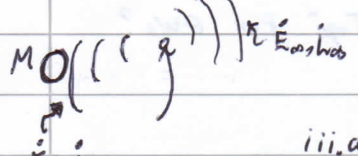
Adiabatic approximation in Schwarzschild

We'll look at one more thing before getting into the details of self-force theory. Without knowing/calculating the actual self-force, we can implement the adiabatic approximation (in Schwarzschild - not in Kerr; more on this later).

The basic idea: i. choose E_0 and L_0 - specifies a geodesic

ii. calculate the GWs at ∞ and at the EH, as sourced by a point mass on the geodesic

$\Rightarrow$ extract $\underbrace{\dot{E}_\infty, \dot{L}_\infty}_{\substack{\text{GW fluxes of E} \\ \text{and L to } \infty}}, \underbrace{\dot{E}_H, \dot{L}_H}_{\substack{\text{GW fluxes of E} \\ \text{and L down into BH}}}$



iii. assume balance laws: $\dot{E}_0 = -\dot{E}_\infty - \dot{E}_H$ (this can be proved using the actual SF)

$\dot{L}_0 = -\dot{L}_\infty - \dot{L}_H$

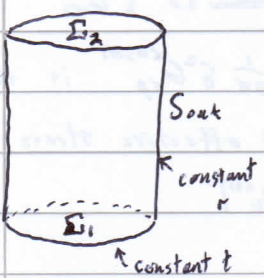
iv. use this to update to new E_0 and L_0

v. repeat as many times as desired

vi. extract phases from $\frac{d\psi^\alpha}{dt} = \Omega_0^\alpha(E_0, L_0)$
(ψ^r and ϕ)

First task: define $\dot{E}_\infty$ and $\dot{L}_\infty$, and $\dot{E}_H$ and $\dot{L}_H$

Consider a matter stress-energy tensor $T^{\alpha\beta}$. We can define the energy in a spatial region Σ to be $E = -\int_\Sigma T^\alpha{}_\beta \xi^\beta_{(t)} d\Sigma_\beta$ ← future-directed volume element



The change in energy between Σ_1 and Σ_2 is

$$E(\Sigma_2) - E(\Sigma_1) = -\int_{S_{out}} T^\alpha{}_\beta \xi^\beta_{(t)} dS_\beta$$ ← outward-directed: $n_\alpha \sqrt{f}\, r^2 dt\, d\Omega$

$n_\alpha = \frac{1}{\sqrt{f}} \delta^r_\alpha$ is radial unit vector

Exercise: prove this

$$= -\int T^r{}_t\, r^2 dt\, d\Omega$$

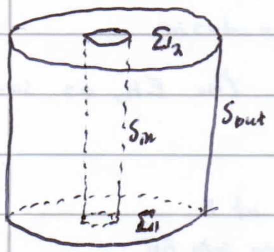
$$= \int \underbrace{\left(-\int T^r{}_t\, r^2 d\Omega\right)}_{\dot{E}} dt$$

$\Rightarrow \dot{E} = -r^2 f \int T_{tr}\, d\Omega$

Analogously, $L(\Sigma) = \int T^{\alpha}{}_{\beta}\, \xi^{\beta}_{(\phi)}\, d\Sigma_\alpha$ and $\dot{L} = r^2 f \int T_{r\phi}\, d\Omega$

So $\dot{E}_\infty = -\lim_{r\to\infty}\left(r^2 f \int_{S_{out}} T_{tr}\, d\Omega\right)$ and $\dot{L}_\infty = \lim_{r\to\infty}\left(r^2 f \int_{S_{out}} T_{r\phi}\, d\Omega\right)$

If we add an inner boundary around the BH, we get



$\dot{E}_H = +\lim_{r\to 2M}\left(r^2 f \int_{S_{in}} T_{tr}\, d\Omega\right)$ and $\dot{L}_H = -\lim_{r\to 2M}\left(r^2 f \int_{S_{in}} T_{r\phi}\, d\Omega\right)$

This is the E and L carried out of the system by a matter field. What is "$T_{\alpha\beta}$" for GWs?

Recall, in a vacuum region, $\delta G_{\alpha\beta}[h] = -\delta^2 G_{\alpha\beta}[h] + O(h^3)$

$\Rightarrow -\delta^2 G_{\alpha\beta}[h]$ is like an effective stress-energy tensor

We can make this more precise: we are interested in how $-\delta^2 G_{\alpha\beta}$ causes slow changes in the system. For example, the BH absorbs GW energy & angular momentum. This causes $h^{(1)}_{\alpha\beta}$ to include slowly varying pieces $h^{\delta M}_{\alpha\beta} \propto \frac{\delta M(\tilde{t})}{r}$ and $h^{\delta J}_{\alpha\beta} \propto \frac{\delta J(\tilde{t})}{r^2}$. Also, outside the orbit, the particle

perturbations due to small corrections to BH parameters

generates perturbations $\propto \frac{E_0(\tilde{t})}{r}$ and $\frac{L_0(\tilde{t})}{r^2}$. In the two-timescale expansion of the EFE, we had $\delta G^{(0,k\gamma)}_{\alpha\beta}[h^{(2,k\gamma)}] + \delta G^{(1,k\gamma)}_{\alpha\beta}[h^{(1,k\gamma)}] = -\delta^2 G^{(0,k\gamma)}_{\alpha\beta}[h^{(1,k\gamma)}]$

$\delta G^{(1,0)}_{\alpha\beta} \sim \delta\dot{M}, \delta\dot{J}, \dot{E}_0, \dot{L}_0$

$\Rightarrow \delta G^{(1,0)}_{\alpha\beta}$ describes the system's slow response to $\delta^2 G_{\alpha\beta}$

$\Rightarrow -\frac{1}{8\pi}\delta^2 G^{(0,0)}_{\alpha\beta}$ is the relevant effective stress-energy

neglect $\frac{\partial}{\partial \tilde{t}}$ terms

$$\therefore\ T^{GW}_{\alpha\beta} = -\frac{1}{8\pi}\left\langle \delta^2 G^{(0)}_{\alpha\beta}\right\rangle = -\frac{1}{8\pi}\frac{1}{4\pi^2}\int_0^{2\pi} d\psi^r \int_0^{2\pi} d\psi^\phi\, \delta^2 G^{(0)}_{\alpha\beta}[h^{(1)}]$$

Note: recall we wrote $h^{(1)}_{\alpha\beta} = \sum_{k^\gamma \in \mathbb{Z}} h^{(1,k^\gamma)}_{\alpha\beta}(x^A, \tilde{t})\, e^{-ik_\gamma \psi^\gamma}$ in the two-timescale expansion.

We can also write the coefficients as $h^{(1,k^\gamma)}_{\alpha\beta}(x^A, J^\mu_0(\tilde{t}))$;

– i.e., their slow-time dependence arises from their dependence on the slowly evolving "constants" of motion

(they also contain terms arising from the slow evolution of the large BH's mass and spin, but those terms are not needed at adiabatic order, because they do not contribute to $f^T_{1,diss}$)

This $h^{(1)}_{\alpha\beta}$ is not the $h^{(1)}_{\alpha\beta}$ sourced by a geodesic orbit.

But, the coefficients $h^{(1,k^\gamma)}_{\alpha\beta}(x^A, J^\mu_0(\tilde{t}))$ are identical, for a given value of J^μ_0, to the coefficients in the expansion for a geodesic, $h^{(1)}_{\alpha\beta} = \sum_{k^\gamma} h^{(1,k^\gamma)}_{\alpha\beta}(x^A, \underset{\uparrow \text{fixed}}{J^\mu_0})\, e^{-ik_\gamma \underbrace{\Omega^\gamma t}_{\text{phases evolve linearly with time for a geodesic}}}$

Since only the coefficients contribute to the adiabatic evolution, this is what allows us to use the $h^{(1)}_{\alpha\beta}$ for a sequence of geodesics.