Quantum Gravity from the QFT perspective

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Lecture 3. Conformal anomaly and effective action

- **Some examples of** $4d$ **conformal theories.**
- **Conformal anomaly and its ambiguities.**
- **Anomaly induced effective action.**
- **Light massive fields case.**

Applications:

- **Vacuum states in the vicinity of a black hole.**
- **Extended Starobinsky model.**

Examples of *4d* conformal theories

- **General scalar action with ξ term**

$$S_{scal} = \int d^4x\sqrt{-g}\,\left\{\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\,\xi R\phi^2 - \frac{f}{4!}\,\phi^4\right\}$$

is invariant under global but not local conformal transformation.

$$g_{\mu\nu}\,\to\,g'_{\mu\nu} = g_{\mu\nu}e^{2\lambda}\,,\qquad \phi\,\to\,\phi' = \phi\,e^{-\lambda}\,,\qquad \lambda = \textbf{const}\,.$$

Only in the case $\xi = \dfrac{1}{6}$

one meets local conformal symmetry

$$g_{\mu\nu}\,\to\,g'_{\mu\nu} = g_{\mu\nu}\,e^{2\sigma},\qquad \phi\,\to\,\phi' = \phi\,e^{-\sigma},$$

$$\sigma = \sigma(x)\,.$$

- **General metric-dilaton theory** *Shapiro & Takata, PLB-1994*

$$S = \int d^4x\sqrt{-g}\,\{A(\phi)\;(\nabla\phi)^2 + B(\phi)R + C(\phi)\}\,.$$

Consider conformal transformation of the metric plus scalar reparametrization

$$g'_{\mu\nu} = g_{\mu\nu}e^{2\sigma(\phi)}, \qquad\qquad \Phi = \Phi(\phi)$$

The well-known particular case is

$$S = \int d^4x\sqrt{-g}\,\left\{-\frac{1}{2}\,\phi\,\Delta_2\,\phi - \frac{f}{4\,!}\,\phi^4\right\}$$

where

$$\Delta_2 = \Box + \frac{1}{6}\,R\,.$$

It is equivalent to Einstein-Hilbert action with a wrong sign

$$S_{EH} = +\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\{R + 2\Lambda\}\,.$$

Massless spinor and vector fields

$$S_{1/2} = \frac{i}{2} \int d^4x \sqrt{-g} \, \{ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \}$$

and

$$S_1 = -\frac{1}{4} \int d^4x \sqrt{-g} \, F_{\mu\nu} F^{\mu\nu} \, .$$

The transformation rules are

$$\psi \to \psi' = \psi \, e^{-3\sigma/2} \, , \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi} \, e^{-3\sigma/2} \, , \quad A_\mu \to A'_\mu = A_\mu \, ,$$

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} \, e^{2\sigma} \qquad \sigma = \sigma(x) \, .$$

Note: the difference between conformal weight and dimension for the vector field is due to

$$A_\mu = A_b \, e^b_\mu \, , \quad e^b_\mu \, e^a_\nu \, \eta_{ab} = g_{\mu\nu} .$$

Direct relation between local & global conformal symmetries.

● **The conformal (Weyl) gravity in the dimension $n=4$ includes only metric field**

$$S_W \,=\, \int d^4x\sqrt{-g}\,C^2\,,$$

It can be easily generalized to an arbitrary dimension

$$C^2(n) \,=\, R^2_{\mu\nu\alpha\beta} - \frac{4}{n-2}\,R^2_{\mu\nu} + \frac{1}{(n-1)(n-2)}\,R^2\,.$$

● **Fourth derivative scalar of the first kind**

$$S_4 \,=\, \int d^4x\sqrt{-g}\;\varphi\,\Delta_4\,\varphi\,,$$

where $\quad \Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,R_{;\mu}\,\nabla^\mu\,.$

The transformation law is $\varphi \to \varphi'\,.$

S.M. Paneitz, MIT preprint - 1983; SIGMA - 2008
R.J. Riegert; E.S. Fradkin & A.A. Tseytlin, PLB - 1984.

● ● **General review of classical conformal theories** ● ●
V.Faraoni, E.Gunzig, P.Nardone, Fund.Cosm.Phys., gr-qc/9811047.

Quantum (Semiclassical) Theory

Introduction: *Birrell & Davies (1980); Buchbinder, Odintsov & I.Sh. (1992); L. Parker & D.J. Toms (2009).*

The most remarkable thing at the quantum level is that the classical conformal invariance is broken by trace anomaly.

Recent reviews: *I.Sh. et al. - gr-qc/0412113, hep-th/0610168 (both very technical), gr-qc/0801.0216.*

The first step is to consistently formulate the action on classical curved background.

In a conformal theory at 1-loop level it is sufficient to consider

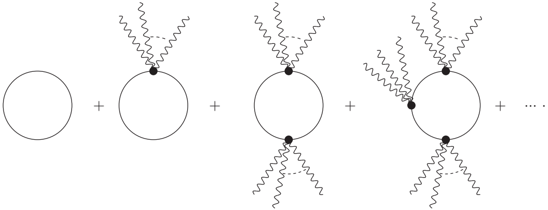
$$S_{conf.\ vac} = \int d^4x\sqrt{-g}\,\big\{a_1 C^2 + a_2 E + a_3 \Box R\big\}\ .$$

QFT in curved space can be renormalizable if we define

$$S_t = S_{min} + S_{non.min} + S_{vac} \,.$$

Renormalization involves fields and parameters like couplings and masses, ξ and vacuum action parameters.

Relevant diagrams for the vacuum sector



All possible covariant counterterms have the same structure as

$$S_{vac} = S_{EH} + S_{HD}\,, \qquad S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R + 2\Lambda)\,,$$

$$S_{HD} = \int d^4x\sqrt{-g}\,\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\}\,.$$

Conformal anomaly

k_Φ **is the conformal weight of the field** Φ.
The Noether identity for the local conformal symmetry

$$\Big[-2\,g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}} + k_\Phi\,\Phi\,\frac{\delta}{\delta\Phi}\Big]\;S(g_{\mu\nu},\,\Phi)\;=\;0$$

produces on shell $\quad -\,\frac{2}{\sqrt{g}}\,g_{\mu\nu}\,\frac{\delta S_{vac}(g_{\mu\nu})}{\delta g_{\mu\nu}}\;=\;T_{(vac)}{}_{\mu}^{\;\mu}\;=\;T_{\mu}^{\mu}\;=\;0\,.$

At quantum level $S_{vac}(g_{\mu\nu})$ **is replaced by the EA** $\Gamma_{vac}(g_{\mu\nu})$.

For free fields only 1-loop order is relevant [here $\varepsilon=(4\pi)^2(n-4)$**]**

$$\Gamma_{div}\;=\;-\,\frac{1}{\varepsilon}\,\int d^4x\sqrt{g}\,\big\{\beta_1 C^2+\beta_2 E+\beta_3\Box R\big\}\,.$$

For the global conf. symmetry the renormalization group tells us

$$\langle T_{\mu}^{\mu}\rangle\;=\;\big\{\beta_1 C^2+\beta_2 E+a'\Box R\big\}\,,$$

where $a'=\beta_3$. **In the local case** a' is **ambiguous**.

The simplest way to derive the conformal anomaly is using dimensional regularization (Duff, 1977).

The expression for divergences

$$\bar{\Gamma}_{div} \;=\; \frac{1}{\varepsilon}\,\int d^4x\sqrt{g}\,\left\{\beta_1 C^2 + \beta_2 E + \beta_3 \Box R\right\}\;.$$

where

$$\begin{pmatrix} \beta_1 \\ -\beta_2 \\ \beta_3 \end{pmatrix} = \frac{1}{360(4\pi)^2}\begin{pmatrix} 3N_0 + 18N_{1/2} + 36N_1 \\ N_0 + 11N_{1/2} + 62N_1 \\ 2N_0 + 12N_{1/2} - 36N_1 \end{pmatrix}$$

The renormalized one-loop effective action has the form

$$\Gamma_R = S + \bar{\Gamma} + \Delta S\,,$$

where $\bar{\Gamma} = \bar{\Gamma}_{div} + \bar{\Gamma}_{fin}$ **is the naive quantum correction to the classical action and** ΔS **is a counterterm.**

ΔS **is an infinite local counterterm which is called to cancel the divergence. It is the only source of non-invariance.**

The anomalous trace is

$$T = \langle T^\mu_\mu \rangle = -\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\, \Gamma_R}{\delta\, g_{\mu\nu}}\Big|_{n=4} = -\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\, \Delta S}{\delta\, g_{\mu\nu}}\Big|_{n=4} .$$

Conformal parametrization of the metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}\,, \qquad \sigma = \sigma(x)$$

where $\bar{g}_{\mu\nu}$ is the fiducial metric with fixed determinant.

There is a useful relation

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, A[g_{\mu\nu}]}{\delta\, g_{\mu\nu}} = \frac{1}{\sqrt{-\bar{g}}}\, \frac{\delta\, A[\bar{g}_{\mu\nu}\, e^{2\sigma}]}{\delta\sigma}\Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu},\, \sigma\to 0,\, n\to 4} \qquad (*)$$

$$\int d^n x \sqrt{-g}\, C^2(n) = \int d^n x \sqrt{-\bar{g}}\, e^{(n-4)\sigma}\, \bar{C}^2(n)\,.$$

Then $$\frac{\delta}{\delta\sigma} \int \frac{d^4x \sqrt{-\bar{g}}}{n-4}\, e^{(n-4)\sigma}\, \bar{C}^2(n)\Big|_{n\to 4} = \sqrt{-g}\, C^2\,.$$

The derivatives of $\sigma(x)$in other terms are irrelevant.

In the simplest case $\sigma = \lambda = const$, we immediately arrive at the expression for T with $a' = \beta_3$.

For global conformal transform this procedure always works,

$$\langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^2} \left(\omega C^2 + bE + c\Box R \right) .$$

However the local case $\sigma(x)$ it is more complicated, e.g.,

$$\frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{-g}\Box R \equiv 0 .$$

We have a conflict between global and local conf. anomalies.

Or a conflict between formulas and intuitive expectations.
M.J. Duff, Class. Quantum. Grav. (1994)

Problem resolved:
M. Asorey, E. Gorbar & I.Sh., CQG 21 (2003).

Anomaly-induced Effective Action (EA) of vacuum

One can use $\langle T_\mu^\mu \rangle$ to obtain equation for the finite 1-loop EA

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} = \langle T_\mu^\mu \rangle = \frac{1}{(4\pi)^2} \left(\omega C^2 + bE + c\Box R \right) .$$

The solution is straightforward
Riegert; Fradkin & Tseytlin, PLB-1984.

It can be generalized for the theory with more background fields, e.g., with vector, torsion or scalar fields.

I.L. Buchbinder, S.D. Odintsov & I.Sh. Phys.Lett. B (1985).
J.A. Helayel-Neto, A. Penna-Firme & I.Sh. Phys.Lett. B (1998);
I.Sh., J. Solà, Phys.Lett. B (2002);
M. Giannotti, E. Mottola, Phys. Rev. D (2009).

The simplest possibility is to parameterize metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}\,, \qquad \sigma = \sigma(x)\,.$$

The solution for the effective action is

$$\bar{\Gamma}_{ind} \;=\; S_c[\bar{g}_{\mu\nu}] \,+\, \frac{1}{(4\pi)^2}\,\int d^4x\,\sqrt{-\bar{g}}\,\{\omega\sigma\bar{C}^2 \tag{1}$$

$$+\, b\sigma(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}) + 2b\sigma\bar{\Delta}_4\sigma \,-\, \frac{1}{12}\,(c + \frac{2}{3}b)[\bar{R} - 6(\bar{\nabla}\sigma)^2 - (\bar{\Box}\sigma)]^2)\}\,,$$

where $S_c[\bar{g}_{\mu\nu}] = S_c[g_{\mu\nu}]$ is an unknown conformal functional, which serves as an integration constant in eq. for Γ_{ind}.

The solution (1) has serious merits:
1) Being simple, 2) Being exact in case $S_c[\bar{g}_{\mu\nu}]$ is irrelevant.
Example: FRW metrics.

An important disadvantage is that it is not covariant or, in other words, it is not expressed in terms of original metric $g_{\mu\nu}$.

Now we obtain the non-local covariant solution and after represent it in the local form using auxiliary fields.
First one has to establish the relations

$$\sqrt{-g}C^2 = \sqrt{-\bar{g}}\bar{C}^2\,, \qquad \sqrt{-\bar{g}}\,\bar{\Delta}_4 = \sqrt{-g}\,\Delta_4\,,$$

$$\sqrt{-g}(E - \frac{2}{3}\Box R) = \sqrt{-\bar{g}}(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma)$$

and also introduce the Green function

$$\Delta_4\, G(x,y) = \delta(x,y)\,.$$

Using these formulas we find, for a functional $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu})$,

$$\frac{\delta}{\delta\sigma}\int_x A\;(E - \frac{2}{3}\Box R)\Big| \;=\; 4\sqrt{-g}\Delta_4\, A\,.$$

where $$\int_x = \int d^4x\,\sqrt{-g(x)}\,, \qquad \Big| = \Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu}}$$

As a consequence, we obtain

$$\frac{\delta}{\delta\sigma(y)} \iint\limits_{x\,y} \frac{1}{4}\, C^2(x)\, G(x,y)\, \Big(E - \frac{2}{3}\Box R\Big)_y \Big|$$

$$= \int d^4x \sqrt{-\bar{g}(x)}\, \bar{\Delta}_4(x)\, \bar{G}(x,y)\, \bar{C}^2(x) \Big| \;=\; \sqrt{-g}\, C^2(y)\,.$$

Hence, the part of Γ_{ind} which is responsible for $T_\omega = -\omega C^2$, is

$$\Gamma_\omega = \frac{\omega}{4} \iint\limits_{x\,y} C^2(x)\, G(x,y)\, (E - \frac{2}{3}\Box R)_y\,.$$

Similarly one can check that the variation $T_b = b\,(E - \frac{2}{3}\Box R)$ is produced by the term

$$\Gamma_b = \frac{b}{8} \iint\limits_{x\,y} (E - \frac{2}{3}\Box R)_x\, G(x,y)\, (E - \frac{2}{3}\Box R)_y\,.$$

Finally, we can use simple relation

$$g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\,\int_x R^2(x)\;=\;-6\,\sqrt{-g}\,\Box R\,.$$

to establish the remaining local constituent of Γ_{ind}

$$\Gamma_c = -\frac{3c+2b}{36(4\pi)^2}\,\int_x R^2(x)\,.$$

The general covariant solution for Γ_{ind} is the sum,

$$\Gamma_{ind} = S_c[g_{\mu\nu}] \;-\; \frac{3c+2b}{36(4\pi)^2}\,\int_x R^2(x)$$
$$+\,\frac{\omega}{4}\,\int_x\int_y C^2(x)\,G(x,y)\,(E-\frac{2}{3}\Box R)_y$$
$$+\,\frac{b}{8}\,\int_x\int_y (E-\frac{2}{3}\Box R)_x\,G(x,y)\,(E-\frac{2}{3}\Box R)_y\,.$$

One can rewrite this expression using auxiliary scalars.

The nonlocal terms can be rewritten in a symmetric form

$$\Big(E-\frac{2}{3}\Box R\Big)_x G(x,y)\Big[\frac{\omega}{4}C^2-\frac{b}{8}(E-\frac{2}{3}\Box R)\Big]_y$$

$$=\frac{b}{8}\int\!\!\int_{x\,y}\Big(E-\frac{2}{3}\Box R-\frac{\omega}{b}C^2\Big)_x G(x,y)\Big(E-\frac{2}{3}\Box R-\frac{\omega}{b}C^2\Big)_y$$

$$-\frac{\omega^2}{8b}\int\!\!\int_{x\,y} C_x^2\,G(x,y)\,C_y^2\,.$$

These form is appropriate for rewriting it via auxiliary fields.
Then we arrive at the local covariant expression for EA

$$\Gamma_{ind}=S_c[g_{\mu\nu}]-\frac{3c+2b}{36(4\pi)^2}\int_x R^2(x)+\int_x\Big\{\frac{1}{2}\,\varphi\Delta_4\varphi-\frac{1}{2}\,\psi\Delta_4\psi$$

$$+\frac{\omega}{8\pi\sqrt{-b}}\,\psi\,C^2+\varphi\,\Big[\frac{\sqrt{-b}}{8\pi}\,(E-\frac{2}{3}\,\Box R)-\frac{\omega}{8\pi\sqrt{-b}}\,C^2\Big]\Big\}\,.$$

The above form of EA is the best one for Γ_{ind}.
I.Sh. and A.Jacksenaev, Phys. Lett. B (1994)

Similar expression has been independently introduced by
P. Mazur & E. Mottola, 1997-1998.

Comments:

1) Imposing boundary conditions on the two auxiliary fields φ **and** ψ **is equivalent to defining boundary conditions for the Green functions** $G(x,y)$.

2) Introducing the new term $\int C_x^2 G(x,y) C_y^2$ **into the action may be viewed as redefinition of the conformal functional** $S_c[g_{\mu\nu}]$.

However, writing the non-conformal terms in the symmetric form, essentially modifies the four-point function. Using ψ **we restore the structure generated by anomaly.**

Recent generalizations:

- **Integration of trace anomaly in 6D.**

F. Ferreira & I.Sh., PLB (2017), arXiv:1702.06892.

- **Quantum effects of chiral fermion**

$$\langle T^\mu_\mu \rangle = -\omega_1 C^2 - bE_4 - c\Box R - \epsilon P_4 \,,$$

where the Pontryagin density term appears,

$$P_4 = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta} R_{\mu\nu\rho\sigma} R_{\alpha\beta}{}^{\rho\sigma}\,, \qquad \epsilon = \frac{i}{48\cdot 16\pi^2}\,.$$

L.Bonora, S.Giaccari, B.de Souza, JHEP (2014), arXiv:1403.2606.

It is an easy exercise to derive the anomaly-induced eff. action

S. Mauro, I.Sh., PLB (2015), arXiv:1412.5002.

First, one can prove the conformal symmetry of this term

$$P_4 = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,C_{\mu\nu\rho\sigma}\,C_{\alpha\beta}{}^{\rho\sigma}\,.$$

After that we immediately arrive at

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int_x R^2 + \int_x \Big\{\frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$

$$+\varphi\left[\frac{\sqrt{-b}}{8\pi}\left(E - \frac{2}{3}\,\square R\right) - \frac{1}{8\pi\sqrt{-b}}\left(\omega C^2 + \epsilon P_4\right)\right] + \frac{1}{8\pi\sqrt{-b}}\,\psi\left(\omega C^2 + \epsilon P_4\right)\Big\}.$$

It is natural to change variables: $\chi = \dfrac{\psi - \varphi}{\sqrt{2}}, \qquad \xi = \dfrac{\psi + \varphi}{\sqrt{2}}.$

Then the total gravitational action becomes

$$\Gamma_{grav} = S_{EH} + S_{HD} + S_c[g_{\mu\nu}] + \int_x \Big\{\xi\Delta_4\chi + k_1\big(E - \frac{2}{3}\,\square R\big)(\xi - \chi)$$

$$+ k_2\,\chi\,\left(\omega C^2 + \epsilon P_4\right) + k_3 R^2\Big\}.$$

The coefficients are, as before,

$$k_1 = \frac{1}{8\pi}\sqrt{-\frac{b}{2}}\,, \qquad k_2 = \frac{1}{8\pi\sqrt{-2b}}\,, \qquad k_3 = -\frac{2b+3c}{36\,(4\pi)^2}\,.$$

The action

$$\Gamma_{grav} = S_{EH} + S_{HD} + S_c[g_{\mu\nu}] + \int_x \Big\{ \underline{\xi\Delta_4\chi} + k_1\big(E - \frac{2}{3}\Box R\big)\big(\xi - \chi\big)$$

$$+ k_2\,\chi\,\big(\omega C^2 + \underline{\epsilon P_4}\big) + k_3 R^2 \Big\}$$

is a special case of the Chern-Simons modified general relativity,

R. Jackiw and S.Y. Pi, Phys. Rev. D 68 (2003), gr-qc/0308071.

A. Lue, L. Wang, M. Kamionkowski, Phys.Rev.Lett. 83 (1999) 1506.

S. Alexander, N. Yunes, Phys.Rept. 480 (2009) 1.

with a special form of the kinetic term.

Applications of the anomaly-induced EA

- **Classification of vacuum states in the vicinity of a black hole**

Anomaly is, in part, responsible for the Hawking radiation
S.M. Christensen, S.A. Fulling, PRD (1977).

The anomaly-induced effective action of gravity enables one to perform a kind of systematic classification of the vacuum states for the quantum fields on the black hole background.

We can distinguish the different vacuum states by choosing different boundary conditions for the auxiliary fields φ and ψ.

R. Balbinot, A. Fabbri & I.Sh., PRL **83***; NPB* **559** *(1999).*

Generalization for the Reissner-Nordstrom black hole,
P.R. Anderson, E. Mottola & R. Vaulin, PRD **76** *(2007).*

At the classical level, the black hole (BH) does not emit radiation, but such emission can take place if we take quantum effects into account.

After being discovered by Hawking (1975)**, the same result has been obtained from analytical estimates of $\langle T_{\mu\nu} \rangle$ for quantum matter fields in a fixed Schwarzschild BH geometry.**

S.M. Christensen & S.A. Fulling, PRD 15 (1977).

Detailed analytical and numerical study, based on the analysis of $\langle T_{\mu\nu} \rangle$ in the classical black hole background:

P. Candelas, PRD **21** *(1980);*
D.N. Page, PRD **25** *(1982);*
M.R. Brown, A.C. Ottewill and D.N. Page, PRD **33** *(1986);*
V.P. Frolov and A.I. Zelnikov, PRD **35** *(1987);*
P.R. Anderson, W.A. Hiscock and D.A. Samuel, PRD **51** *(1995). ...*

A fundamental property is the existence of three different vacuum quantum states.

i) **The Boulware $|B\rangle$ state reproduces the Minkowski vacuum $|M\rangle$ in the limit $r \to \infty$, where $\langle B|T_{\mu\nu}|B\rangle \sim r^{-6}$.**
On the horizon this quantity is divergent in a free falling frame.

ii) **For Unruh vacuum $|U\rangle$ the value $\langle U|T_{\mu\nu}|U\rangle$ is regular on the future event horizon but not on the past one. Asymptotically in the future $\langle U|T_{\mu\nu}|U\rangle$ has the form of a flux of radiation at the Hawking temperature $T_H = 1/8\pi M$.**
This vacuum state is the most appropriate to discuss evaporation of black holes formed by gravitational collapse of matter.

iii) **The Israel-Hartle-Hawking $|H\rangle$** state
$\langle H|T_{\mu\nu}|H\rangle$ **for $r \to \infty$ describes a thermal bath of radiation at T_H.**

The existence of three vacuum states reflects distinct positions of observers and the construction of different *in* and *out* modes with respect to the corresponding coordinates.

The main difference between classical and quantum theories is that, in the first case we know how to transform the relevant quantities when we change the coordinate system.

The natural question is how to perform a transition between different vacuum states $|H\rangle$, $|B\rangle$ and $|U\rangle$ **?**

The anomaly-induced effective action doesn't make any reference to a particular quantum state, but it includes the conformal invariant functional $S_c[g_{\mu\nu}]$ – a source of uncertainty.

Strategy: one has to fix the extended set of boundary conditions, including the ones for the auxiliary scalars φ and ψ.

The procedure for identifying the vacuum state is as follows:

1) Solving equations for φ and ψ.
The solutions always depend on the set of integration constants.

2) One has to find "appropriate" boundary conditions to identify $\langle V|T_{\mu\nu}|V\rangle$ for the given vacuum state $|V\rangle = (|B\rangle,\ |U\rangle,\ |H\rangle)$.

3) Use

$$\langle T_{\mu\nu}\rangle \quad \longrightarrow \quad \frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{ind}}{\delta g_{\mu\nu}} = \langle S_{\mu\nu}\rangle\,,$$

where of course $\langle S\rangle = \langle T\rangle$.

The general solution is $\varphi(r,t) = d \cdot t + w(r)$, **where** $w(r)$ **satisfies the equation**

$$\frac{dw}{dr} = \frac{B}{3}r + \frac{2MB}{3} - \frac{A}{6} - \frac{\alpha}{72M} + \frac{1}{r-2M}\Big(\frac{4}{3}BM^2 + \frac{C}{2M} - AM - \frac{\alpha}{24}\Big) - \frac{C}{2Mr}$$

$$- \left[\frac{\alpha M}{r^3} + \frac{24AM - \alpha}{144\,M^2}\right] \frac{r^2 \ln r}{r-2M} + \frac{(24AM - a)\big(r^3 - 8M^3\big)\ln(r-2M)}{3r(r-2M)48M^2}\,.$$

(d,A,B,C) **are constants which specify the homogeneous solution** $\Box^2\varphi = 0$ **and hence the quantum state.**

For ψ **we have a similar solution, but with** (d',A',B',C').

Due to the independence of φ **and** ψ. **the two sets are independent on each other.**

In case of a Boulware state $|B\rangle$ we request

$$|B\rangle \to |M\rangle \quad \text{when} \quad r \to \infty\,.$$

In the Minkowski vacuum we can safely set $\varphi = \psi = 0$.

This asymptotic conditions enables one to arrive at the asymptotic expressions

$$\langle B|S_\mu^\nu|B\rangle \,\to\, \frac{\alpha^2-\beta^2}{2\,(24)^2\,(2M)^4\,\big(1-2M/r\big)^2} \times \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}$$

for $r \to 2M$ and

$$\langle B|S_\mu^\nu|B\rangle \propto \mathcal{O}\big(r^{-6}\big) \qquad \text{for} \qquad r \to \infty\,.$$

This behavior fits perfectly will with the ones observed within other methods.

Unruh vacuum case

Choosing another values of the integration constants we meet the following asymptotic behavior near the horizon $r \to 2M$:

$$\langle U|S_a{}^b|U\rangle \sim \frac{\alpha^2-\beta^2}{2(48M^2)^2}\begin{pmatrix} 1/f & -1 \\ 1/f^2 & -1/f \end{pmatrix},$$

regular on the future horizon, $a, b = r, t$. **The asymptotic form**

$$r \to \infty \qquad \langle U|S_\mu{}^\nu|U\rangle \to \frac{\alpha^2-\beta^2}{2r^2(24M)^2}\begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

These results are in exact agreement with the standard ones on the Hawking radiation: *B.S. DeWitt, Phys. Rep.* **C19** *(1975) 297.*

once the luminosity L **of the radiating BH is identified with**

$$\frac{L}{4\pi} = \frac{(\alpha^2-\beta^2)}{2(24M)^2}\,.$$

Cosmological application: Starobinsky Model.

Starobinsky model based on quantum effects.

Fischetti, Hartle and Hu (1978);
Starobinsky, (1980-1983) ;
Mukhanov, Chibisov, (1982) ;
Anderson, Vilenkin, ... (1983-1986)
Hawking, Hertog and Real, (2001) .

Modified (or extended) Starobinsky model

Fabris, Pelinson, Solà, A. Starobinsky, T. Paula-Neto, I.Sh., ...

Cosmological Model based on the action

$$S_{total} \,=\, -\,\frac{M_P^2}{16\pi}\,\int d^4x\sqrt{-g}\,(R+2\Lambda)+S_{matter}+S_{vac}+\bar{\Gamma}_{ind}\,.$$

Equation of motion for $a(t),\quad dt=a\,(\eta)\,d\eta,\quad k=0$

$$\frac{\ddot{\ddot{a}}}{a}+\frac{3\dot{a}\dddot{a}}{a^2}+\frac{\ddot{a}^2}{a^2}-\left(5+\frac{4b}{c}\right)\frac{\ddot{a}\dot{a}^2}{a^3}-\frac{M_P^2}{8\pi c}\left(\frac{\ddot{a}}{a}+\frac{\ddot{a}^2}{a^2}-\frac{2\Lambda}{3}\right)\,=\,0\,,$$

$k=0,\pm 1$. **Particular solutions (Starobinsky, PLB-1980)**

$$a(t)\,=\,a_0e^{Ht}\,,\qquad\quad k=0\,,$$

where Hubble parameter $H=\dot{a}/a$ **is**

$$H^2\,=\,-\,\frac{M_P^2}{32\pi b}\,\left(1\pm\,\sqrt{1+\frac{64\pi b}{3}\,\frac{\Lambda}{M_P^2}}\,\right).$$

A. Pelinson, I.Sh., F. Takakura, NPB (2003).

For $0 < \Lambda \ll M_P^2$ there are two solutions:

$$H \approx \sqrt{\Lambda/3}\,; \qquad\qquad (IR)$$

$$H \approx \sqrt{-\frac{M_P^2}{16\pi b} - \frac{\Lambda}{3}} \approx \frac{M_P}{\sqrt{-16\pi b}}\,. \qquad\qquad (UV)$$

Perturbations of the conformal factor

$$\sigma(t) \to \sigma(t) + y(t).$$

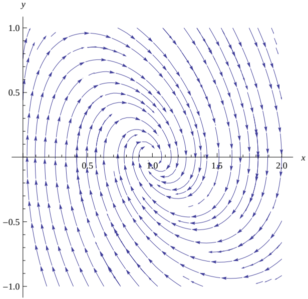
The criterion for a stable (UV) inflation is

$$c > 0 \quad \Longleftrightarrow \quad N_1 < \frac{1}{3}N_{1/2} + \frac{1}{18}N_0\,,$$

in agreement with Starobinsky (1980).

The original Starobinsky model is based on the unstable case and involves special choice of initial data. This situation can be improved further by using the stable version and an appropriate transition scheme.

In the unstable phase there are very different solutions, some of them violent (hyperinflation). **How can we know that the transition from stable to unstable phase really happens?** *A. Pelinson et al, NPB(PS) (2003).* **Phase portrait of a stable case:**



$$\text{Starobinsky (1980)}: \; x = \left(\frac{H}{H_0}\right)^{\frac{3}{2}}, \quad y = \frac{\dot{H}}{2\sqrt{H_0^3\, H}}, \quad dt = \frac{dx}{3H_0\, x^{2/3}\, y}.$$

Simple test of the unstable version of Starobinsky Model.
A.Pelinson, I.Sh. et al., IRGA-NPB(PS)- 2003.,

Consider late Universe, $k = 0, \; H_0 = \sqrt{\Lambda/3}$.

Only photon is active, $N_0 = 0\,, \qquad N_{1/2} = 0\,, \qquad N_1 = 1\,.$

Graviton typical energy is $H_0 \approx 10^{-42}\,GeV$, $\Longrightarrow$ **all massive particles (even neutrino)** $m_\nu \geq 10^{-12}\,GeV$ **decouple from gravity.** $c < 0 \quad \Longrightarrow$ **today inflation is unstable.**

Stability for the small $H = H_0$ **case:** $H \to H_0 + const \cdot e^{\lambda t}$

$$\lambda^3 + 7H_0\lambda^2 + \left[\frac{(3c-b)4{H_0}^2}{c} - \frac{M_P^2}{8\pi c}\right]\lambda - \frac{32\pi b{H_0}^3 + M_P^2 H_0}{2\pi c} = 0\,.$$

The solutions are $\lambda_1 = -4H_0\,, \quad \lambda_{2/3} = -\frac{3}{2}H_0 \pm \frac{M_P}{\sqrt{8\pi|c|}}\,i\,.$

$\Lambda > 0$ **protects our world from quantum corrections!**

Transition. **Suppose at UV** $(H \gg M_F)$ **there is SUSY, e.g. MSSM,**

$$N_1 = 12\,, \qquad N_{1/2} = 32\,, \qquad N_0 = 104\,.$$

This provides stable inflation, because

$$\frac{1}{3}\,N_{1/2} + \frac{1}{18}\,N_0 > N_1 \qquad \Longrightarrow \qquad c > 0\,.$$

For realistic SUSY model inflation is independent on initial data.

Fine!

But why should inflation end? **Already for MSM** $(\,N_{1,\,1/2,\,0} = 12, 24, 4)$, $c < 0$, **inflation is unstable.**

Natural interpretation:
I.Sh. Int.J.Mod.Ph.D. (2002); A. Pelinson et al NPB (2003).

All sparticles are heavy $\Rightarrow$ **decouple when** H **becomes smaller than their masses.**

Direct calculations confirmed that the transition $c > 0 \Longrightarrow c < 0$ **is smooth, indicating a possibility of a smooth graceful exit.**

• Using anomaly for deriving EA of massive fields.

Why the energy scale H decreases during inflation?

In the exponential phase Hubble parameter $H(t) = const$.

Another unclear point: Using anomaly-induced EA for massive fields is not a correct approximation.

Maybe all difficulties can be solved if taking masses of the fields into account?

Consider a reliable Ansatz for the EA of massive fields.
J.Solà, I.Sh. PLB - 2002;
also A.Pelinson, I.Sh. & F.Takakura, Nucl.Ph. 648B (2003).

In part, it is based on
R.D.Peccei, J.Solà, C.Wetterich, Ph.Lett. B 195 (1987) 183
and S. Deser, Ann. Phys. **59** *(1970) 248.*

The idea is to construct the conformal formulation of the SM and use it to derive EA for massive fields.

Conformal formulation of massive theory

The conformally non-invariant terms:

$$m_s^2\,\varphi^2\,,\quad m_f\,\bar{\psi}\psi\,,\quad \textbf{and}\quad L_{EH}\,=\,-\,\frac{1}{16\pi G}\,(R+2\Lambda)\,.$$

Replacing dimensional parameters by the new scalar χ:

$$m_{s,f}\to\frac{m_{s,f}}{M}\,\chi\,,\quad M_P^2\to\frac{M_P^2}{M^2}\,\chi^2\,,\quad \Lambda\to\frac{\Lambda}{M^2}\,\chi^2\,.$$

M **is related to a scale of conformal symmetry breaking. Massive terms get replaced by Yukawa and** $(scalar)^4$ **type interactions with** χ**. In the IR** $\chi\sim M$**.**

In the gravity sector

$$\mathcal{L}_{EH}^{*}=-\frac{M_P^2}{16\pi\,M^2}\,\Big\{\big[R\chi^2+6\,(\partial\chi)^2\big]\,+\,\frac{2\Lambda\,\chi^4}{M^2}\Big\}$$

in order to provide local conformal invariance.

The new theory is conformal invariant

$$\sigma = \sigma(x), \qquad \begin{cases} \chi \to \chi\, e^{-\sigma}\,, & g_{\mu\nu} \to g_{\mu\nu}\, e^{2\sigma} \\ \varphi \to \varphi\, e^{-\sigma}\,, & \psi \to \psi\, e^{-3/2\,\sigma} \end{cases}$$

The conformal symmetry comes together with a new scalar χ, absorbing conformal degree of freedom. Fixing $\chi \to M$ we come back to original formulation.
The conformal anomaly becomes

$$\langle T \rangle \,=\, -\Big\{ wC^2 + bE + c\Box R \;+\; \frac{f}{M^2}\,[R\chi^2 + 6(\partial\chi)^2] + \frac{g}{M^4}\,\chi^4 \Big\},$$

f **and** g **are** β**-functions for** $(16\pi G)^{-1}$ **and** $\rho_\Lambda = \Lambda/8\pi G$ **,**

$$f \,=\, \sum_i \frac{N_f}{3\,(4\pi)^2}\, m_f^2\,, \qquad \tilde{f} = \frac{16\pi f}{M_P^2}\,,$$

$$g \,=\, \frac{1}{2(4\pi)^2} \sum_s N_s\, m_s^4 - \frac{2}{(4\pi)^2} \sum_f N_f\, m_f^4\,,$$

N_f **and** N_s **are multiplicities of the fields.**

Anomaly-induced EA in terms of $g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}$ and $\chi = \bar{\chi} \cdot e^{-\sigma}$

$$\bar{\Gamma} \,=\, S_c[\bar{g}_{\mu\nu}, \bar{\chi}] \,+\, \int d^4x \sqrt{-\bar{g}} \Big\{ w\sigma \bar{C}^2 + b\sigma (\bar{E} - \frac{2}{3}\bar{\nabla}^2 \bar{R}) + 2b\sigma \bar{\Delta}\sigma$$

$$+\, \frac{f}{M^4}\, \sigma [\bar{R}\bar{\chi}^2 + 6(\partial\bar{\chi})^2] \,+\, \frac{g}{M^4}\, \bar{\chi}^4 \sigma \,\Big\} - \frac{3c+2b}{36} \int d^4x \sqrt{-g} R^2 \,.$$

This may be seen as a local version of Renormalization Group.
In curved space-time RG corresponds to the scaling

$$g_{\mu\nu} \to g_{\mu\nu} \cdot e^{-2\tau} \quad \Longrightarrow \quad \Gamma[e^{-2\tau} g_{\alpha\beta}, \Phi_i, P, \mu] = \Gamma[g_{\alpha\beta}, \Phi_i(\tau), P(\tau), \mu] \,.$$

In the leading-log approximation we meet an
RG - improved classical action of vacuum

$$S_{vac}[g_{\alpha\beta}, P(\tau), \mu] \,, \quad \textbf{where} \quad P(\tau) = P_0 + \beta_P \, \tau \,.$$

The equivalence in all terms which do not vanish for $\sigma = \tau$.

Cosmological implications

$$S_t = S_{matter} + S^*_{EH} + S_{vac} + \bar{\Gamma}\,.$$

The equation of motion for $\Lambda = 0,\ \ g = 0$

$$a^2\,\dddot{a} + 3\,a\,\dot{a}\,\ddot{a} - \left(5 + \frac{4b}{c}\right)\dot{a}^2\,\ddot{a} + a\,\ddot{a}^2 - \frac{M_P^2}{8\pi c}\left(a^2\ddot{a} + a\dot{a}^2\right)\,[1 - \tilde{f}\cdot\ln a] \,=\, 0\,,$$

Let us solve by $M_P^2 \to M_P^2\,[1 - \tilde{f}\cdot\ln a]$**,**

$$\dot{\sigma} = H = H_o\sqrt{1 - \tilde{f}\sigma(t)}\,, \qquad H_o = \frac{M_P}{\sqrt{-16b}}\,.$$

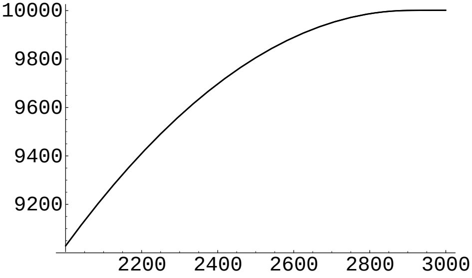
This leads to the simple solution

$$\sigma(t) \,=\, H_o\,t \,-\, \frac{H_0^2}{4}\,\tilde{f}\;t^2\,.$$

Remarkably, this formula fits with the numerical solution with a wonderful 10^{-6} **precision!**

$\tilde{f} > 0 \ \ \Rightarrow$ **we arrive at the tempered inflation!!**

Anomaly-induced inflation slows down if taking masses of quantum fields into account.



$$\sigma(t) = \ln a(t) \approx H_0 t - \frac{H_0^2}{4} \tilde{f} \, t^2 \,, \qquad H_0 \propto M_P$$

The total amount of e-folds may be as large as 10^{32}, but only 65 last ones, where $H \propto M_*$ (SUSY breaking scale) are relevant.

From the formal QFT viewpoint, there is no solution, because for the transition period, when

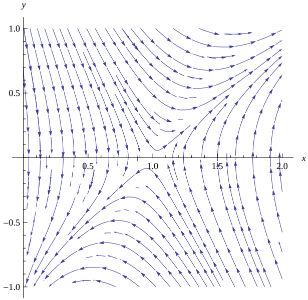
$$H \sim \text{masses of quantum matter fields}$$

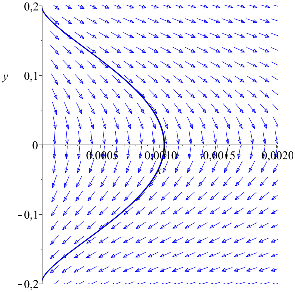
we have no method, approach, idea or approximation to perform calculations, except for dS space, which is useless here.

The simplest, purely phenomenological approach is to take a final point of the stable tempered inflation epoch ... and use it as initial point for the unstable phase. Where we are going to end up in this way?

A. Pelinson, Tibério de Paula Netto, I.Sh., A. Starobinsky,
From stable to unstable anomaly-induced inflation.
Eur. Phys. J. (2016), arXiv:1509.08882.

The qualitative output of this phenomenological approach is positive, in the sense that the final point of the stable inflation (related to SUSY breakdown) belongs to the "right" integration curve of the unstable inflation.





One can check that this curve really ends up at the classical radiation-dominated solution.

This result gives us a chance to have a consistent inflation based on QFT results.

We have seen that the anomaly-induced corrections remove initial cosmological singularity. What about other cases?

There are indications that the black hole singularity at $r = 0$ disappears if the semiclassical effects are taken into account.
Frolov & Vilkovisky, PLB (1980).
Frolov & I.Sh., PRD (2009).
Lu, Perkins, Pope, Stelle, PRL-2015, arXiv:1502.01028; PRD-2015.

Singularities represent a "WINDOW" to QG. The semiclassical effects may CLOSE IT, making observation of QG impossible.

Conclusions.

- Integrating conformal anomaly is very efficient and extremely economic and explicit way to derive EA.

- The conformal symmetry can not be exact, it is only a useful approximation. And its effectiveness is mainly restricted to the one-loop level.

- In order to arrive at some applications one is forced to deal with the non-conformal massive quantum fields.

- The success of anomaly-based approach is closely related to the fact we know very well how to deal with divergences and hence control UV limit of QFT in curved space.

- Currently it is unclear how to go beyond the UV limit. This problem represents the most challenging and very difficult part of the semiclassical approach.

Exercises and references

1. Verify the conformal invariance of massless fermion field in $4D$ and consider a generalization to an arbitrary dimension.

Refs.: Book by Parker and Toms, arXiv:1611.02263.

2. Explore whether it is possible to recover the anomaly (or at least part of it) in $4D$ from the non-local form factors. Discuss the ambiguity of anomaly and anomaly-induced action related to the R^2 term.

Refs.: [2.1] A.O. Barvinsky, Yu.V. Gusev, G.A.Vilkovisky, V.V.Zhitnikov, Nucl. Phys. B439 (1995) 561.
[2.2] hep-th-0307187. [2.3] arXiv:0801.0216.

3. Discuss the anomaly in $2D$ and derive the corresponding induced effective action (Polyakov action).

4. Prove that the one-loop divergences in the conformal invariant theory are conformal invariant. Starting from this point make a classification of the terms in trace anomaly into three groups: topological structure, surface terms, conformal terms.

Refs.: [4.1] Book by Birrell and Davies.
[4.2] S. Deser, and A. Schwimmer, hep-th-9302047.
[4.3] arXiv:0801.0216, arXiv:1702.06892, arXiv:1812.01140.

5. Derive the anomaly and anomaly-induced action in the case when the set of external fields includes metric and also electromagnetic field.

Refs.: [5.1] arXiv:0906.3837.
[5.2] arXiv:0812.0351.