- Is roundoff error a problem?

post-Newtonian black holes

- Start with energy, e.g. as function of separation R or orbital frequency ω: $E(R)$, $E(\omega)$. Kepler: $\omega^2 R^3 = G M$.

- PN expansion:

$$\omega^2(R) = \frac{GM}{R^3}\left(1 + f_1(R)\left(\frac{v}{c}\right)^2 + f_2(R)\left(\frac{v}{c}\right)^4 + \ldots\right)$$

- Compute energy loss P=-dE/dt to some order in v/c, e.g. at leading order quadrupole formula (see GR text books like Wald)

- To compute the rate of change of any quantity X (e.g. X=ω, R) we write

$$\frac{dX}{dt} = \frac{\frac{dE}{dt}}{\frac{dE}{dX}}$$

- To lowest (Newtonian/quadrupole)order:

$$E(R) = m_1 + m_2 - M\frac{\eta}{2}\frac{M}{R}$$

$$E(\omega) = m_1 + m_2 - M\frac{\eta}{2}\left(\frac{(M\omega)^2}{G}\right)^{\frac{1}{3}}$$

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G^4}{c^5}\eta^2\left(\frac{v}{c}\right)^{10}\left(1 + O(v^2) + \dots\right)$$

- Here v is the velocity parameter, η the symmetric mass ratio:

$$v = (GM\omega)^{1/3}$$

$$\eta = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

- For GW science, we also need the phase

$$\frac{d\phi}{dt} = \omega$$

exact solution

$$R(t) = \left(\frac{256}{5} \eta M^3 \right)^{\frac{1}{4}} (t_c - t)^{\frac{1}{4}}$$

- Generalize your ODE code to handle systems of ODEs, solve the 2-body problem in GR for point particles in the "Newtonian orbits + quadrupole formula energy loss" approximation.
- Carry out a convergence test and evaluate the numerical error.
- How much energy is radiated in the inspiral?