Problem: CMB acoustic peaks

a) The scale of the acoustic peaks in the CMB power spectrum is set by the comoving distance that a sound wave could travel between the time when the perturbation was created ($t \approx 0$) and the epoch of recombination ($t = t_*$). The sound speed results from the competition between restoring forces and inertia. For a simple fluid, $c_s^2 = dp/d\rho$. Before recombination, the photons and baryons can be assumed to be locked together. For the photons, $p_\gamma = \rho_\gamma c^2/3$, but for the baryons $p_b \approx 0$.

Compute the sound speed of the combined fluid, $c_s^2 = (dp/da)/(d\rho/da)$ in terms of redshift and cosmological parameters. Remember that the photons have their pressure and density scale by a^{-4}, while the baryon density scales only as a^{-3}.

Show that for $\Omega_b h^2 = 0.02$ and $\Omega_{\rm rad} h^2 = 4.2 \times 10^{-5}$ the approximation that $c_s \approx c/\sqrt{3}$ is good to $\sim 30\%$ for $z > 1000$.

For the rest of the problem, assume $c_s = c/\sqrt{3}$.

b) Demonstrate that the position $\ell_{\rm acoustic}$ of the acoustic peaks depends primarily on Ω_K and only slightly on Λ and h, by varying the relevant parameters. Assume $\Omega_m h^2 = 0.14$ and $\Omega_{\rm rad} h^2 = 4.2 \times 10^{-5}$.

Remember that the first acoustic peak is observed at an angular scale of

$$\theta = \frac{r_s}{d_A}, \tag{1}$$

where r_s is the physical size of the sound horizon at decoupling and d_A is the angular diameter distance.

The angular diameter distance takes different values for the different geometries of the Universe:

$$\begin{aligned}
d_A &= \frac{a}{H_0\sqrt{|\Omega_K|}} \sinh\left(\chi H_0 \sqrt{\Omega_K}\right), \quad (\Omega_K > 0) \\
d_A &= a\chi, \quad (\Omega_K = 0) \\
d_A &= \frac{a}{H_0\sqrt{|\Omega_K|}} \sin\left(\chi H_0 \sqrt{-\Omega_K}\right), \quad (\Omega_K < 0)
\end{aligned}$$

Here $\chi(z) = \int_0^z dz'/H(z'))$ is the comoving distance to redshift z.