The Sound of Space-Time: exercises on EMRI modelling

Adam Pound

December 5, 2018

1. (a) Use the symmetries of the Riemann tensor to show that $\nabla_\mu R^\mu{}_{\alpha\beta\gamma} = 0$ if $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor of a vacuum metric.

   (b) Show that if $\delta R_{\alpha\beta}[h] = 0$ and $G_{\alpha\beta}[g] = 0$, then $\delta G_{\alpha\beta}[h] = 0$.

   (c) Use the above two results, along with the expression

$$\delta R_{\alpha\beta}[h] = -\frac{1}{2}\Box h_{\alpha\beta} - \frac{1}{2}\nabla_\alpha\nabla_\beta(g^{\mu\nu}h_{\mu\nu}) + \nabla^\mu\nabla_{(\alpha}h_{\beta)\mu},$$

   to show that $\delta G_{\alpha\beta}[h]$ always identically vanishes for a gauge perturbation $h_{\alpha\beta} = 2\nabla_{(\alpha}\xi_{\beta)}$, where ξ^μ is an arbitrary (smooth) vector field. This shows that $\delta G_{\mu\nu}[h] = \delta G_{\mu\nu}[h']$ if $h_{\mu\nu}$ and $h'_{\mu\nu}$ are related by a gauge transformation.

2. The Killing tensor of Kerr, given by $K_{\alpha\beta} = 2\Sigma\ell_{(\alpha}n_{\beta)} + r^2 g_{\alpha\beta}$, satisfies $\nabla_{(\alpha}K_{\beta\gamma)} = 0$, where the parentheses denote symmetrization over all three indices. Show that the Carter constant $C = K_{\alpha\beta}u^\alpha u^\beta$ is constant along a geodesic, where u^α is the geodesic's four-velocity.

3. In this question we'll use a simple scalar toy model to examine the quasicircular inspiral of a small object into a Schwarzschild black hole at leading, adiabatic order. As discussed in the lectures, at adiabatic order we can approximate the motion as a smooth sequence of geodesics, which in this case are circular orbits described (in Schwarzschild coordinates) by $z^\mu = (t, r_0, \pi/2, \Omega_0 t)$, where $\Omega_0 = \sqrt{\frac{M}{r_0^3}}$. The geodesics' four-velocity is $u^\mu = u^t(1, 0, 0, \Omega_0)$, where $u^t = 1/\sqrt{1 - 3M/r_0}$.

   (a) Show that the point-particle stress-energy tensor,

$$T^{\mu\nu} = \mu \int u^\mu u^\nu \frac{\delta^4(x^\gamma - z^\gamma(\tau))}{\sqrt{-g}} d\tau,$$

   can be written as

$$T^{\mu\nu} = \frac{\mu u^\mu u^\nu}{u^t r_0^2}\delta(r - r_0)\sum_{lm} Y^*_{lm}(\pi/2, \Omega_0 t) Y_{lm}(\theta, \phi).$$

Here μ is the particle's mass. As a toy model, we will consider instead the scalar charge distribution

$$\rho = \frac{q}{u^t r_0^2}\delta(r - r_0)\sum_{lm} Y^*_{lm}(\pi/2, \Omega_0 t)Y_{lm}(\theta, \phi).$$

(b) Assume that both r (the radius at which the field $h_{\mu\nu}$ is evaluated) and r_0 are large compared to M, such that the spacetime is approximately flat. The linearized EFE in the Lorenz gauge then reads

$$\eta^{\mu\nu}\partial_\mu\partial_\nu \bar{h}^{\alpha\beta} = -16\pi T^{\alpha\beta},$$

where $\eta_{\mu\nu}$ is the Minkowski metric and we work in Cartesian coordinates (t, x^a) (related to r, θ, ϕ in the usual, flat-space way). In our toy model, we'll consider the analogous equation for a scalar field,

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi = -4\pi\rho.$$

Based on the form of the source, motivate the ansatz

$$\varphi = \sum_{lm} R_{lm}(r)e^{-im\Omega_0 t}Y_{lm}(\theta, \phi).$$

Use this ansatz to write the field equation as an ordinary differential equation for $R_{lm}(r)$.

(c) Solve your differential equation subject to the following boundary conditions:

- when $r \gg r_0$, $R_{lm} \sim \frac{e^{im\Omega_0 r}}{r}$. This is the form of an outgoing wave, since it implies $R_{lm}e^{-im\Omega_0 t} \sim \frac{e^{-im\Omega_0(t-r)}}{r}$.
- at $r = 0$, R_{lm} is regular. The physical boundary condition for the original problem (prior to approximating $E_{\alpha\beta}[\bar{h}]$ as $\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{\alpha\beta}$) is that waves are down-going at the horizon. But since the wavelength is very large, with $\lambda \sim r_0 \gg M$, the black hole has a very small effect on the wave propagation.

(Hint: for $m \neq 0$ you should find that the homogeneous solutions to your ODE are spherical Bessel functions.)

(d) The gravitational wave fluxes to infinity are

$$\dot{E}_\infty = -\lim_{r\to\infty}\int T^{GW}_{tr} r^2 d\Omega,$$
$$\dot{L}_\infty = \lim_{r\to\infty}\int T^{GW}_{r\phi} r^2 d\Omega,$$

where $T^{GW}_{\alpha\beta}$ is the effective stress-energy tensor given in lecture, which in this situation reduces to $T^{GW}_{\alpha\beta} = \frac{1}{32\pi}\langle h^{\mu\nu}{}_{;\alpha}h_{\mu\nu;\beta} - \frac{1}{2}h_{,\alpha}h_{,\beta}\rangle$; here the angular parentheses denote an average over one period,

$\frac{\Omega_0}{2\pi}\int_0^{2\pi/\Omega_0} dt$. In our toy model, we'll replace this with $T^{GW}_{\alpha\beta} = \frac{1}{32\pi}\langle\varphi_{,\alpha}\varphi_{,\beta}\rangle$. Use this to calculate $\dot{E}_\infty$ and $\dot{L}_\infty$, and verify that $\dot{E}_\infty = \Omega_0\dot{L}_\infty$. (This relationship holds true in general for circular orbits; it doesn't depend on the details of our toy model.) Hint: use the leading-order (for $r \gg r_0$) expansion of R_{lm} to simplify the calculations.

(e) The particle's energy is $E_0 = -\mu u_t$. Use this and the energy balance law $\dot{E}_0 = -\dot{E}_\infty$ to obtain $\frac{dr_0}{dt}$, keeping only the dominant term in a large-r_0 expansion. (Here we neglect the fluxes down the BH horizon, as they are very small in this scenario.)

(f) Use your result for $\frac{dr_0}{dt}$ to calculate the coordinate time Δt it takes for r_0 to decrease from $100M$ to $50M$. How many orbital cycles does the particle complete in this time? Express both results in terms of the parameter $\epsilon = q^2/(\mu M)$; in the toy model, this plays the same role that the mass ratio μ/M plays in the actual problem.

4. Consider a unit vector $n^i = x^i/r$, where $x^i = (x, y, z)$ are Cartesian coordinates and $r = \sqrt{\delta_{ab}x^a x^b}$. Prove the identities $\partial_i r = n_i$, $n^i\partial_i\hat{n}^L = 0$, and $\partial_i n^i = 2/r$, where $n^i = \frac{x^i}{r}$ and indices are raised and lowered with the Euclidean metric δ_{ij}. Use these identities, along with the eigenvalue equation $\partial^i\partial_i\hat{n}^L = -\frac{l(l+1)}{r^2}\hat{n}^L$, to prove $\partial^i\partial_i(r^p\hat{n}_L) = r^{p-2}[p(p+1) - l(l+1)]\hat{n}^L$.

5. Consider a point mass μ in Minkowski space. If we approximate its worldline γ as a geodesic, then in Fermi normal coordinates centered on γ, the first-order singular field of the particle is

$$\bar{h}^S_{\alpha\beta} = \frac{4\mu\delta^t_\alpha\delta^t_\beta}{r},$$

where r is the spatial distance to the particle. Since a geodesic of Minkowski is imply a straight line, we can adopt a new inertial Cartesian coordinate system (t, x^a) in which the particle sits at a constant spatial position x^a_p, such that

$$\bar{h}^S_{\alpha\beta} = \frac{4\mu\delta^t_\alpha\delta^t_\beta}{|x^a - x^a_p|}.$$

(a) The physical field satisfies $\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{h}_{\mu\nu} = 0$ for $x^a \neq x^a_p$. In this problem we can take the puncture $\bar{h}^P_{\mu\nu}$ to be precisely $\bar{h}^S_{\mu\nu}$. Show that the residual field $\bar{h}^{\mathcal{R}}_{\mu\nu} = \bar{h}_{\mu\nu} - \bar{h}^P_{\mu\nu}$ satisfies $\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{h}^{\mathcal{R}}_{\mu\nu} = 0$ for all x^a. If we impose static boundary conditions (i.e., $\partial_t h_{\mu\nu} = 0$), what is the force exerted by $h^{\mathcal{R}}_{\mu\nu}$ on the point mass? What if our only boundary condition is regularity at $x^a = 0$?

(b) The physical field also satisfies $\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$, where $T_{\mu\nu}$ is the stress-energy tensor of the point mass μ. Show that the general solution to this equation is $\bar{h}_{\mu\nu} = \bar{h}^S_{\mu\nu} + \bar{h}^R_{\mu\nu}$, where $h^R_{\mu\nu}$ is a smooth homogeneous solution.

(c) Show that (independent of the choice of boundary conditions) $h^R_{\mu\nu}$ can be calculated on the particle using the mode-sum formula $\bar{h}^R_{\mu\nu}(t, x^a_p) = \sum_l [\bar{h}^l_{\mu\nu}(t, x^a_p) - B_{\mu\nu}]$, where $\bar{h}^l_{\mu\nu}(t, x^a_p) = \sum_{m=-l}^{l} \bar{h}^{lm}_{\mu\nu}(t, x^a_p) Y_{lm}(\theta_p, \phi_p)$, and the regularization parameter is $B_{\mu\nu} = \frac{4\mu}{r_p} \delta^t_\mu \delta^t_\nu$, with $r_p = \sqrt{\delta_{ij} x^i_p x^j_p}$.