EXERCISE SET Lecture 2

1. **Modified Growth of Structure**
   Consider the following linear Einstein equations for f(R) gravity

   $\mathbf{E_{00}}$:

   $$\left[k^2\Phi + 3\mathcal{H}\left(\dot{\Phi} + \mathcal{H}\Psi\right)\right] + \frac{3}{2}\dot{\mathcal{H}}\frac{\delta f_R}{F} - \frac{3}{2}\mathcal{H}\frac{\delta \dot{f}_R}{F} - \frac{k^2}{2}\frac{\delta f_R}{F} + \left(3\dot{\Phi} + 6\mathcal{H}\Psi\right)\frac{\dot{F}}{2F} = -\frac{a^2}{2M_P^2}\frac{\rho}{F}\delta$$

   $\mathbf{E_{0i}}$:

   $$\left[\dot{\Phi} + \mathcal{H}\Psi\right] - \frac{1}{2}\frac{\delta \dot{f}_R}{F} + \frac{1}{2}\mathcal{H}\frac{\delta f_R}{F} + \frac{1}{2}\frac{\dot{F}}{F}\Psi = \frac{a^2}{2M_P^2}\frac{(\rho + P)}{F}u + \frac{\dot{F}}{F}\left(2\dot{\Phi} + 2\mathcal{H}\Psi + \dot{\Psi}\right) = \frac{1}{F}\frac{a^2}{M_P^2}\delta P$$

   $\mathbf{E_{ij}}$:

   $$k^2\left(\Phi - \Psi\right) - k^2\frac{\delta f_R}{F} = 0\,,$$

   where dots indicate derivatives w.r.t. conformal time. Set $\delta f_R = f_{RR}\delta R$, take the quasi-static limit (which also gives $\delta R = 2k^2/a^2\ (\Psi + 2\Phi)$), and derive the Poisson equation, i.e.

   $$k^2\Psi = -\frac{a^2}{2M_P^2}\mu(a,k)\rho\delta\,, \tag{1}$$

   finding an explicit expression for $\mu(a,k)$ (in terms of the expansion history, f_R and f_{RR}) .

2. **Degrees of Freedom**
   Consider the following action for gravity:

   $$S = \int d^4x\sqrt{-g}\,f(R,\mathcal{G})\,, \tag{2}$$

   where $\mathcal{G} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is the Gauss-Bonnet term. Will this theory introduce ghost spin-2 fields? Derive the equations of motion for $g_{\mu\nu}$. Specialize to the background and using $\mathcal{G} = 24H^2(H^2 + dH/dt))$ and $R = 6\left(2H^2 + dH/dt\right)$, derive the first Friedmann equation.

   extra (involved calculation): Find the condition under which, regardless of the background, there is only one propagating scalar degree of freedom.