Quantum Gravity from the QFT perspective

Ilya L. Shapiro

Universidade Federal de Juiz de Fora, MG, Brazil

Partial support: CNPq, FAPEMIG

ICTP-SAIFR/IFT-UNESP – 1-5 April, 2019

Contents of the mini-course

- **GR and its limits of applicability. Planck scale. Semi-classical approach. Renormalizability in curved space-time.**

- **Renormalization group. Decoupling: what remains in the IR? Conformal anomaly and anomaly-induced effective action. Applications: Starobinsky model.**

- **Gauge-invariant renormalization in QG. Power counting. Quantum GR vs Higher derivative QG. Can we live with/without ghosts? More HD's and more ghosts. Superrenormalizable QG.**

- **Alternative approaches: Effective approach to QG and Induced Gravity paradigm.**

- **Effective approach in curved space-time. Cosmological constant problem and running in cosmology.**

Bibliography

N.D. Birell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, 1982).

I.L. Buchbinder, S.D. Odintsov, I.Sh., *Effective Action in Quantum Gravity* (IOPP, 1992).

I.Sh., *Effective Action of Vacuum: Semiclassical Approach.* Class. Quant. Grav. 25 (2008) 103001 (Topical review); hep-th/0801.0216.

L. Parker, D.J. Toms, *Quantum Field Theory in Curved Spacetime, Quantized Fields and Gravity,* (Cambridge, 2009).

I.L. Buchbinder, I.Sh. *Introduction to quantum field theory with applications to quantum gravity,* (Oxford Univ. Press, book in preparation, 2020).

I.Sh., *Primer in Tensor Analysis and Relativity,* (Springer-NY, 2019, to be published).

Lecture 1. GR and its limits of applicability. Semi-classical approach.

GR and singularities.

Dimensional approach and Planck scale.

Quantum Field Theory in curved space and its importance.

Formulation of classical fields in curved space.

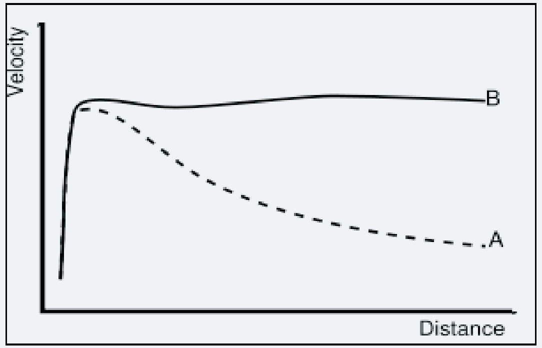
Quantum theory with linearized parametrization of gravity.

Local momentum representation. Covariance.

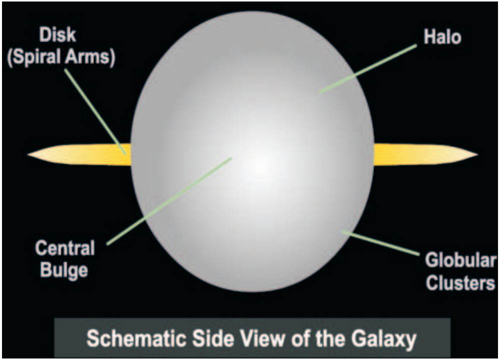
Classical Gravity – Newton's Law,

$$\vec{F}_{12} = -\frac{G\,M_1\,M_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12} \quad \text{or} \quad U(r) = -G\,\frac{M_1\,M_2}{r}\,.$$

Newton's law work well from laboratory up to the galaxy scale.



For galaxies one needs, presumably, to introduce a HALO of Dark Matter, which consists from particles of unknown origin,



or modify the Newton's law - MOND,

$$F = F(\vec{r}, \vec{v})\,.$$

The real need to modify Newton gravity was because it is not relativistic while the electromagnetic theory is.

- Maxwell 1868 ... • Lorentz 1895 ... • **Einstein 1905**

Relativity: instead of **space + time**, **there is a unique space-time** M_{3+1} **(Minkowski space)**. Its coordinates are

$$x^\mu = (ct, x, y, z) .$$

The distances (intervals) are defined as

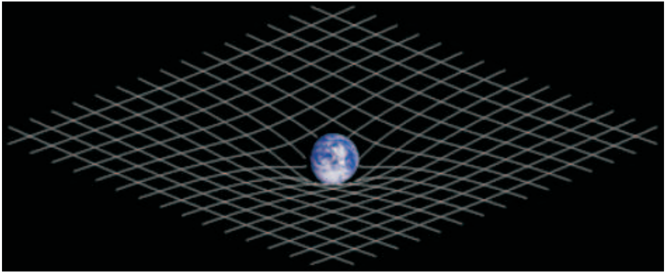
$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 .$$

How to incorporate gravity?

The Minkowski space is flat, as the surface of a table.

GR (A. Einstein, 1915): Gravitation = space-time metric.

- **Geometry shows matter how to move.**
- **Matter shows space how to curve.**



General Relativity and Quantum Theory

General Relativity (GR) is a complete theory of classical gravitational phenomena. It proved to be valid in the wide range of energies and distances.

The basis of the theory are the principles of equivalence and general covariance.
There are covariant equations for the matter (fields and particles, fluids etc) and Einstein equations for the gravitational field $g_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G\, T_{\mu\nu} - \Lambda\, g_{\mu\nu}\,.$$

We have introduced Λ**, cosmological constant (CC) for completeness.**

The most important solutions of GR have specific symmetries.

1) Spherically-symmetric solution. Planets, Stars, Black holes.
2) Isotropic and homogeneous metric. Universe.

Spherically-symmetric solution of Schwarzschild.

This solution corresponds to the spherical symmetry in the static mass distribution and in the classical solution.
The metric may depend on the distance r and time t, but not on the angles φ and θ.

For the sake of simplicity we suppose that there is a point-like mass in the origin of the spherical coordinate system. The solution can be written in the standard Schwarzschild form

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - r_g/r} - r^2 d\Omega \,.$$

where $r_g = 2GM$.

Performing a $1/r$ expansion we arrive at the Newton potential

$$\varphi(r) = -\frac{GM}{r} + \frac{G^2M^2}{2r^2} + ...$$

Schwarzschild solution has two singularities:
At the gravitational radius $r_g = 2GM$ and at the origin $r = 0$.

The first singularity is coordinate-dependent, indicating the existence of the horizon.

Light or massive particles can not propagate from the interior of the black hole to an outside observer. The $r = r_g$ horizon looks as singularity only if it is observed from the "safe" distance.

An observer can change his coordinate system such that no singularity at $r = r_g$ will be observed.

On the contrary, $r = 0$ singularity is physical and indicates a serious problem.

Indeed, the Schwarzschild solution is valid only in the vacuum and we do not expect point-like masses to exist in the nature. The spherically symmetric solution inside the matter does not have singularity.

However, the object with horizon may be formed as a consequence of the gravitational collapse, leading to the formation of physical singularity at $r = 0$.

After all, assuming GR is valid at all scales, we arrive at the situation when the $r = 0$ **singularity becomes real.**

Then, the matter has infinitely high density of energy, and curvature invariants are also infinite. Our physical intuition tells that this is not a realistic situation.

Something must be modified.

Standard cosmological model

Another important solution of GR is the one for the homogeneous and isotropic metric (FLRW solution).

$$ds^2 \,=\, dt^2 \,-\, a^2(t)\cdot\left(\frac{dr^2}{1-kr^2}+r^2d\Omega\right)\,,$$

Here r **is the distance from some given point in the space (for homogeneous and isotropic space-time. The choice of this point is not important).** $a(t)$ **is the unique unknown function,**

$k=(0,1,-1)$ **defines the geometry of the space section** M^3 **of the** 4**-dimensional space-time manifold** M^{3+1}.

Consider only the case of the early universe, where the role of k **and** Λ **is negligible and the radiation dominates over the matter.**

Radiation-dominated epoch

is characterized by the dominating radiation with the relativistic relation between energy density and pressure $p = \rho/3$ and $T^\mu_\mu = 0$. Taking $k = \Lambda = 0$, we meet the Friedmann equation

$$\dot{a}^2 = \frac{8\pi G}{3}\,\frac{\rho_0 a_0^4}{a^4},.$$

Solving it, we arrive at the solution

$$a(t) = \Big(\frac{4}{3}\cdot 8\pi G\rho_0 a_0^4\Big)^{1/4} \times \sqrt{t}\,,$$

This expression becomes singular at $t \to 0$. Also, in this case the Hubble constant

$$H = \dot{a}/a = \frac{1}{2t}$$

also becomes singular, along with ρ_r and with components of the curvature tensor.

The situation is qualitatively similar to the black hole singularity.

Applicability of GR

The singularities are significant, because they emerge in the most important solutions, in the main areas of application of GR.

Extrapolating backward in time we find that the use of GR leads to a problem, while at the late Universe GR provides a consistent basis for cosmology and astrophysics. The most natural resolution of the problem of singularities is to assume that

- **GR is not valid at all scales**.

At the very short distances and/or when the curvature becomes very large, the gravitational phenomena must be described by some other theory, more general than the GR.

But, due to success of GR, we expect that this unknown theory coincides with GR at the large distance & weak field limit.

The most probable origin of the deviation from the GR are quantum effects.

Need for quantum field theory in curved space-time.

Let us use the dimensional arguments.

The expected scale of the quantum gravity effects is associated to the Planck units of length, time and mass. The idea of Planck units is based on the existence of the 3 fundamental constants:

$$c = 3 \cdot 10^{10}\, cm/s, \qquad \hbar = 1.054 \cdot 10^{-27}\, erg \cdot sec,$$
$$G = 6.67 \cdot 10^{-8}\, cm^3/sec^2\, g.$$

One can use them uniquely to construct the dimensions of

length $l_P = G^{1/2}\, \hbar^{1/2}\, c^{-3/2} \approx 1.4 \cdot 10^{-33}\, cm;$

time $t_P = G^{1/2}\, \hbar^{1/2}\, c^{-5/2} \approx 0.7 \cdot 10^{-43}\, sec;$

mass $M_P = G^{-1/2}\, \hbar^{1/2}\, c^{1/2} \approx 0.2 \cdot 10^{-5} g \approx 10^{19}\, GeV.$

One can use these fundamental units in different ways.

In particular physics people use to set $c = \hbar = 1$ and measure everything in GeV. Indeed, for everyday life it may not be nice.

E.g., you have to schedule the meeting "just $10^{27}\, GeV^{-1}$ from now", but " 15 minutes" may be appreciated better.

However, in the specific area, when all quantities are (more or less) of the same order of magnitude, GeV units are useful.

One can measure Newton constant G in GeV.
Then $G = 1/M_P^2$ and $t_P = l_P = 1/M_P$.

Now, why do not we take M_P as a universal measure for everything? Fix $M_P = 1$, such that $G = 1$. Then everything is measured in the powers of the Planck mass M_P.

"20 grams of butter" $\equiv$ "10^6 of butter"

Warning: sometimes you risk to be misunderstood !!

Status of QFT in curved space

One may suppose that the existence of the fundamental units indicates fundamental physics at the Planck scale.

It may be Quantum Gravity, String Theory ...
We do not know what it really is.

So, which concepts are certain?

Quantum Field Theory and Curved space-time definitely are.

Therefore, our first step should be to consider QFT of matter fields in curved space.

Different from quantum theory of gravity, QFT of matter fields in curved space is renormalizable and free of conceptual problems.

However, deriving many of the most relevant observables is yet an unsolved problem.

Formulation of classical fields on curved background

● We impose the principles of locality and general covariance.

● Furthermore, we require the most relevant symmetries of a theory in flat spacetime (specially gauge invariance) in flat space-time to hold on in the curved spacetime theory.

● It is also natural to forbid the introduction of new parameters with the inverse-mass dimension.

These set of conditions leads to a simplest consistent quantum theory of matter fields on the classical gravitational background.

● The form of the action of a matter field is fixed except the values of a few parameters which remain arbitrary.

● The procedure which we have described above, leads to the so-called non-minimal actions.

Along with the nonminimal scheme, there is a more simple, minimal one. According to it one has to replace

$$\partial_\mu \to \nabla_\mu \,, \quad \eta_{\mu\nu} \to g_{\mu\nu} \,, \quad d^4x \to d^4x\sqrt{-g} \,.$$

Below we consider the fields with spin zero (scalar), spin $1/2$ **(Dirac spinor) and spin** 1 **(massless vector)**.

The actions for other possible types of fields (say, massive vectors or antisymmetric $b_{\mu\nu}$ **, spin** 3/2 **, etc), can be constructed using the same approach.**

Scalar field

The minimal action for a real scalar field is

$$S_0 = \int d^4x \sqrt{-g}\,\Big\{ \frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi - V_{min}(\varphi) \Big\}\,,$$

where $$V_{min}(\varphi) = -\frac{1}{2}\,m^2\varphi^2 - \frac{\lambda}{4!}\,\varphi^4$$

is a minimal potential term. Possible nonminimal structure is

$$S_{non-min} \;=\; \frac{1}{2}\,\int d^4x\,\sqrt{-g}\,\xi\,\varphi^2\,R\,.$$

The new quantity ξ is called nonminimal parameter.

Since the non-minimal term does not have derivatives of the scalar field, it should be included into the potential term, and thus we arrive at the new definition of the classical potential.

$$V(\varphi) \;=\; -\,\frac{1}{2}\,\big(m^2 + \xi R\big)\,\varphi^2 \;+\; \frac{f}{4!}\,\varphi^4\,.$$

In case of the multi-scalar theory the nonminimal term is

$$\int d^4x \sqrt{-g}\, \xi_{ij}\, \varphi^i \varphi^j R\,.$$

Further non-minimal structures involving scalar are indeed possible, for example

$$\int R^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi\,.$$

However, these structures include constants of inverse mass dimension, therefore do not fit the principles declared above.

In fact, these terms are not necessary for the construction of consistent quantum theory.

Along with the non-minimal term, our principles admit some terms which involve only metric. These terms are conventionally called **"vacuum action"** and their general form is the following

$$S_{vac} = S_{EH} + S_{HD}$$

where $$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \, \{R + 2\Lambda\} \, .$$

is the Einstein-Hilbert action with the CC

S_{HD} **includes higher derivative terms. The most useful form is**

$$S_{HD} = \int d^4x \sqrt{-g} \left\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\right\},$$

where $$C^2(4) = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + 1/3 \, R^2$$

is the square of the Weyl tensor in $n = 4$,

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4 \, R_{\alpha\beta}R^{\alpha\beta} + R^2$$

is the integrand of the n=4 Gauss-Bonnet topological invariant.

In $n = 4$ case some terms in the action

$$S_{vac} = S_{EH} + S_{HD}$$

gain very special properties.

S_{HD} **includes a conformal invariant $\int C^2$, topological and surface terms, $\int E$ and $\int \Box R$.**

The last two terms do not contribute to the classical equations of motion for the metric.

Moreover, in the FRW case $\int C^2 = const$ and only $\int R^2$ is relevant!

However, as we shall see later on, all these terms are important, for they contribute to the dynamics at the quantum level, e.g., through the conformal anomaly.

The basis E, C^2, R^2 is, in many respects, more useful than $R^2_{\mu\nu\alpha\beta}$, $R^2_{\alpha\beta}$, R^2, and that is why we are going to use it here.

For the Dirac spinor the minimal procedure leads to the expression

$$S_{1/2} = i \int d^4x \sqrt{-g}\, \left(\bar{\psi}\, \gamma^\alpha \nabla_\alpha \psi - im\, \bar{\psi}\psi \right) ,$$

where γ^μ and ∇_μ are γ-matrices and covariant derivatives of the spinor in curved space-time.

Let us define both these objects.

The definition of γ^μ requires the tetrad (vierbein)

$$e_a^\mu \cdot e^{\nu a} = g^{\mu\nu} , \quad e_\mu^a \cdot e^{\mu b} = \eta^{ab} .$$

Now, we set $\gamma^\mu = e_a^\mu\, \gamma^a$, where γ^a is usual (flat-space) γ-matrix.

The new γ -matrices satisfy Clifford algebra in curved space-time

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} .$$

The covariant derivative of a Dirac spinor $\nabla_\alpha \psi$ should be consistent with the covariant derivative of tensors. Consider

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{i}{2}\, w_\mu^{ab}\, \sigma_{ab}\, \psi\,,$$

w_μ^{ab} **is usually called spinor connection and**

$$\sigma_{ab} = \frac{i}{2}\,(\gamma_a \gamma_b - \gamma_b \gamma_a)\,.$$

The conjugated expression is

$$\nabla_\mu\, \bar{\psi} = \partial_\mu\, \bar{\psi} - \frac{i}{2}\, \bar{\psi}\, w_\mu^{ab}\, \sigma_{ab}\,.$$

In order to establish the form of the spinor connection, consider the covariant derivative acting on the vector $\bar{\psi}\gamma^\alpha\psi$.

$$\nabla_\mu(\bar{\psi}\gamma^\alpha\psi) = \partial_\mu(\bar{\psi}\gamma^\alpha\psi) + \Gamma^\alpha_{\mu\lambda}\bar{\psi}\gamma^\lambda\psi\,.$$

The solution has the form

$$w_\mu^{ab} = -w_\mu^{ba} = \frac{1}{2}\,\left(e^b_\alpha \partial_\mu\, e^{\alpha a} + \Gamma^\alpha_{\lambda\mu}\, e^b_\alpha\, e^{\lambda a}\right)\,.$$

The minimal generalization for massless Abelian vector field A_μ is straightforward

$$S_1 \,=\, \frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}\,F^{\mu\nu}\,,$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$

In the non-Abelian case we have very similar structure.

$$A_\mu \to A^a_\mu\,, \qquad F_{\mu\nu} \to G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\,.$$

In both Abelian and non-Abelian cases the minimal action keeps the gauge symmetry. The non-minimal covariant terms for spins 1/2 and 1 have inverse mass dimension and the vacuum terms are the same as before.

Interaction with external gravity does not spoil gauge invariance of a fermion or charged scalar coupled to a gauge field. Also, the Yukawa interaction can be obtained via the minimal procedure, $\int d^4x\sqrt{-g}\varphi\bar{\psi}\psi$.

SSB in curved spacetime

Consider how the spontaneous Symmetry breaking (SSB) changes when it is used in curved spacetime.

$$S = \int d^4x\sqrt{-g}\,\Big\{g^{\mu\nu}\,\partial_\mu\varphi^*\,\partial_\nu\varphi + \mu_0^2\,\varphi^*\varphi + \lambda(\varphi^*\varphi)^2 - \xi\,R\,\varphi^*\varphi\Big\}.$$

The vacuum expectation value for scalar is defined from

$$-\Box v + \mu_0^2\,v + \xi R\,v - 2\lambda v^3 = 0\,. \qquad (1)$$

If the scalar is minimal $\xi = 0$, the SSB is standard and

$$v_0^2 = \frac{\mu_0^2}{2\,\lambda}.$$

For the general case of a nonzero ξ and a non-constant scalar curvature R one meets $v(x) \neq const$. Let us look for a solution in the form of a power series in curvature or, equivalently, in ξ,

$$v(x) = v_0 + v_1(x) + v_2(x) + \dots .$$

In the first order

$$-\Box v_1 + \mu^2 v_1 + \xi R\, v_0 - 6\lambda v_0^2\, v_1 = 0.$$

The solution of this equation has the form

$$v_1 = \frac{\xi\, v_0}{\Box - \mu^2 + 6\lambda v_0^2}\, R = \frac{\xi\, v_0}{\Box + 4\lambda v_0^2}\, R,$$

If one replaces the solution for $v(x)$ back into the action, the induced low-energy action of vacuum follows,

$$S_{ind} = \int d^4x \sqrt{-g}\, \Big\{ g^{\mu\nu}\, \partial_\mu v\, \partial_\nu v + (\mu_0^2 + \xi R)\, v^2 - \lambda\, v^4 \Big\}$$

with the induced Newton and cosmological constants

$$\frac{1}{16\pi\, G_{ind}} = -\,\xi\, v_0^2\,, \qquad \frac{\Lambda_{ind}}{8\pi\, G_{ind}} = -\,\mu_0^2\, v_0^2 + \lambda\, v_0^4 = -\,\lambda\, v_0^4.$$

On the top of that, we get many non-local terms, e.g.,

$$S_{ind} = \int d^4x \sqrt{-g}\, \Big\{ \lambda v_0^4 + \xi R v_0^2 + \xi^2\, v_0^2\, R\, \frac{1}{\Box + 4\,\lambda v_0^2}\, R + \dots \Big\}.$$

The quantization in curved space

can be performed by means of the path integral approach.

The generating functional of the connected Green functions $W[J, g_{\mu\nu}]$ is defined as

$$e^{iW[J,g_{\mu\nu}]} = \int d\Phi \, e^{iS[\Phi,g]+i\Phi J} \, ,$$

$d\Phi$ is the invariant measure of the functional integral and $J(x)$ are independent sources for the fields $\Phi(x)$.

- **The classical action is replaced by the Effective Action (EA)**

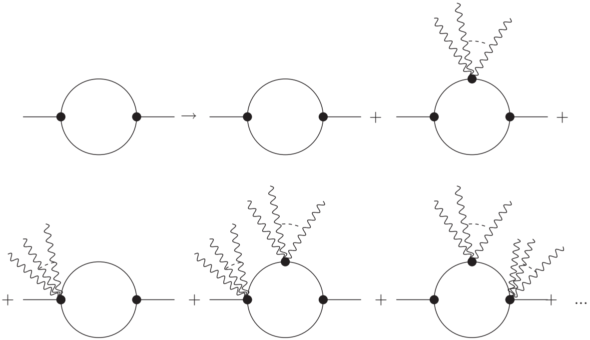
$$\Gamma[\Phi, g_{\mu\nu}] = W[J(\Phi), g_{\mu\nu}] - J(\Phi) \cdot \Phi \, , \qquad \Phi = \frac{\delta W}{\delta J} \, ,$$

which depends on the mean fields Φ and on $g_{\mu\nu}$.

The QFT in curved space, as it is formulated above, is renormalizable and consistent.

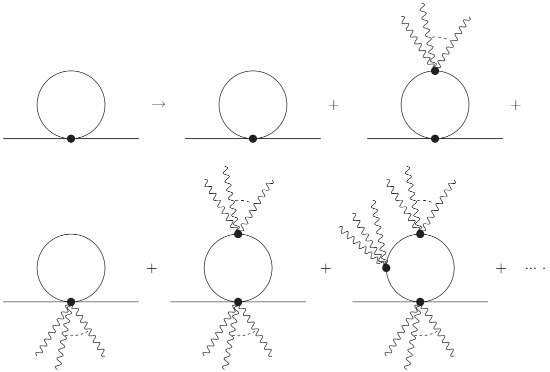
The main difference with QFT in flat space is that in curved space EA depends on the background metric, $\Gamma[\Phi, g_{\mu\nu}]$

In terms of Feynman diagrams, one has to consider graphs with internal lines of matter fields & external lines of both matter and metric. In practice, one can consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.



An important observation is that
all those “new” diagrams with $h_{\mu\nu}$ legs have superficial degree of divergence equal or lower that the “old” flat-space diagrams.

Consider the case of scalar field which shows why the nonminimal term is necessary



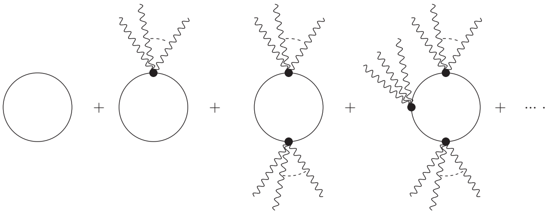
In general, the theory in curved space can be formulated as renormalizable. One has to follow the prescription

$$S_t = S_{min} + S_{non.min} + S_{vac} \,.$$

Renormalization involves fields and parameters like couplings and masses, ξ and vacuum action parameters.

Introduction: *Buchbinder, Odintsov & I.Sh. (1992).*

Relevant diagrams for the vacuum sector



All possible covariant counterterms have the same structure as

$$S_{vac} \,=\, S_{EH} \,+\, S_{HD}$$

In curved space the Effective Action (EA) depends on metric

$$\Gamma[\Phi] \to \Gamma[\Phi, g_{\mu\nu}]\,.$$

Feynman diagrams: one has to consider grafs with internal lines of matter fields and external limes of both matter and metric. In practice, one can consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

Is it possible to get EA for an arbitrary background in this way? Perhaps not. But it is sufficient to explore renormalization!

An important aspect is that the general covariance in the non-covariant gauges can be shown in the framework of mathematically rigid Batalin-Vilkovisky quantization scheme:

- *P. Lavrov and I.Sh., Phys. Rev. D81 (2010).*

Strong arguments supporting locality of the counterterms follow from the “quantum gravity completion” consideration.

Still, it would be very nice to have an explicitly covariant method of deriving counterterms at all loop orders.

Riemann normal coordinates.

A.Z.Petrov, Einstein Spaces. (Pergamon Press, 1969).
T.S. Bunch & L. Parker, Phys. Rev. D20 (1979) 2499. + InSpire.

Consider manifold $M_{3,1}$ and choose a point with coordinates x'^{μ}. The normal coordinates $y^\mu = x^\mu - x'^\mu$ satisfy several conditions.

The lines of constant coordinates are geodesics which are completely defined by the tangent vectors

$$\xi^\mu = \frac{dx^\mu}{d\tau}\Big|_{x'}, \qquad \tau(x') = 0$$

and τ is natural parameter along the geodesic. Moreover, we request that metric at the point x' be the Minkowski one $\eta_{\mu\nu}$. For an arbitrary function $A(x)$

$$A(x'+y) = A' + \frac{\partial A}{\partial y^\alpha}\Big| \, y^\alpha + \frac{1}{2}\, \frac{\partial^2 A}{\partial y^\alpha \partial y^\beta}\Big| \, y^\alpha y^\beta + \dots ,$$

where the line indicates $y^\mu = 0$.

Direct calculations show that

$$\Gamma^{\lambda}_{\alpha\beta}(x) \,=\, \frac{2}{3}\,{R'}^{\lambda}{}_{(\alpha\beta)\nu}\, y^{\nu} - \frac{1}{2}\,{R'}^{\lambda}{}_{\nu\alpha(\mu\,;\,\beta)}\, y^{\mu}\, y^{\nu} \,+\, ... \,,$$

r.h.s. depends only on the tensor quantities at the point $y = 0$.
From this follows the expansion for the metric

$$g_{\alpha\beta}(y) = \eta_{\alpha\beta} - \frac{1}{3}\,R'_{\alpha\mu\beta\nu}y^{\mu}y^{\nu} - \frac{1}{6}R'_{\alpha\nu\beta\lambda;\mu}y^{\mu}y^{\nu}y^{\lambda} \,+\, ... \,.$$

and $\quad R_{\mu\rho\nu\sigma}(y) = R'_{\mu\rho\nu\sigma} + R'_{\mu\rho\nu\sigma;\lambda}y^{\lambda} + ...$

The most fortunate feature of these series is that coefficients are curvature tensor and its covariant derivatives at one point $y = 0$.

We gain a tool for deriving local quantities, e.g., counterterms. The covariance is guaranteed by construction!

The procedure is as follows:

- **Introduce local momentum representation at the point** $y = 0$.
- **Develop Feynman technique in the momentum space.**
- **Calculate diagrams with the new propagators and vertices.**

Everything is manifestly covariant with respect to the transformations in the point x'.

Example. Scalar field propagator. The bilinear operator

$$\hat{H} = -\frac{1}{\sqrt{-g}}\,\frac{\delta^2 S_0}{\delta\varphi(x)\,\delta\varphi(x')}\,.$$

It has the form $\quad \hat{H} = \left(\Box - m^2 - \xi R\right)_x\,.$

Expanding $\hat{H}$ in normal coordinates in $\mathcal{O}(R)$

$$\hat{H} = \partial^2 - m^2 - \xi R + \frac{1}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}y^{\alpha}y^{\beta}\partial_{\mu}\partial_{\nu} - \frac{2}{3}R^{\alpha}_{\beta}y^{\beta}\partial_{\alpha} + \,\dots\,.$$

The equation for the propagator is

$$\hat{H}\,G(x,x') = -\,\delta(x,x')\,.$$

which leads to the following expression:

$$G(k) = \frac{1}{k^2+m^2} + \frac{1}{3}\,\frac{(1-3\xi)R}{(k^2+m^2)^2} - \frac{2}{3}\,\frac{R^{\mu\nu}k_{\mu}k_{\nu}}{(k^2+m^2)^3} + \mathcal{O}\left(\frac{1}{k^3}\right)\,.$$

One can continue this expansion to further orders in curvature.

It is clear that higher orders in an expansion

$$G(k) = \frac{1}{k^2+m^2} + \frac{1}{3}\,\frac{(1-3\xi)R}{(k^2+m^2)^2} - \frac{2}{3}\,\frac{R^{\mu\nu}k_\mu k_\nu}{(k^2+m^2)^3} + \mathcal{O}\left(\frac{1}{k^3}\right)$$

will always produce less divergences when replaced into internal line of the loop Feynman diagram.

The same effect occurs in the expansion in y^α for vertices.

For instance, any divergent diagram in renormalizable flat-space QFT has

$$d + D \leq 4\,,$$

where D is superficial degree of divergence and d is number of derivatives on external lines.

Clearly, the terms with background curvatures will have smaller $d+D$ and the maximal number of metric derivatives in vacuum diagrams is four.

The described method is explicitly covariant.

Combining the information from the two methods

- **Usual Feynman technique with external $h_{\mu\nu}$;**
- **Local momentum representation.**

The necessary counterterms in curved space are covariant local expressions constructed from matter fields and metric.

Consider a theory power-counting renormalizable in flat space.

- **Using Feynman technique with external $h_{\mu\nu}$ tails we observe an increase of the number of propagators and vertices**
$\Longrightarrow$ **superficial degree of divergence decrease.**

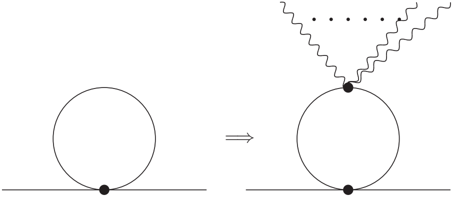
- **Using local momentum representation: the new terms always have some extra negative powers of momenta k, compensated by the background curvatures and their derivatives**
$\Longrightarrow$ **superficial degree of divergence decrease.**

Therefore, independent of the approach, the new counterterms do not have $O(1/mass)$ -factors and the theory remains power-counting renormalizable in curved space.

Types of the counterterms:

- **Minimal, e.g.,** $m^2\varphi^2, \ (\nabla\varphi)^2, \ i\bar{\psi}\gamma^\mu\nabla_\mu\psi$.
- **Non-minimal in the scalar sector,** $R\varphi^2$.

E.g., the quadratically divergent diagram



in the $\lambda\varphi^4$ **theory produces log. divergences corresponding to** $\int d^4\sqrt{-g}R\varphi^2$ **counterterm.**

- **Vacuum terms** $\Lambda, R, R^2, C^2,$ **etc.**

Renormalization doesn't depend on the choice of the metric!

Renormalization in matter fields sector

It is possible to perform renormalization in curved space in a way similar to the one in flat space.

Counterterms are controlled by symmetries & power counting.

In the simple case of the scalar $\lambda\varphi^4$-theory,

$$S_0 = \int d^4x \sqrt{-g}\,\Big\{ \frac{1}{2}\, g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi + \frac{1}{2}\,\big(m^2 + \xi R\big)\,\varphi^2 - \frac{f}{4!}\,\varphi^4 \Big\},$$

we meet, in dim. regularization, the following counterterms:

$$\Delta S_{scal} = \int d^nx \sqrt{-g}\;\mu^{n-4}\,\sum_{k=1}^{10}\alpha_k\left(\frac{1}{n-4}\right) \times \mathcal{L}_k\,,$$

where $\mathcal{L}_7 = (\nabla\varphi)^2\,, \quad \mathcal{L}_8 = m^2\varphi^2\,,$

$$\underline{\mathcal{L}_9 = R\varphi^2}\,, \quad \mathcal{L}_{10} = \varphi^4\,, \quad \mathcal{L}_{1,\dots,6} = \mathcal{L}_{1,\dots,6}(g_{\mu\nu})\,.$$

$\alpha_k(x)$ **are polynomials of the order equal to the loop order.**

The situation is similar for any theory which is renormalizable in flat space: only $\xi R\varphi^2$ counterterms represent a new element in the matter sector.

Moreover, due to covariance, multiplicative renormalization factors, e.g., Z_1, in

$$\varphi_0 = \mu^{\frac{n-4}{2}} Z_1^{1/2} \varphi \,,$$

are exactly the same as in the flat space.

The renormalization relations for the scalar mass m and nonminimal parameter ξ have the form

$$m_0^2 = Z_2 \, m^2 \,, \qquad \xi_0 - \frac{1}{6} = \tilde{Z}_2 \left(\xi - \frac{1}{6}\right) + Z_3 \,.$$

At one loop we have, also,

$$\tilde{Z}_2 = Z_2 \,, \qquad Z_3 = 0 \,.$$

So, in principle, we even do not need to perform a special calculation of renormalization for ξ at the 1-loop order.

Renormalization in the vacuum sector

Remember the action of vacuum is $S_{vac} = S_{EH} + S_{HD}$,

where $$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, \{R + 2\Lambda\} \,.$$

and $$S_{HD} = \int d^4x \sqrt{-g}\, \{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\} \,,$$

The possible counterterms are:

$$\Delta S_{vac} = \int d^nx \,\sqrt{-g}\, \mu^{n-4} \sum_{k=1}^{6} \hat{\alpha}_k \left(\frac{1}{n-4}\right) \times \hat{\mathcal{L}}_k \,,$$

where $\hat{\mathcal{L}}_\Lambda = 1\,, \quad \hat{\mathcal{L}}_G = R\,, \quad \hat{\mathcal{L}}_1 = C^2\,,$

$$\hat{\mathcal{L}}_2 = E\,, \quad \hat{\mathcal{L}}_3 = \Box R\,, \quad \hat{\mathcal{L}}_4 = R^2\,.$$

$\alpha_k(x)$ **are polynomials of the order equal to the loop order.**

General situation

● The theory such as SM, GUT etc, which is renormalizable in flat space, can be formulated as renormalizable in curved space

● The action of the theory can be divided into following three sectors:
1. Minimal matter sector;
2. Non-minimal matter sector;
3. Vacuum (metric-dependent) sector.

● The renormalization satisfies the hierarchy

$$1. \Longrightarrow 2. \Longrightarrow 3.$$

In the minimal sector it is identical to the one in flat space.

● The conformal invariance is supposed to hold in the one-loop counterterms, $\xi = 1/6, \quad a_4 = 0$.

Exercises and references

1. Derive the propagator and vertex of scalar model with φ^3 interaction in the linear parametrization of gravity on flat background in the first (second, if you have a lot of force) approximation in $h_{\mu\nu}$.

2. Repeat the calculations for covariant derivative of fermion.

Refs.: [2.1] Book by Parker and Toms.
[2.2] arXiv:1611.02263.

3. Make the calculations for expansion of metric in normal coordinates up to the second order in curvature. Derive propagator of a scalar field. Use this result to calculate the effective potential in the massive φ^4 theory.

Refs.: [3.1] A.Z.Petrov, *Einstein Spaces.* (Pergamon, Oxford, 1969).
[3.2] T.S. Bunch and L. Parker, Phys. Rev. D20 (1979) 2499.
[3.3] arXive:1107.2262.

4. Consider the power counting for the curved space QFT in $2D$. Do we need a non-minimal scalar to have renormalizable theory in this case? What is the minimal necessary form of the vacuum action?