Problem set – Large-Scale Structure Lectures

① We derived in class the equations for matter perturbations, namely:

$$\dot{\delta} + \frac{1}{a}\vec{\nabla}\cdot\vec{v} = 0 \quad (1)$$

$$\dot{\vec{v}} + H\vec{v} = -\frac{1}{a}\frac{\vec{\nabla}\delta p}{\bar{\rho}} - \frac{1}{a}\vec{\nabla}\phi \quad (2)$$

$$\frac{1}{a^2}\vec{\nabla}^2 = 4\pi G\left(\delta\rho + 3\delta p\right) \ , \quad (3)$$

where we use units such that $c = 1$. In class we also showed that in a pure (cold) matter-dominated Universe the solution to these equations is a power-law, such that $\delta \sim t^{\alpha}$, with two possible values for the power index: $\alpha_g = +2/3$ (the growing mode) and $\alpha_g = -1$ (the decaing mode). Therefore, in a matter-dominated Universe the dominating mode is then $\delta(\vec{x};t) = a(t)/a(t_i)\,\delta(\vec{x};t_i)$.

Solve these equations in a non-expanding Universe, in the case where we em have some pressure perturbations, so $\delta p = c_s^2\delta\rho \neq 0$. Since there will be spatial derivatives in this case, use the Fourier decomposition to make some statement about a mode $\tilde{\delta}(\vec{k})$, and how the behaviour of that mode depends on k. In particular, obtain the solution for large scales ($k \to 0$) and for small scales ($k \to \infty$).

② Now, suppose that the Universe is not dominated by matter, but that there is some dark energy component that is relevant for the background expansion. It turns out that the perturbations of dark energy are too small to have any effect on structure formation, so their pressure perturbations can be neglected, which means that dark energy really only enters the above equations through the evolution of the backgroun expansion. Assume that this dark energy component is determined by its equations of state, $p_{de}/\rho_{de} = w$, which can be a function of time.

In that case we can again solve the equations above, and it is still true that there are no spatial derivatives in the linear equations for matter perturbations. It is common to express the growth of density fluctuations in terms of a *growth function* $D(t)$ which determines the deviation from a pure matter-dominated Universe, and is normalized to unity at $t = t_0$ ($z = 0$), i.e.:

$$\delta(\vec{x};t) = D(t) \times a(t)\, \delta(\vec{x};t_0) \ .$$

(a) Show that the growth function obeys the equation, in terms of $N \equiv \log a$:

$$\frac{d^2 D}{dN^2} + \left(\frac{5}{2} - \frac{3}{2}\, w\, \Omega_{de} \right) \frac{dD}{dN} + \frac{3}{2}(1-w)\, \Omega_{de}\, D = 0 \ ,$$

where both w and Ω_{de} are functions of time.

(b) Show that in a Λ-dominated Universe the matter perturbations are *frozen*

(c) Show that if $w < -1$ the perturbations in fact *decay* with time. How do you explain these last two results?