High energy physics processes in astrophysics

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The main goal of the present lectures is to outline the main mechanisms and processes through which high-energy particles evolve in configuration and energy space in typical galactic and extragalactic environments. The important concept of collisionless diffusion of charged particles will be tackled, followed by a description of particle interactions and loss mechanisms in the Galactic and extragalactic medium. These interactions are an important ingredient entering the propagation equation and contribute shaping the spectrum of high-energy charged particles, whose population observed at the Earth (top of the atmosphere) is known as "Cosmic Rays". The neutral byproducts of the charged energetic particle interactions (gamma rays and neutrinos) point back to their production point, and are themselves signals one can look for and study... but that will be covered elsewhere!

The main reference is [1]. For the cosmic ray propagation part, classical texts like [2] remain useful. These lecture notes also include other references (including reviews and articles), but have not undergone a deep proof-reading, nor are fully referenced, sorry. They should not be considered fully reliable, but just help you to navigate through the literature.

Contents

I. PROLEGOMENA

A. On Units

As in particle physics, astroparticle physics practitioners tend to use *natural units* (where $c = k_B = \hbar = 1$), so that e.g. energy, mass, momenta, temperature, inverse length and inverse time have the same unit. Typically, energy is measured in eV (and multiples) for microscopic applications or ergs for macroscopic/astrophysics scales; distances are given in parsecs (and multiples), cross sections in barns (and multiples), etc. Some rough conversions are

- 1 s= 3×10^{10} cm=1.5×10^{15} eV^{-1}
- 1 J= 10^7 erg=6.25×10^{18} eV
- 1 pc=3×10^{16} m
- 1 barn=10^{-24} cm^2
- 1 G=10^{-4} T$\simeq$ 0.07 eV^2

On the other hand, differently from some convention frequent among quantum gravity *aficionados*, in astroparticle we retain $G_N \equiv M_P^{-2} = (1.22 \times 10^{19}\,\mathrm{GeV})^{-2}$. If you are unfamiliar with these units, practice a bit!

Exercises

- Compute your typical body temperature (assuming you are still alive) and your mass in eV.
- Check the working frequency of your mobile phone. Rephrase it into eV.
- Compute your height and age in eV^{-1}.
- Compute your density (estimated with $\mathcal{O}$(10%) error!) in eV^4.

Finally, in agreement with most astrophysical literature, the Gaussian electromagnetic convention is used (with the 4π's in Maxwell equations, not in Coulomb's or Biot-Savart's laws). The charge of the positron is $e \simeq \sqrt{\alpha} \simeq \sqrt{1/137} \simeq 0.085$.

B. The Galactic and extragalactic environments

These environments are *extremely rarified*. Typical benchmark densities of matter are:

- $\lesssim$1 cm^{-3} in the Galaxy.

- $\lesssim 10^{-6}$ cm^{-3} on average, in a cosmological setting

They are also *much bigger* than terrestrial scales. Typical benchmark distances are:

- Several kpc, for Galactic objects (Earth-Gal. Center is $\simeq$8 kpc).
- hundreds of Mpc, or Gpc, for cosmologically distant objects (at low redshift, the relation $d \simeq 40(z/0.01)\,\mathrm{Mpc}$ holds).

They also evolve over *very long timescales*. Typical benchmark timescales are:

- $\sim$230 Myr for a solar orbit in the Milky Way.
- $\sim$14 Gyr, age of the Universe.

1. *Magnetic fields*

The interstellar medium (ISM) of our Galaxy, as well as external ones, is magnetized. This is most clearly revealed by radio observations (Faraday rotation, synchrotron radiation) but also via polarized light emission (due to dust grains) or, in some cases, via Zeeman splitting. A magnetic field coherent (at least) up to scales comparable to several kpc-long Galactic structures, such as spiral arms, has been detected. Radio observations of external Galaxies as well as our own, notably via synchrotron emission (see below), indicate several kpc thick magnetized halos embedding the stellar disks, as *alfajores* embed their *dulce de leche*. The ISM magnetic field extends to small scales down to at least the typical pc-scale distance between neighbouring stars, where the field orientation is believed to fluctuate following the turbulence of the ISM. Typical inferred intensities of Galactic fields are in the 1-10μG range, where $1\,G$ is the order of magnitude of the Earth magnetic field.

There are also indications that the extragalactic medium is magnetized, with magnetic fields exceeding 10^{-19} G at least [3], and not exceeding ~ 2 nG in the truly extragalactic medium [4]. Fields reaching μG in proximity of Galaxy clusters have been inferred. For a relatively recent review and ample references on Galactic and cosmic magnetism, including the microphysics and simulations, see for instance [5].

2. *Photon fields*

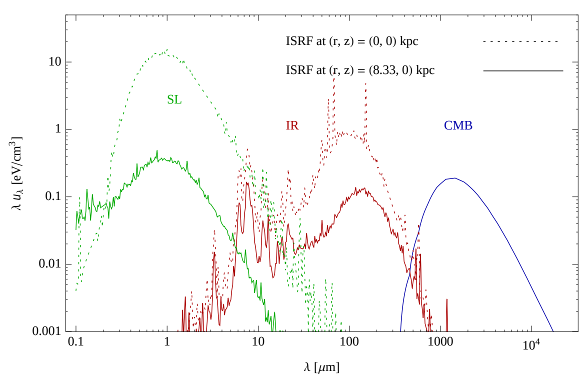

FIG. 1: Typical photon densities in the Galactic regions, from [7].

The most important extragalactic ones are: The CMB (blackbody of cosmological origin and temperature of about 2.7 K) pervades the whole universe (Galaxy and extragalactic sky alike). While today its energy density is only $\sim 0.3\,$eV/cm^3, it scales with redshift as $(1+z)^4$ (number density of photons as $(1+z)^3$); the extragalactic background light (EBL), mostly due to starlight and dust reflection, pervades the extragalactic medium; a recent determination of its spectral energy density can be found in [6]. Other backgrounds also exist (e.g. radio) but are less important for what follows.

For the Galactic environment, besides the CMB, UV, optical, and IR backgrounds are important (comparable or larger than the CMB, in energy density) and definitely non-homogeneously distributed (peaking towards inner Galaxy), see Fig. 1.

C. *Collisional* random motions

The mean-free-path ℓ (rate Γ) is the average distance (inverse timescale) travelled by a particle before interacting. If σ is the cross-section of the interaction process for a particle moving at velocity β in an environment with target density n, its mean-free-path and interaction rate are

$$\boxed{\ell = \frac{1}{\sigma\, n}, \quad \Gamma = \sigma\, \beta\, n = \frac{\beta}{\ell}} \tag{1}$$

For an opaque source of radius R, the optical depth is

$$\tau = \frac{R}{\ell}\,. \tag{2}$$

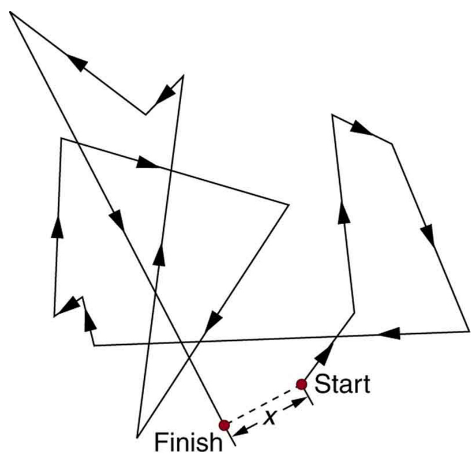

FIG. 2: Typical displacement vectors in a random motion.

How many scatterings before escaping a source with optical depth τ? One has to require that the particle moves a distance X away from its initial position equal to R. Its motion will be a "random motion", with vectors of average length ℓ and random direction after each bounce, see Fig. 2, so that

$$\langle X \rangle_N = \left\langle \sum \vec{r}_i \right\rangle = 0\,, \tag{3}$$

since the vectors are randomly directed and of comparable lengths, but

$$\langle X^2 \rangle_N = \left\langle \left(\sum_i \vec{r}_i \right) \cdot \left(\sum_j \vec{r}_j \right) \right\rangle = \sum_i \langle \vec{r}_i^{\,2} \rangle + \ell^2 \sum_{i \neq j} \langle \cos\theta_{ij} \rangle = \ell^2 N \tag{4}$$

so that

$$\langle X^2 \rangle_N = R^2 \text{ when } N = \tau^2 \tag{5}$$

How long does it take to escape? Obviously

$$t_{\rm esc} = \Gamma^{-1} N = \frac{\tau^2}{\Gamma} = \frac{\tau\, R}{\beta}\,. \tag{6}$$

The law of Eq. 4 is the discrete version of a diffusive propagation, with N proportional to the time elapsed via the constant Γ. We can thus guess the continuum limit

$$\langle X^2 \rangle(t) = \ell^2 \Gamma t = \ell \beta t\,, \tag{7}$$

or $X^2 \propto K\, t$ with the diffusion coefficient K roughly given by $K \propto \ell\beta$ (We have been omitting numerical constants depending on the space dimensions, see after Eq. (51) below for a more rigourous calculation).

II. THE DIFFUSIVE COSMIC RAY PROPAGATION

A. Motion in a constant field

In order to gain an intuitive understanding of the CR movement, let us start with the description of the evolution in a constant (large scale) field of intensity $B_0 \equiv |\mathbf{B}_0|$, neglecting (small scale) field fluctuations. The trajectory of a particle of charge $q = Z|e|$ of mass m moving with velocity $\mathbf{v}$ (associated to Lorentz factor $\gamma(v)$) obeys the EoM

$$\frac{\mathrm{d}(m\gamma\mathbf{v})}{\mathrm{d}t} = q\mathbf{v}\times\mathbf{B}_0 \Rightarrow m\gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v}\times\mathbf{B}_0\,, \tag{8}$$

since (**Exercise**: prove it, from $\gamma = (1 - v_j v^j)^{-1/2}$)

$$\frac{\mathrm{d}(m\gamma\mathbf{v})}{\mathrm{d}t} = m\gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + m\gamma^3\mathbf{v}\frac{\mathrm{d}(\mathbf{v}\cdot\mathbf{a})}{\mathrm{d}t} = m\gamma\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,, \tag{9}$$

the last step following from the fact that in a $\mathbf{B}_0$ field, the acceleration is always orthogonal to the velocity. This is equivalent to the fact that γ is constant, which we can see also from

$$m\gamma\frac{\mathrm{d}(\mathbf{v}\cdot\mathbf{v})}{\mathrm{d}t} = 2m\gamma\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 2\mathbf{v}\cdot(q\mathbf{v}\times\mathbf{B}_0) = 0\,. \tag{10}$$

So, both v and $p = m\gamma v$ are constant. The EoM also imply that the velocity component parallel to $\mathbf{B}_0$ is constant. If we choose the z direction aligned with $\mathbf{B}_0$, this means $v_z =$const. and $p_z = m\gamma v_z =$const. If we denote with μ (the cosine of the angle (dubbed *pitch angle*) formed by the particle momentum $\mathbf{p}$ with the magnetic field direction, this means that $\mu = p_z/p$ is also constant. Hence, the momentum component in the $x-y$ plane, $p_\perp \equiv \sqrt{p_x^2+p_y^2} = \sqrt{1-\mu^2}p$ is also constant. In the $x-y$ plane, the particle gyrates with a radius given by equating the acceleration in the $x-y$ plane to the centripetal acceleration

$$\frac{\mathrm{d}v_\perp}{\mathrm{d}t} = \frac{q}{m\gamma}v_\perp\,B_0 \;\text{ and }\; \frac{\mathrm{d}v_\perp}{\mathrm{d}t} = \frac{v_\perp^2}{r} \Rightarrow r = \frac{m\gamma\,v_\perp}{q\,B_0}\,. \tag{11}$$

With respects to the non-relativistic *Larmor radius or gyroradius* r_g, *angular gyrofrequency or cyclotron frequency* ω_g, and *gyrofrequency* ν_g defined as

$$r_g \equiv \frac{v_\perp}{\omega_g}\,,\quad \omega_g \equiv \frac{q\,B_0}{m}\,,\quad \nu_g = \frac{\omega_g}{2\pi} \equiv \frac{q\,B_0}{2\pi m} = 2.8\,\mathrm{Hz}\,Z\left(\frac{m_e}{m}\right)\left(\frac{B_0}{\mu\mathrm{G}}\right)\,, \tag{12}$$

the relativistic generalizations are thus

$$r_L = \gamma r_g = \sqrt{1-\mu^2}\frac{\mathcal{R}}{B_0} \simeq 10^{-6}\sqrt{1-\mu^2}\frac{\mathcal{R}}{\mathrm{GV}}\frac{\mu\mathrm{G}}{B_0}\mathrm{pc}\,, \tag{13}$$

where we introduced the rigidity $\mathcal{R}$, or momentum over charge (measured typically in GV), and

$$\Omega = \frac{\omega_g}{\gamma} = \frac{q\,B_0}{E} \simeq 10^{-2}\,Z\frac{B_0}{\mu G}\frac{\mathrm{GeV}}{E}\mathrm{rad/s}\,. \tag{14}$$

Note that the timescales or equivalently spatial scales of this movement are very small for Galactic astrophysics standards. . .

Exercise

. . . up to which energy? Compute the energy at which a proton and a iron nucleus gyroradius exceeds the kpc scale, if $B_0 = 3\,\mu$G. If I told you that the skymap of CRs at $E \simeq 10^{19}$ eV looks roughly isotropic, what would you infer about the source locations?

The gyrating motion means that (with a suitable choice of initial time)

$$v_x = v_\perp\cos(\Omega\,t)\,,\quad v_y = v_\perp\sin(\Omega\,t)\,,\quad v_z = v\mu = \mathrm{const.} \tag{15}$$

and equivalenty

$$x = x_g + r_L\sin(\Omega\,t)\,,\quad y = y_g - r_L\cos(\Omega\,t)\,,\quad z = z_g + v_z\,t = z_g + v\,\mu\,t\,, \tag{16}$$

with the point $\{x_g, y_g, z(t)\}$, around which the particle rotates, which is called *guiding center*.

B. Heuristic derivation of the diffusive propagation

Let us now add an ensemble of *small-scale, stochastic perturbations* to the B-field, orthogonal to its regular value, i.e. $|\delta\mathbf{B}| \ll |\mathbf{B}_0|$ and $\delta\mathbf{B} \perp \mathbf{B}_0$. For clarity and simplicity, the perturbation has been chosen so that p_z—or better $\mu = p_z/p$—is affected by the perturbation since the new term yields the leading non-vanishing force, while the $x-y$ trajectory is left unchanged. Also, assume the magnetostatic limit for the field perturbation, which should capture the dominant effect in the ultrarelativistic limit for the CR (under which one also has $E \simeq p$). Under these conditions, for a single plane wave perturbation of wavenumber k and phase ψ, $\delta\mathbf{B} = \{\cos(-kz+\psi), \sin(-kz+\psi), 0\}$ one gets

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{q\sqrt{1-\mu^2}\,|\delta\mathbf{B}|}{E}[\cos(\Omega\,t)\cos(-kz+\psi) - \sin(\Omega\,t)\sin(-kz+\psi)] = C\cos[w\,t+\psi]\;, \tag{17}$$

where at the second step we defined the quantities $C \equiv q\sqrt{1-\mu^2}\,|\delta\mathbf{B}|/E$ and $w \equiv (\Omega - k\,v\mu)$. Below, we assume a *random phase approximation* for the perturbations; *ensemble-averaging* over phase ψ:

$$\left\langle \frac{\mathrm{d}\mu}{\mathrm{d}t} \right\rangle_\psi \propto \int_0^{2\pi} \mathrm{d}\psi \cos(A+\psi) = 0\,. \tag{18}$$

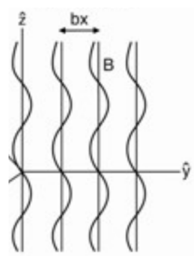

FIG. 3: Example of the perturbed B-field, with static perturbation orthogonal to the background.

If we compute however the variance after some finite interval of time (from sime initial time t_0, henceforth omitted),

$$\Delta\mu^2(t) = C^2 \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t'' \cos[wt'+\psi]\cos[t''+\psi]\;. \tag{19}$$

we can express the integrand as a sum of cosine of the sum and the difference of the arguments, i.e. use

$$\cos(A)\cos(B) = \frac{\cos(A-B)+\cos(A+B)}{2}\,, \tag{20}$$

the piece depending on the sum ensemble-averages to zero; but the one depending on the difference survives

$$\langle \Delta\mu^2(t) \rangle = \frac{C^2}{2} \int^t dt' \int^t \mathrm{d}t''\; \cos[w(t''-t')]\,, \tag{21}$$

or

$$\frac{\mathrm{d}\langle\Delta\mu^2\rangle}{\mathrm{d}t} = \frac{C^2}{2}\left(\int^t \mathrm{d}t''\; \cos[w(t''-t)] + \int^t \mathrm{d}t'\; \cos[w(t-t')] \right) = \frac{C^2}{2}\int_{-t}^{t} \mathrm{d}s\; \cos[w(s-t)]\,. \tag{22}$$

In the limit $\Delta t \gg \Omega^{-1}$, the domain of integration covers many periods, so that

$$\int_{-t}^{t} \mathrm{d}s\; \cos[w(s-t)] \simeq \int_{-\infty}^{\infty} \mathrm{d}s\, \frac{e^{iw(s-t)}+e^{-iw(s-t)}}{2} = \pi\delta(w)\,,\text{ given that } \int_{-\infty}^{\infty} \mathrm{d}x\, e^{i\,x\,y} = 2\pi\delta(y)\,. \tag{23}$$

This leads to the following (time independent) time-derivative of the variance

$$\frac{\mathrm{d}\langle\Delta\mu^2\rangle}{\mathrm{d}t} \simeq \pi C^2\delta(w) = \pi\,(1-\mu^2)\Omega\frac{|\delta\mathbf{B}|^2}{B_0^2} k_{\mathrm{res}}\delta\,(k-k_{\mathrm{res}})\;, \quad k_{\mathrm{res}} \equiv \frac{\Omega}{v\mu}\,. \tag{24}$$

We see that on average μ remains constant, but its variance linearly grows with time: this is the typical behaviour of a *diffusive* process, see Eq. (7). We also deduce that the diffusion process is (quasi)resonant [there are corrections to that at least due to the fact that $1/(\Delta t\Omega)$ is not zero!], with the momentum direction along the regular field only changing if the CR finds a fluctuation whose wavelength matches the gyroradius of the CR in the field, modulo a geometric projection. In Fig. 4 we provide a cartoon interpretation of this effect: a CR "surfs" along field lines whose fluctuations have very long wavelengths, "ignores" fluctuations of the field at too small scales compared with its Larmor radius, but undergoes a significant deflection with respect to its unperturbed trajectory if the perturbation matches r_L. It is straightforward

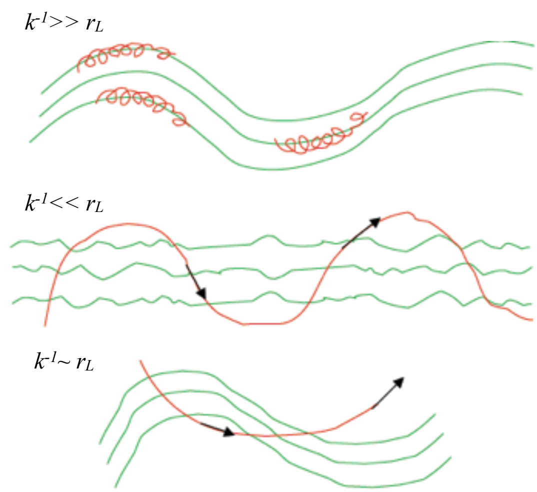

FIG. 4: Sketch of the resonant CR diffusion mechanism.

to generalize the previous derivation to an ensemble of perturbations with different wavelengths. From the analogy with Eq. (7), it makes sense to introduce the diffusion coefficient of the (cosine of the) pitch angle, $D_{\mu\mu}$, according to

$$D_{\mu\mu}(k) \equiv \frac{1}{2}\left\langle\frac{\mathrm{d}\Delta\mu^2}{\mathrm{d}t}\right\rangle_\psi \equiv (1-\mu^2)\nu_{\theta\theta} = \frac{1}{4}(1-\mu^2)\Omega\frac{1}{B_0^2}\int dx e^{ikx}\delta B^2(x)\,. \tag{25}$$

which essentially means that the typical frequency $\nu_{\theta\theta}$ (or inverse timescale $\tau_{\theta\theta}^{-1}$) over which the pitch angle (as opposed to its cosine) of CRs whose resonant wavenumber is $k_{\rm res}$ changes by order one is given by the intuitive formula

$$\boxed{\nu_{\theta\theta}(k_{\rm res}) \sim \Omega\left(\frac{\delta B}{B_0}\right)^2(k_{\rm res})\,.} \tag{26}$$

We will not need precise numerical factors, also because this simplified approach cannot account for quantitative subtleties of the CR propagation problem.

For a relatively broad distribution of magnetic field fluctuation power, we can thus expect CRs to lose complete memory of their initial velocity with respect to the regular magnetic field over a few gyroperiods, which is a very short timescale if compared to astrophysical times, see Eq. (14). It can also be readily checked that this timescale is significantly shorter than the collisional timescale for a typical interstellar plasma with order of magnitude number density of $\mathcal{O}(1)\,\mathrm{cm}^{-3}$. Despite the simplifications and unavoidable limitations of our treatment above, it correctly captures two features of the CR propagation phenomenon in an astrophysical medium such as the Galactic ISM:

- the CR movement is essentially a diffusive process.
- it is a *collisionless* (rather than a collisional) diffusion. Namely, CR do not scatter on other particles, but on inhomogeneities (or more in general, "waves") of the magnetic field.

A long-standing observation, coherent with the diffusive character of CR propagation, concerns the over-abundance of "fragile" nuclei such as Li-Be-B (or sub-Fe species) in CR with respect to elemental ratios of solar system material, see Fig. 5: due to their small nuclear binding energy, these nuclei are easily burned in stellar thermonuclear processes and are only present in traces in conventional astrophysical environments, such as the ISM. Their sizable presence among CR nuclear fluxes is interpreted as the consequence of spallation of heavier nuclei, such as C or O—which are common in the

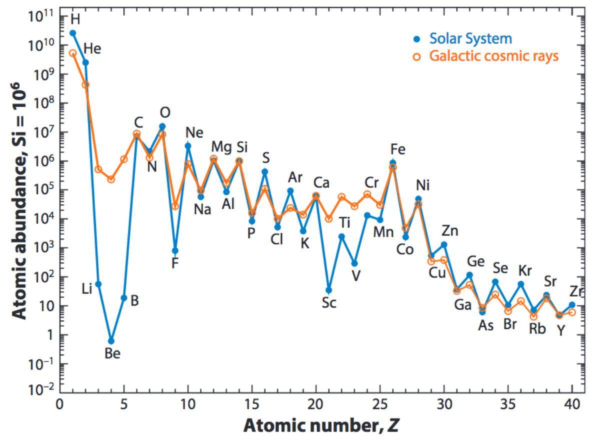

FIG. 5: Relative abundances of CR species, compared to Solar System values, normalized to Si $= 10^6$. From [8].

ISM—onto the ISM gas during their propagation. This is also why such nuclei are also referred to as "secondary" CR species. However, given the known density of the ISM medium of $\mathcal{O}(1)\,\mathrm{cm}^{-3}$, the abundances of these species at the tens of percent of their progenitors requires CR residence times in the Galaxy which are many order of magnitude larger than the ballistic crossing time. Later on, we shall illustrate with a simple dynamical model how secondary over primary ratios can be used to constrain propagation parameters.

C. The CR transport equation for magnetostatic perturbations, more formal approach

The above problem can be reformulated in a slightly more formal way in terms of a transport equation. CRs propagating in an externally assigned field, however complicated, obey hamiltonian dynamics, and their phase space distribution f obeys Liouville equation. By the way, since the number of particles N is (classically) invariant and the phase space volume $\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}$ is invariant under Lorentz transformation (see e.g. arXiv:1105.2120), the phase space density (such that $\mathrm{d}N = f(\mathbf{x},\mathbf{p})\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}$ is a number of particles) is also Lorentz invariant. In this language, the approximation of the previous section can be seen as splitting the distribution function $f = \langle f\rangle + \delta f$, where the first term is the solution to the ensemble-averaged equation (over B-field fluctuations $\delta\mathbf{B}$), so that by definition $\langle\delta f\rangle = 0$. The ensemble-average version of the Liouville equation writes

$$\partial_t\langle f\rangle + \mathbf{v}\cdot\nabla_{\mathbf{x}}\langle f\rangle - (\mathbf{\Omega}\times\mathbf{p})\cdot\nabla_{\mathbf{p}}\langle f\rangle = \langle(\delta\mathbf{\Omega}\times\mathbf{p})\cdot\nabla_{\mathbf{p}}\delta f\rangle\,, \tag{27}$$

where we introduced the gyrofrequency vector of the ensemble averaged field, $\mathbf{\Omega}\equiv q\langle\mathbf{B}\rangle/E$, as well as the one associated to fluctuations, $\delta\mathbf{\Omega}\equiv q\delta\mathbf{B}/E$. Note that the third term at the LHS also writes as $E^{-1}qB\,p_\perp\,\nabla_{\mathbf{p}_\perp}$ and it describes the gyration around the direction of $\mathbf{B}_0$. It is also instructive to rewrite this term as $-i\,\mathbf{\Omega}\cdot\mathbf{L}\langle f\rangle$, introducing the the angular momentum operator, $\mathbf{L}\equiv -i\mathbf{p}\times\nabla_{\mathbf{p}}$. This formulation also justifies rewriting the RHS as $\langle i\,\delta\mathbf{\Omega}\cdot\mathbf{L}\delta f\rangle$. Often, different ways to look at the CR propagation problem reduce to different ways to approximate the RHS, which can be thought of as a "collisional" term for the ensemble-averaged distribution function. Physically, the discussion of the previous section allows one to describe it as a relaxation to an isotropic distribution function, in the frame of the scattering centers, over a timescale $\nu_{\theta\theta}^{-1}$. This can be heuristically used to argue that

$$\langle i\,\delta\mathbf{\Omega}\cdot\mathbf{L}\delta f\rangle \simeq -\nu_{\theta\theta}\left(\langle f\rangle - \frac{n}{4\pi}\right)\,, \tag{28}$$

where we introduced *the isotropic mean* $n\equiv\int d\Omega_{\mathbf{p}}\langle f\rangle$. This is known as *BGK Ansatz*, from [9]. In general, we expect that $\langle f\rangle$ is dominated by the isotropic part, and in fact it can be consistently checked that the leading anisotropic component is the dipolar one, with the higher order multipoles being suppressed. Under a more general Ansatz allowing for a dipolar flux term, $\langle f\rangle = (4\pi)^{-1}(n + 3\,\hat{\mathbf{p}}\cdot\mathbf{j})$, with $\mathbf{j}\equiv\int d\Omega_{\mathbf{p}}\hat{\mathbf{p}}\langle f\rangle$ (we assume the relativistic limit $v\simeq c=1$, $p=E$) the transport equation with the approximation of Eq. (28) reduces to

$$\partial_t n + \nabla_{\mathbf{x}}\mathbf{j} = 0\,, \tag{29}$$

$$\partial_t\mathbf{j} + \frac{1}{3}\nabla_{\mathbf{x}}n + \mathbf{\Omega}\times\mathbf{j} \simeq -\nu_{\theta\theta}\mathbf{j}\,. \tag{30}$$

(31)

Since we argued that the relaxation time $\nu_{\theta\theta}^{-1}$ is much shorter than the astrophysical flux evolution time, in the second equation we can neglect $\partial_t \mathbf{j}$ with respect to the RHS, and the equation reduces to $j_i \simeq -K_{ij}\partial_j n$ (*Fick's law*), yielding

$$\partial_t n = \partial_i (K_{ij}\partial_j n)\,, \tag{32}$$

where the *spatial* diffusion tensor is

$$K_{ij} = \frac{1}{3\nu_{\theta\theta}} \frac{\nu_{\theta\theta}^2 \delta_{ij} + \nu_{\theta\theta}\Omega_k \epsilon_{ijk} + \Omega_i \Omega_j}{\nu_{\theta\theta}^2 + \Omega^2}\,, \tag{33}$$

with eigenvalues $1/3\nu_{\theta\theta}$ and $1/3(\nu_{\theta\theta} \pm i\Omega)$ corresponding to diffusion parallel and perpendicular to the magnetic field, respectively. Note that *the spatial diffusion coefficient is inversely proportional to the pitch angle scattering diffusion coefficient*, which is physically reasonable since the easier it is to change momentum direction, the harder it is to propagate away from the original direction in physical space. Also, since in the limit of weak turbulence one has $\nu_{\theta\theta} \ll |\Omega|$, this result confirms that the predominant diffusion is parallel to the background field. Later on, we will derive the above results more carefully, where we will drop the magnetostatic approximation and will account for $v < 1$.

It is instructive to arrive at a similar equation starting from the description of collisionless pitch angle diffusion, keeping this time the factors of v:

$$\langle i\,\delta\mathbf{\Omega}\cdot\mathbf{L}\delta f\rangle \simeq \frac{\partial}{\partial\mu}\left(D_{\mu\mu}\frac{\partial\langle f\rangle}{\partial\mu}\right)\,. \tag{34}$$

The resulting 1D transport equation for the gyrophase-averaged distribution function $\langle f\rangle = \langle f\rangle(t,z,\mu)$ is of the Fokker-Planck type,

$$\frac{\partial\langle f\rangle}{\partial t} + v\,\mu\frac{\partial\langle f\rangle}{\partial z} = \frac{\partial}{\partial\mu}\left(D_{\mu\mu}\frac{\partial\langle f\rangle}{\partial\mu}\right)\,. \tag{35}$$

By definying the zeroth and first moment of $\langle f\rangle$ with respect to the pitch angle,

$$f_0(t,z) = \frac{1}{2}\int_{-1}^{+1} d\mu\langle f\rangle\,, \tag{36}$$

$$f_1(t,z) = \frac{v}{2}\int_{-1}^{+1} d\mu\,\mu\langle f\rangle\,, \tag{37}$$

and the quantities

$$K_0 \equiv \frac{v}{4}\int_{-1}^{+1} \mathrm{d}\mu\frac{1-\mu^2}{D_{\mu\mu}}(1+\mu)\,, \quad K \equiv \frac{v^2}{8}\int_{-1}^{+1}\mathrm{d}\mu\frac{(1-\mu^2)^2}{D_{\mu\mu}}\,, \tag{38}$$

one can prove that

$$\frac{\partial f_0}{\partial t} \simeq \frac{\partial}{\partial z}\left[K\frac{\partial f_0}{\partial z}\right]\,, \tag{39}$$

confirming that the spatial diffusion coefficient $K \propto D_{\mu\mu}^{-1}$, in addition to (reinstating $c \neq 1!$)

$$\frac{f_1}{f_0} \sim -\frac{K}{c\,f_0}\nabla f_0 \sim \mathcal{O}\left(\frac{K}{c\,H}\right) \ll 1 \tag{40}$$

which is always smaller than 1 as long one the diffusive approximation is fulfilled (see exercise below). This ratio can be used as an estimator of the anisotropy level expected in the CR flux.

Exercise: Follow in detail the proof of Eq.s (39),(40).

Integrating Eq. (35) over μ between -1 and $+1$, the diffusion term does not contribute because $D_{\mu\mu} \propto (1-\mu^2)$ and hence $D_{\mu\mu}(\mu = \pm 1) = 0$ (no current across boundaries), so we obtain a continuity (or current conservation) condition,

$$\partial_t f_0 = -\partial_z f_1\,. \tag{41}$$

We can integrate again Eq. (35) over μ' between -1 and μ; multiplying times $(1-\mu^2)/D_{\mu\mu}$, one obtains

$$\frac{1-\mu^2}{D_{\mu\mu}}\frac{\partial}{\partial t}\int_{-1}^{\mu}\mathrm{d}\mu'\langle f\rangle+\frac{1-\mu^2}{D_{\mu\mu}}\int_{-1}^{\mu}\mathrm{d}\mu'\,\mu'\,v\frac{\partial\langle f\rangle}{\partial z}=(1-\mu^2)\frac{\partial\langle f\rangle}{\partial\mu}\,. \tag{42}$$

Since we are estimating the first anisotropy correction, at the LHS of Eq. (42) we can approximate $\langle f\rangle\simeq f_0$, thus

$$(1-\mu^2)\frac{\partial\langle f\rangle}{\partial\mu}\simeq\frac{1-\mu^2}{D_{\mu\mu}}\frac{\partial f_0}{\partial t}(1+\mu)-\frac{(1-\mu^2)^2}{D_{\mu\mu}}\frac{v}{2}\frac{\partial f_0}{\partial z}\,, \tag{43}$$

Also, since $-2\,\mu=\partial(1-\mu^2)/\partial\mu$, from the definition of f_1 one has

$$f_1=\frac{v}{4}\int_{-1}^{1}\mathrm{d}\mu'(1-\mu'^2)\frac{\partial\langle f\rangle}{\partial\mu'}\,, \tag{44}$$

which, multiplying Eq. (43) times $v/4$, integrating over μ between -1 and $+1$, and remembering Eq. (41), leads to

$$f_1=K_0\frac{\partial f_0}{\partial t}-K\frac{\partial f_0}{\partial z}=-K_0\frac{\partial f_1}{\partial z}-K\frac{\partial f_0}{\partial z}\,, \tag{45}$$

where

$$K_0\equiv\frac{v}{4}\int_{-1}^{+1}\mathrm{d}\mu\frac{1-\mu^2}{D_{\mu\mu}}(1+\mu)\,,\quad K\equiv\frac{v^2}{8}\int_{-1}^{+1}\mathrm{d}\mu\frac{(1-\mu^2)^2}{D_{\mu\mu}}\,. \tag{46}$$

Using again Eq. (41) and plugging in the second equality in Eq. (45) yields

$$\frac{\partial f_0}{\partial t}=\frac{\partial}{\partial z}\left[K_0\frac{\partial f_1}{\partial z}+K\frac{\partial f_0}{\partial z}\right]\,. \tag{47}$$

Now, the first term at the RHS is of the order $v^2(\langle f\rangle-f_0)$, while the second term of order $v^2\,f_0$ and thus dominant, leading to Eq.s (39), i.e. the diffusive equation for the isotropic part of the (ensemble-averaged) distribution function. Similarly, Eq. (45) leads to Eq. (40).

1. Solving the diffusion problem

Eq. (32), supplemented by a source term Q at the RHS, is the paradigmatic case of a *diffusion equation*. In our case, diffusion is expected to be strongly anisotropic as long as $\delta B\ll B_0$. However this expectation holds over the scales where the large-scale field B_0 does not change. If however this happens over scales much smaller than the distance from the sources (say, of $\mathcal{O}(100)$ pc vs. several kpc) the diffusive propagation is often modeled as an effective isotropic diffusion coefficient, $K_{ij}\to K\delta_{ij}$. A good fit to actual data yields is obtained for $K=0.3(\mathcal{R}/10\mathrm{GV})^{0.5}$kpc^2/Myr [10]. We will not indulge in techniques to solve similar equations, which deserve a course of its own. Let us simply recall here that, if K is only dependent upon energy,

$$\frac{\partial n}{\partial t}-K\nabla^2 n=Q(p,t,\mathbf{x})\,, \tag{48}$$

can be solved with the Green's function technique, i.e. if we know $G_p(t,t',\mathbf{x},\mathbf{x}')$ such that

$$\frac{\partial G_p}{\partial t}-K(p)\nabla^2 G_p=\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\,, \tag{49}$$

and satisfying appropriate boundary conditions, then

$$n(p,t,\mathbf{x})=\int\mathrm{d}^3x'\mathrm{d}t'\,G_p(t,t',\mathbf{x},\mathbf{x})Q(p,t',\mathbf{x}')\,.d \tag{50}$$

In the simplest case of *free escape boundary* (i.e. vanishing at infinity), we can take the Fourier transform of Eq. (49) and obtain

$$G_p=\int\frac{\mathrm{d}^3k\mathrm{d}\omega}{(2\pi)^4}\frac{\exp\left[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i\omega(t-t')\right]}{K(p)k^2-i\omega}=\frac{1}{(4\pi K\,(t-t'))^{n/2}}\exp\left[-\frac{(\mathbf{x}-\mathbf{x}')^2}{4K\,(t-t')}\right]\ \text{if }t>t',\ 0\,\text{otherwise}\,.d \tag{51}$$

n being the spatial dimensionality, and the latter step follows from taking the residue in the ω integral. Hence, we see that the mean square displacement from the initial position is $\langle r^2\rangle \equiv \langle \sum_i^n (x_i - x_i')^2\rangle = 2\,n\,K\,t$. For actual problems obeying e.g. different geometric boundary conditions, a generalization of the above procedure applies (see e.g. [2]).

Exercise For a source at distance d whose CR flux arrives to us diffusively, argue that $6\,K/d$ is the effective average velocity. For $d \simeq H \simeq 3$kpc (a typical Galactic distance), assuming the value of the average Galactic diffusion coefficient is $K = 0.3(\mathcal{R}/10\mathrm{GV})^{0.5}$kpc^2/Myr, compute the rigidity at which the diffusion approximation breaks down, since the diffusion speed nominally exceeds the speed of light.

D. The CR transport equation: moving scattering centers

A major limitation of the previous derivations is that the magnetic field fluctuations are considered static in the Lab (Galactic) frame. In practice, the plasma is moving with respect to this frame, for instance with a large scale, coherent movement: plasma winds and shocks are common both in the Solar System and in the Galactic environment. But this is not the whole story: A gas is known to support longitudinal "sound" waves (with frequencies well below the gas collisional frequency) which, in the ideal gas limit, are dissipationless. Similarly, the ISM—as any other magnetized plasma—can support a number of collective excitation modes, with frequencies well below the cyclotron ones. Perhaps the most peculiar waves in a magnetized medium are the *Alfvén waves*, propagating along the magnetic field direction, with the ion motion as well as the magnetic field perturbation perpendicular to it [1], and the restoring force provided purely by magnetic tension, with field lines behaving just like plucked strings. A simple linearization of MHD equations shows that these waves propagate at the characteristics speed $v_A = B_0/\sqrt{4\pi\rho}$, with ρ the mass density of the medium. In the ideal MHD limit, they are dissipationless. Physically, their long-lived nature is also understandable since—unlike sound waves—they are not associated to a gas compression, which cannot thus radiate, channelling away its energy.

The need for some extra ingredient is also obvious for a very fundamental reason: in magnetostatics, no change of the modulus of the momentum can take place, so that no acceleration of particles can occur. The movement of magnetic scattering centers in the lab frame, however, offers such a possibility: a purely elastic scattering *in the plasma* (or wave) frame is inelastic in the Lab frame, where the magnetic perturbation moving with velocity $\mathbf{V}$ is associated to an *electric* field $\mathbf{E} = \mathbf{B}\times\mathbf{V}$.

Let us denote with $\mathbf{V} \equiv \mathbf{u} + \delta\mathbf{v}$ the velocity of the scattering centers, itself the sum of the average velocity of the plasma frame $\mathbf{u}$ plus the deviation from it due to the microscopic motion of the scattering centers with respect to the plasma frame. As previously argued, we can write down an effective Boltzmann Equation for $\langle f\rangle$ (to avoid cluttering, below this will be referred to simply as f), with the average velocity entering the force term at the LHS, and the processes responsible for diffusion acting at the microscopic level described via a collisional term at the RHS. We have $\mathbf{F} = -\mathbf{\Omega}\times(\mathbf{p} - E\,\mathbf{u})$ (i.e. $F_i = -\epsilon_{ijk}\Omega_j(p_k - E\,u_k)$), and

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \mathbf{F}\cdot\nabla_{\mathbf{p}} f = \left.\frac{\partial f}{\partial t}\right)_c . \tag{52}$$

As argued above, it makes sense to approximate f with an expansion in moments, neglecting the directional anisotropy at multipoles higher than the dipole

$$f(\mathbf{x},\,\mathbf{p}) \simeq \frac{1}{4\pi} n(\mathbf{x},p) + \frac{3}{4\pi}\mathbf{w}(\mathbf{x},p)\cdot\hat{\mathbf{p}} \tag{53}$$

where $\hat{\mathbf{p}} \equiv \mathbf{p}/p$ and (spatial coordinate dependence will be left implicit henceforth)

$$n \equiv \int \mathrm{d}\Omega_{\mathbf{p}} f\,, \quad \mathbf{w} \equiv \int \mathrm{d}\Omega_{\mathbf{p}}\hat{\mathbf{p}} f\,, \tag{54}$$

and we introduced the flux $\mathbf{w}$, related to the usual current definition by $\mathbf{j} = v\,\mathbf{w} = p\,\mathbf{w}/E$. By replacing this approximation into Eq. (52), we obtain

$$\frac{\partial n}{\partial t} + \frac{p}{E}\nabla\cdot\mathbf{w} + \left(\frac{E}{p^2}\frac{\partial}{\partial p}(p^2\mathbf{w})\right)\cdot(\mathbf{\Omega}\times\mathbf{u}) = \int \mathrm{d}\Omega_{\mathbf{p}}\left.\frac{\partial f}{\partial t}\right)_c , \tag{55}$$

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{1}{3}\frac{p}{E}\nabla n + (\mathbf{\Omega}\times\mathbf{u})\frac{E}{3}\frac{\partial n}{\partial p} + \mathbf{\Omega}\times\mathbf{w} = \int \mathrm{d}\Omega_{\mathbf{p}}\hat{\mathbf{p}}\left.\frac{\partial f}{\partial t}\right)_c . \tag{56}$$

---

[1] The perturbation considered in previous sections had this property!

To deal with the collisional terms, it is worth starting in the frame where the scatterers are on average at rest ($\mathbf{V} = 0$). In this frame, we argued that the "collisional" processes drive the distribution towards isotropy, i.e.

$$\left.\frac{\partial f}{\partial t}\right)_c \simeq -\nu_{\theta\theta}\left(f - \frac{n}{4\pi}\right) = -\frac{3}{4\pi}\nu_{\theta\theta} w_i(p)\frac{p_i}{p} = -\frac{3}{4\pi}\sigma_{\theta\theta}\, n_s\, |v_{\rm rel}| w_i(p)\frac{p_i}{p}\,. \tag{57}$$

where at the last step we used a symbolic notation for the collision frequency $\nu_{\theta\theta} = \sigma_{\theta\theta} n_s |v_{\rm rel}|$ in terms of cross section $\sigma_{\theta\theta}$ with scattering centers of density n_s in order to highlight the appearance of the relative velocity $|v_{\rm rel}|$ (numerically equal to p/E in this frame). Note that in order for this expression to be valid, the a-priori tensorial structure of the interaction should simplify to a scalar form. An elastic scattering taking place in the scatterer frame is also assumed: no emission processes are accounted for, valid e.g. when scattering against infinitely heavy targets, but that is not strictly true in a plasma.

The actual result we need can be obtained with a boost to the Lab frame: henceforth a bar will indicate quantities in the plasma frame. Although the phase space distribution f is Lorentz invariant, so that $f(\mathbf{p}) = \bar{f}(\bar{\mathbf{p}})$, the separation into n and $\mathbf{w}$ is not, so we need to derive their transformation properties.

To leading order in $\mathbf{V}$ (we are only considering slowly moving, hence non-relativistic, scattering centers)

$$p_i = \bar{p}_i + \bar{E} V_i\,, \tag{58}$$

from which it follows the transformation law

$$\bar{f}(\bar{\mathbf{p}}) = f(\mathbf{p}) \Rightarrow \bar{f}(\bar{\mathbf{p}}) = f(\bar{\mathbf{p}}) + \bar{E}\, V_i \frac{\partial f(\bar{\mathbf{p}})}{\partial \bar{p}_i}\,, \tag{59}$$

$$\bar{w}_i(\bar{\mathbf{p}}) \equiv \int {\rm d}\bar{\Omega}_{\bar{\mathbf{p}}} \frac{\bar{p}_i}{\bar{p}} \bar{f}(\bar{\mathbf{p}}) = \int {\rm d}\bar{\Omega}_{\bar{\mathbf{p}}} \frac{\bar{p}_i}{\bar{p}}\left[f(\bar{\mathbf{p}}) + \bar{E} V_j \frac{\partial f(\bar{\mathbf{p}})}{\partial \bar{p}_j}\right]\,. \tag{60}$$

with the latter leading to

$$\bar{w}_i(\bar{\mathbf{p}}) = w_i(\bar{\mathbf{p}}) + \frac{\bar{E} V_i}{3}\frac{\partial n}{\partial \bar{p}}\,, \tag{61}$$

which is actually valid up to second order in V.

At leading order, one has

$$\bar{p}_i = p_i - E V_i\,, \quad w_i(\bar{p}) = w_i(p) - E\frac{V_j p_j}{p}\frac{\partial w_i}{\partial p}\,, \quad n(\bar{p}) = n(p) - E\frac{V_j p_j}{p}\frac{\partial n}{\partial p}\,. \tag{62}$$

In the above formulae we replaced $\partial/\partial\bar{p}$ with $\partial/\partial p$ and $\bar{E}$ with E, since they only enter in terms which are already suppressed by a power of V, hence would only be responsible for effects of higher order in V.

Provided that the momentum-dependence of the collisional frequency is negligible with respect to the momentum dependence of the CR distribution moments (which are indeed steep power-laws), we can treat it as a constant in the change of frame, and the collisional term of Eq. (57) in the Lab frame writes

$$\left.\frac{\partial f}{\partial t}\right)_c = -\frac{3}{4\pi}\sigma_{\theta\theta}\, n_s\, |\bar{v}_{\rm rel}| w_i(\bar{p})\frac{\bar{p}_i}{\bar{p}} \to -\frac{3}{4\pi\, E}\sigma_{\theta\theta}\, n_s\, \left[w_i(\bar{p})\bar{p}_i\right]_{\rm Lab}\,. \tag{63}$$

By expressing $[w_i(\bar{p})\bar{p}_i]_{\rm Lab}$ in terms of Lab variables via Eq.s (61,62), and then integrating over the solid angle via the relations i)-iv), we obtain the RHS of the Eqs (64,65) in the compact form

$$\left[\int {\rm d}\Omega_{\mathbf{p}} \left.\frac{\partial f}{\partial t}\right)_c\right]_{\rm Lab} = \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^3\left(w_i V_i + \frac{1}{3} E\, V^2 \frac{\partial n}{\partial p}\right)\right]\sigma_{\theta\theta}\, n_s\,, \tag{64}$$

$$\left[\int {\rm d}\Omega_{\mathbf{p}} \frac{p_i}{p}\left.\frac{\partial f}{\partial t}\right)_c\right]_{\rm Lab} = -\frac{p}{E}\left(w_i + \frac{1}{3} E\, V \frac{\partial n}{\partial p}\right)\sigma_{\theta\theta}\, n_s\,, \tag{65}$$

where, for practical purposes, one can approximate $\sigma_{\theta\theta}\, n_s \approx \nu_{\theta\theta} E/p$, since the constancy of $\nu_{\theta\theta}$ across the Lab and plasma frame is anyway only approximate. Note that the legitimacy of the expansions used crucially relies on the assumed hierarchy $|\mathbf{w}| \sim \mathcal{O}(V/c) n \ll n$ (which can be checked a posteriori), so that e.g. terms proportional to $V\,\mathbf{w}$ are kept and are of the same order of terms $V^2 n$, but terms of the order $V^2\,\mathbf{w}$ are neglected.

If the velocity of the scattering centers $\mathbf{V}$ obeys some distribution $h(\mathbf{V})$, the final result should be obtained by performing the average of the above RHS over h. By denoting with $\mathbf{u}$ the average velocity and $\delta\mathbf{v}$ the deviation with respect to the average, we get

$$\left[\int \mathrm{d}\Omega_{\mathbf{p}} \left.\frac{\partial f}{\partial t}\right)_c\right]_{\rm Lab} = \frac{\sigma_{\theta\theta}\, n_s}{p^2}\frac{\partial}{\partial p}\left[p^3\left(w_i u_i + \frac{E}{3}u^2\frac{\partial n}{\partial p}\right)\right] + \frac{\sigma_{\theta\theta}\, n_s}{p^2}\frac{\partial}{\partial p}\left[p^3\frac{E}{3}\langle\delta v^2\rangle\frac{\partial n}{\partial p}\right], \tag{66}$$

$$\left[\int \mathrm{d}\Omega_{\mathbf{p}} \frac{p_i}{p}\left.\frac{\partial f}{\partial t}\right)_c\right]_{\rm Lab} = -\frac{p}{E}\left(w_i + \frac{1}{3}E\,u\frac{\partial n}{\partial p}\right)\sigma_{\theta\theta}\, n_s\,. \tag{67}$$

We can now write the final result as

$$\frac{\partial n}{\partial t} + \frac{p}{E}\frac{\partial w_i}{\partial x_i} + \left(\frac{E}{p^2}\frac{\partial}{\partial p}(p^2 w_i)\right)\epsilon_{ijk}\Omega_j u_k = \frac{\sigma_{\theta\theta}\, n_s}{p^2}\frac{\partial}{\partial p}\left[p^3\left(w_i u_i + \frac{E}{3}u^2\frac{\partial n}{\partial p} + \frac{E}{3}\langle\delta v^2\rangle\frac{\partial n}{\partial p}\right)\right], \tag{68}$$

$$\frac{\partial w_i}{\partial t} + \frac{1}{3}\frac{p}{E}\frac{\partial n}{\partial x_i} - \left(w_j + \frac{E}{3}\frac{\partial n}{\partial p}u_j\right)\epsilon_{ijk}\Omega_k = -\sigma_{\theta\theta}\, n_s\,\frac{p}{E}\left(w_i + \frac{E}{3}u_i\frac{\partial n}{\partial p}\right). \tag{69}$$

The usual way to obtain a single equation for n consists in assuming that in Eq. (69), $\partial_t w_i \approx 0$ compared to the much larger frequency present at the RHS. This allows one to derive

$$\left(w_j + \frac{E}{3}u_j\frac{\partial n}{\partial p}\right)(\nu_{\theta\theta}\delta_{ij} - \epsilon_{ijk}\Omega_k) = -\frac{1}{3}\frac{p}{E}\frac{\partial n}{\partial x_i}\,, \tag{70}$$

i.e.

$$w_i = -\frac{E}{p}K_{ij}\frac{\partial n}{\partial x_j} - \frac{E}{3}u_i\frac{\partial n}{\partial p}\,, \tag{71}$$

where (note the factor p/E, so that the tensor K is defined as conventionally in terms of the current $\mathbf{j}$, rather than $\mathbf{w}$)

$$K_{ij} = \frac{1}{3\nu_{\theta\theta}}\left(\frac{p}{E}\right)^2\frac{\nu_{\theta\theta}^2\delta_{ij} + \nu_{\theta\theta}\Omega_k\epsilon_{ijk} + \Omega_i\Omega_j}{\nu_{\theta\theta}^2 + \Omega^2}\,, \tag{72}$$

which can then be replaced in the equation for n to eventually obtain:

$$\boxed{\frac{\partial n}{\partial t} - \frac{\partial}{\partial x_i}K_{ij}\frac{\partial n}{\partial x_j} + u_i\frac{\partial n}{\partial x_i} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\left(p\frac{\partial n}{\partial p}\right) = \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 K_{pp}\frac{\partial n}{\partial p}\right),} \tag{73}$$

where we defined

$$K_{pp} \equiv \frac{\nu_{\theta\theta}E^2\langle\delta v^2\rangle}{3}\,. \tag{74}$$

For more details, see some general textbook references [2, 11, 12]. Compared to the previously derived Eq. (32), we see three new terms, identified by color-code:

- **convection/advection:** It accounts for spatial transport due to large scale movements like Galactic winds. This therm is usually considered mostly perpendicular to the Galactic plane and antisymmetric with respect to it, $u(-z) = -u(z)$. For typical values of $u = \mathcal{O}(10)\,\mathrm{km\,s^{-1}}$ it is relevant at $\mathcal{O}(1)$ GeV, but its relevance decreases at higher energies.

- **adiabatic energy losses/gains:** Diverging flows lead to adiabatic energy losses, while converging flows to energy gains. While this effect may be important in propagation models with non-uniform galactic winds, it is particularly crucial as a way to *accelerate particles* i.e. in our factorized production-propagation scenario, as a way to generate Q. More details on this in Pasquale Blasi's lectures.

- **reacceleration:** If we adopt for the spatial diffusion coefficient its parallel value, we arrive at the estimate $K_{zz}K_{pp} = p^2\langle\delta v^2\rangle/9$ where $\langle\delta v^2\rangle$ is typically set to the square of the the Alfvèn velocity v_A. Acceleration in the ISM via this term cannot be the main source of CR acceleration because e.g. of the observed energy dependence of secondary-to-primary ratios. However this term is important in determining the momentum shape of CR fluxes below a few GeV, and this "turbulent" acceleration may also play an important role at sources. More details on this in Blasi's lectures.

Modulo a few extra terms related to continuous and catastrophic losses that will be introduced in the following, Eq. (73) above is used in almost all phenomenological treatments of Galactic CR propagation. Needless to say, this is still an approximation. The two major limitations of this treatment are:

- The requirement that the scattering centers are moving non-relativistically. This is crucial in obtaining a hierarchy between the isotropic and anisotropic part of the CR distribution, the latter in turn being dominated by the dipole term. This is certainly inadequate in relativistic environments (such as gamma-ray bursts, or most candidate sources for ultra-high energy cosmic rays), where a technical difficulty is that the distribution function retains a non-trivial angular dependence.

- At a more conceptual level, we have performed some further approximations. First, we assumed that a kinetic description in terms of the single-particle distribution function is suitable, i.e. we neglected particle-particle correlations terms (entering higher-order equations of the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy), implicitly adopting the so-called *plasma approximation*. More importantly, even the single particle distribution functions f_a (for the species a) are ruled by Maxwell-Vlasov equations

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\frac{\partial f_a}{\partial \mathbf{x}} + \frac{q_a}{m_a}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f_a}{\partial \mathbf{v}} = 0\,. \quad (75)$$

$$\nabla\times\mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi\left(\mathbf{j}_{\rm ext} + \sum_b q_b \int {\rm d}\mathbf{p}\,\mathbf{v}_b f_b\right) \quad (76)$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (77)$$

$$\nabla\cdot\mathbf{B} = 0 \quad (78)$$

$$\nabla\cdot\mathbf{E} = 4\pi\left(\rho_{\rm ext} + \sum_b q_b \int {\rm d}\mathbf{p} f_b\right) \quad (79)$$

$$(80)$$

where $\rho_{\rm ext}$ and $\mathbf{j}_{\rm ext}$ represent external charge densities and currents due to non-dynamical components of the medium. In general, we see that the CR themselves contribute to generate the fields in which they propagate. This coupling has been ignored. A simple way to convince oneself that some coupling must be accounted for, in order to guarantee the consistency of the previous description, is to think about the isotropisation process. If CR tend to be isotropized in the frame of the plasma, where does the initial CR momentum go? In fact, it constitute a current, acting as one of the sources of the e.m. waves of similar type to the ones onto which CRs scatter. More details on this feedback in Blasi's lectures.

III. GENERALITIES ON (RELATIVISTIC) TWO-BODY COLLISIONS

In a two-body collision, the following quantity is a relativistic invariant (square of the COM energy):

$$s = (p_a+p_b)^2 = m_a^2 + m_b^2 + 2\left(E_aE_b - \mathbf{p}_a\mathbf{p}_b\right) = m_a^2+m_b^2+2E_aE_b\left(1-\beta_a\beta_b\cos\vartheta\right) \quad (81)$$

If equated to the (square) of the sum masses of the final state, this allows one to estimate threshold energies in the Lab frame.

Exercise Calculate the minimum energy in the lab for a CR proton, hitting another one at rest, to produce one antiproton. *Hint:* remembering that baryon number is conserved, what is the lightest final state containing an antiproton, i.e. what is the lightest X in $pp\to\bar{p}X$?

Another useful invariant is the (square) of the momentum transferred

$$t = (p_a-p_c)^2 = (p_b-p_d)^2\,. \quad (82)$$

For the case of a electron scattering with another charged particle (a proton, for instance) one has

$$t = (p_e-p_e')^2 = 2m_e^2 - 2E_eE_e'\left(1-\beta_e\beta_e'\cos\vartheta\right) \to -4E_eE_e'\sin^2\vartheta/2\,, \quad (83)$$

where the last step is valid in the high-energy limit. Note that its value is negative, while the squared four-momentum of any real photon has $q^2=0$ (it's a virtual photon!). If you remember the strong angular dependence of Rutherford scattering,

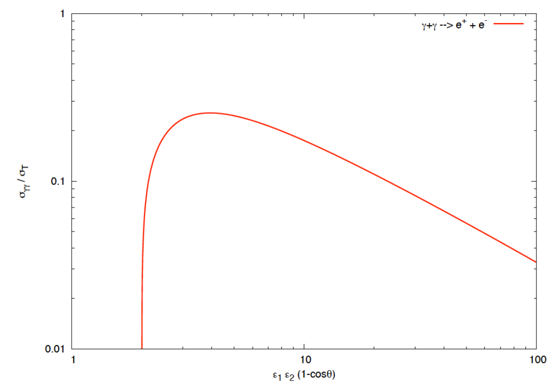

FIG. 6: Energy dependence of the pair-production cross-section.

with $d\sigma/d\Omega \propto Z_1 Z_2 \alpha/\sin^4\theta/2$, this implies $d\sigma/d\Omega \propto 1/t^2$: the cross-section is dominated by small-angle scatterings! On the other hand, if one wants to produce a particle of mass M in a collision, a rough criterion must be $|t| > M^2$. So, the 'bulk" of the cross-section (what is mostly relevant for cosmic ray physics in the atmosphere, for instance) is dominated by small angle scatterings; machines like LHC focus instead on large-angle scatterings, which are associated to a la large exchanged momentum. Besides the relative rarity of very energetic CR, this is also the reason why CRs and high-energy collider physics have largely complementary targets.

A. One simple application: Pair production and the "gamma-ray horizon"

If we specialize Eq. (81) to the case $\gamma + \gamma \to e^+e^-$ for heads-on collision, we get

$$4E_\gamma \epsilon = (2m_e)^2 \Longrightarrow E_\gamma > \frac{m_e^2}{\epsilon}\,. \tag{84}$$

The cross-sections behaves as

$$\sigma_{\gamma\gamma}(\beta) = \frac{3\pi\sigma_T}{16}\left(1-\beta^2\right)\left[2\beta\left(\beta^2-2\right)+\left(3-\beta^4\right)\ln\left(\frac{1+\beta}{1-\beta}\right)\right]\,, \tag{85}$$

where β is either lepton velocity in the COM frame. Graphically, it peaks at about $\sigma_T/4$ at about twice the threshold energy, see Fig. 6.

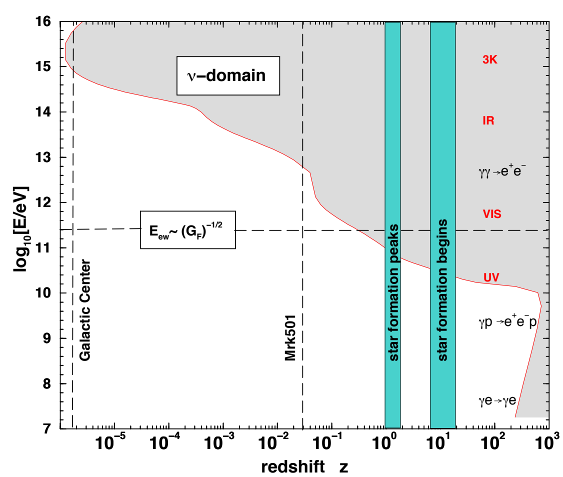

FIG. 7: Gamma-ray horizon (for a Hubble constant of H_0=60 km s^{-1} Mpc^{-1} and a cold dark matter cosmology) due to various absorption processes (predominantly $\gamma - \gamma$ pair production in the extragalactic radiation field). The shaded area is invisible for gamma-ray astronomy [13].

Exercise Compute the energy of background photons at threshold for pair-production for incoming photons of 1 TeV or 1 PeV energy. What type of "light" are these bands corresponding to? In the case of CMB, compute the mean-free path of a photon of typical PeV energies.

The exercise above is meant to illustrate the fact that the gamma-ray sky is not always optically thin, with the $e^{\pm}$ pair production mechanism constituting a serious limitation for the remote extragalactic sky already at hundred of GeV, for the near extragalactic sky at about 10 TeV, and even for Galactic objects in the PeV range (see the "photon horizon in Fig. 7)

IV. SYNCHROTRON RADIATION

In the discussion of Sec. II, we have explicitly neglected any "non-elastic" process. But, as we saw, in the presence of a magnetic field such as the one threading the ISM, a charged particle follows a helicoidal trajectory, with a typical gyroradius given by Eq. 13 and a typical gyrofrequency given by Eq. 14.

But a rotating charge is actually accelerated, and we know from basic electrodynamics that an accelerated charge radiates, see Fig. 8 for a cartoon. In the non-relativistic limit, the radiated power (or $-\mathrm{d}E/\mathrm{d}t$) is described by the Larmor formula

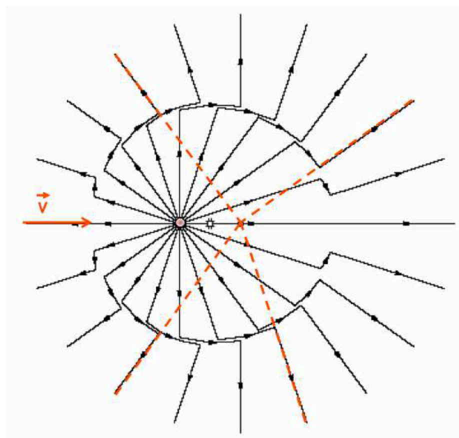

FIG. 8: Electric field due to a charge, initially in uniform motion, that is stopped within a short time δt. Nearby, the field had time to adjust and points to where the charge is (red dot). Far away, the field points to where the charge would be if it had not been stopped (orange point). There is then a region of space where the electric field has to change direction, corresponding to the propagating e.m. wave: its front propagates radially outwards, and the field of the wave (the $\mathbf{\Delta E}$ of the disturbance) is orthogonal to the propagating direction, null along the direction of the braking (which is the initial velocity one, where the two fields are already aligned), and the width of this region is $c\delta t$.

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}=\frac{q^2a^2\sin^2\phi}{4\pi}\Rightarrow P=\frac{2}{3}q^2a^2\,, \tag{86}$$

ϕ being the angle with respect to the acceleration vector (dependence consistent with cartoon in Fig. 8) [2]. We shall assume familiarity with it, without re-deriving it (for more details on this and related topics in this and the following section, see also [14]). Note however how the dependence upon the square of the acceleration follows trivially from dimensional analysis (and the impossibility of dependence upon position or velocity because of Galilean or Poincaré invariance). The appearance of e^2 is also trivially following from the basic e.m. interaction vertex.

In fact, this formula is also valid in the relativistic limit, provided that a^2 is now replaced by the four-acceleation (or, four-force over mass) $a^2\to a_\mu a^\mu$. This is because the radiated power is a Lorentz-invariant, as one can guess from the fact that dE and dt are both the 0-th components of four-vectors, transforming the same way.

---

[2] This derives from

$$\mathrm{d}P=\mathbf{S}\cdot d\vec{A}=\mathbf{S}\cdot\hat{\mathbf{n}}R^2\mathrm{d}\Omega\,, \tag{87}$$

R being the distance from the radiating source, $\hat{\mathbf{n}}$ the normal to the crossed surface, and the Poynting vector

$$\mathbf{S}=\frac{\mathbf{E}\times\mathbf{B}}{4\pi}=\frac{|\mathbf{E}|^2}{4\pi}\hat{\mathbf{S}}\,, \tag{88}$$

to be computed from the electrodynamics of a radiating charge moving non-relativistically, yielding $|\mathbf{E}|^2\sim q^2a^2/R^2$.

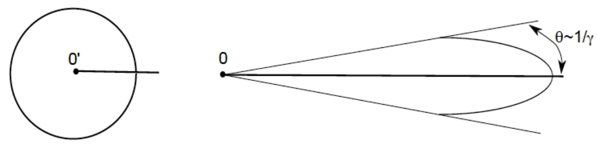

FIG. 9: Illustration of the beaming effect in the lab (unprimed frame) for an isotropic emission in the source (primed).

A charge q moving with Lorentz factor γ with respect to the frame of the static magnetic field views an induced electric field equal to

$$\mathbf{E}' = -\gamma \mathbf{v} \times \mathbf{B}\,, \tag{89}$$

associated to the acceleration in the electron frame

$$\mathbf{a} = -\frac{q}{m}\gamma \mathbf{v} \times \mathbf{B}\,, \tag{90}$$

hence the synchrotron power (energy radiated away per unit time)

$$\boxed{P_s = \frac{2q^4\gamma^2}{3\,m^2}v^2 B^2 \sin^2\theta\,.} \tag{91}$$

i.e., proportional to γ^2 in the relativistic limit, as well as B^2, itself proportional to the energy density stored in the B-field.

1. *Beaming effect from relativistic motion*

Remember the time dilation and space contraction by the same factor $\gamma(v)$ in a (primed) frame moving at velocity v with respect to the initial one, depending upon Lorentz transformation:

$$\boxed{t' = \gamma(t - \beta x) \quad x' = \gamma(x - \beta t)}\,. \tag{92}$$

From which one derives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma(\mathrm{d}x' + \beta \mathrm{d}t')}{\gamma(\mathrm{d}t' + \beta \mathrm{d}x')} = \frac{u'_x + \beta}{1 + \beta u'_x}\,, \tag{93}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\gamma(\mathrm{d}t' + \beta \mathrm{d}x')} = \frac{u'_y}{\gamma(1 + \beta u'_x)}\,. \tag{94}$$

This implies the aberration formula

$$\tan\theta \equiv \frac{u_y}{u_x} = \frac{u'_y}{\gamma(u'_x + \beta)} = \frac{u'\sin\theta'}{\gamma(\beta + u'\cos\theta')}\,. \tag{95}$$

This means that a photon ($u = c$) emitted at $\theta' = 0$ travels at $\theta = 0$, while a photon emitted at $\theta' = \pi/2$ travels at $\sin\theta \simeq \theta \simeq 1/\gamma$, as illustrated in Fig. 9.

2. *Characteristic frequency of emission*

To compute the spectrum of photons emitted in a time-varying emission process, one takes the Fourier transform of the signal. In the non-relativistic limit, one has simply a circular uniform motion, thus emission at the single frequency ν_g given (cyclotron line) given by Eq. (12). In the relativistic case, a major alteration is due to beaming (See sec. IV 1) the frequency spectrum is different because of sizable boost effects affecting the radiation timing: One must take into account that only during the fraction of the orbit within the beaming angle the emission is observed. The timescale of the observer

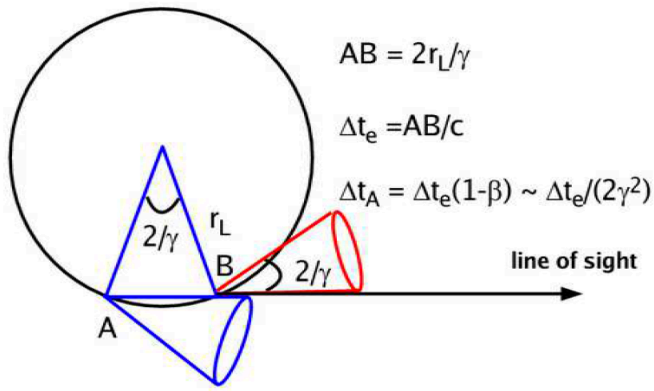

FIG. 10: Beaming effect.

is further different from the timescale of the emission. The time difference from the passage at point B and and point A is simply given by

$$\Delta t \equiv t_B - t_A = \frac{AB}{v} = r_L \frac{2}{\gamma}\frac{1}{v}\,. \tag{96}$$

For an observer at distance well beyond B along the direction AB, the duration of signal detected is

$$\Delta t_{\rm obs} = t_B + \delta t_B^{\rm prop} - (t_A + \delta t_A^{\rm prop}) = \Delta t - AB/c = \frac{AB}{v}(1-\beta) = \frac{AB}{v}\frac{(1-\beta^2)}{1+\beta} \simeq \frac{AB}{v}\frac{1}{2\,\gamma^2} \simeq \frac{r_L}{\gamma^3}\,. \tag{97}$$

As a poor man's proxy for a Fourier transform, we can estimate the typical frequency of the radiation (known as *synchrotron* radiation) as the inverse of the above, hence

$$\nu_s \simeq \frac{\gamma^3}{r_L} = \gamma^2 \frac{\omega_g}{2\pi} \Rightarrow E_s \simeq 500\,\mu{\rm eV}\frac{B}{\mu G}\left(\frac{E_e}{\rm GeV}\right)^2\,. \tag{98}$$

The more complete calculation leads to a power emitted per unit frequency which is not monochromatic, rather with a behaviour $\sim \nu^{1/3}e^{-\nu/\nu_c}$, with the critical frequency ν_c of the same order as the ν_s above.

What about the spectrum from an ensemble of particles, say an electron population? For an electron spectrum of energy-differential number density $n(E) \propto E^{-\alpha}$, we have the following link

$$\nu_s = \left(\frac{E}{m_e}\right)^2 \nu_g \Rightarrow dE = \frac{m_e}{2\sqrt{\nu\nu_g}}d\nu\,. \tag{99}$$

The scaling of the synchrotron spectrum emitted with the B field and the index α can be estimated simply as follows (remember that $\nu_g \propto B$). In particular, the specific emissivity (power per unit solid angle per unit volume per unit frequency) is

$$\epsilon(\nu)d\nu = \frac{1}{4\pi}P_s n(E)dE \propto E^2 B^2 E^{-\alpha} dE \propto \nu^{\frac{1-\alpha}{2}} B^{\frac{1+\alpha}{2}} d\nu\,. \tag{100}$$

So, from the slope typically in the radio-microwaves band ($(1-\alpha)/2$) one can infer the slope of the parent electron distribution ($-\alpha$).

Exercise Gamma-ray spectra of pulsed emissions from several pulsars show a cutoff at an energy of a few GeV. Assuming that this is curvature radiation from a compact object of the size of (few times the) neutron star radius (of the order of 15 km or so), estimate the energy of electrons emitting those photons. (Compare with the energy associated to a unipolar inductor, with B field of 10^{12} Gauss and rotation period of 1 s, of the order of $\omega B R^2/2$)

V. INVERSE COMPTON

A. Thomson cross-section

Let's start from the classical problem of interaction of a electromagnetic wave with a charged particle, computing the power that this particle re-radiates. The acceleration is (Lorentz force over mass) $q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$, which for $v \ll 1$ and considering that the electric and magnetic field of the wave have the same amplitude, gives and acceleration $q|\mathbf{E}|$ along the direction of the field. For a sinusoidal wave $E = E_0 \sin(\omega t + \phi)$ along some direction, the acceleration is along the same direction and of intensity $a = qE/m$, so that the Larmor formula gives for the (time-averaged) emitted power

$$\langle P \rangle = \frac{2}{3} q^2 \langle a^2 \rangle = \frac{2}{3} \frac{q^4}{m^2} \frac{E_0^2}{2} \,. \tag{101}$$

In scattering theory, the cross-section is the ratio of the radiated power to incident flux,

$$\sigma = \frac{\langle P \rangle}{|\langle \mathbf{S} \rangle|} = \frac{8\pi}{3} \frac{q^4}{m^2} \equiv \sigma_T \quad (\text{if } q = \pm e) \tag{102}$$

with σ_T is the well-known Thomson cross-section, and where the Poynting vector (incident energy per unit time and unit surface) is

$$|\langle \mathbf{S} \rangle| = \frac{|\langle \mathbf{E} \times \mathbf{B} \rangle|}{4\pi} = \frac{E_0^2}{8\pi} \,. \tag{103}$$

Note that in terms of u is the energy density in the e.m. field,

$$u = \left\langle \frac{|\mathbf{E}|^2}{8\pi} + \frac{|\mathbf{B}|^2}{8\pi} \right\rangle = \frac{E_0^2}{8\pi} \,, \tag{104}$$

one can also write

$$\langle P \rangle = \sigma_T \, u \,, \tag{105}$$

which clearly suggests a more microscopic interpretation (scattering rate—remember $c = 1$!—onto the number of particles/photons in the field, times the average energy of the photons). Compare with Eq. (91): can you justify the interpretation of synchrotron power as up-scattering of 'virtual' photons associated to an external magnetic field?

1. Energy loss rate

The formula derived for the emission power in the non-relativistic case

$$\langle P \rangle = \sigma_T \, \tilde{u} \,, \tag{106}$$

is actually valid more in general since, loosely speaking, it's a dE/dt and energy and time transform the same way. The energy density $\tilde{u}$ is the energy density in the frame comoving with the electron. In the lab frame where the electron is moving at β, it can be expressed by remembering that $[u] = [\epsilon \times n]$, and that the number density n transforms as a time since both dtd^3x (four-volume) and nd^3x (number of particles) are relativistic invariants. So, u transforms as the square of the energy,

$$\tilde{u} = u\gamma^2 (1 - \beta \cos\alpha)^2 \,, \tag{107}$$

which, for a isotropic radiation field, yields the angular average

$$\langle \tilde{u} \rangle = u\gamma^2 \left(1 + \frac{\beta^2}{3} \right) \tag{108}$$

since

$$\langle \cos\alpha^2 \rangle = \frac{1}{2} \int_{-1}^{+1} \mathrm{d}(\cos\alpha) \cos^2\alpha = \frac{1}{2} \frac{2}{3} \,. \tag{109}$$

The energy lost by the electrons per unit time being the difference of the scattered power minus incoming power $\sigma_T \, u$ (impinging photons had some energy, too!), we have

$$\boxed{-\frac{dE}{dt} = \sigma_T \, u \left[\gamma^2 \left(1 + \frac{\beta^2}{3} \right) - 1 \right] = \frac{4}{3} \gamma^2 \beta^2 \, u \sigma_T \simeq \frac{4}{3} \gamma^2 \, u \sigma_T \,.} \tag{110}$$

If u is interpreted as the energy density of both photon fields and magnetic field, this formula describes both synchrotron and IC losses (in the Thomson regime)!

Also, it is useful to define (not only for IC, but more generally!) a characteristic loss timescale,

$$\boxed{\tau_{\rm loss} \equiv \frac{E}{-{\rm d}E/{\rm d}t}}\,, \tag{111}$$

which is often a quick way to compare the relative importance of various processes.

Exercise Consider a (globally neutral) plasma of electrons and protons (remember: the two species are in tight e.m. coupling!), spherically symmetric of mass M and radius R, kept in equilibrium by the balance of gravity and radiation pressure. What is the luminosity (called Eddington luminosity) supporting such a "star" on the verge of disruption? Does our Sun satisfy that limit? If yes, it is close to it? If not, can you explain why? If L is the stellar luminosity (energy per unit time emitted), the momentum (or energy, in natural units for a photon) per unit time (i.e. the Force) and unit surface (i.e. the Pressure) is

$$\frac{L}{4\pi R^2} \tag{112}$$

so that the effective force felt by the tightly bound $e-p$ system is the one felt by the electrons thanks to their "effective surface" (scattering cross-section)

$$\sigma_T \frac{L}{4\pi R^2} \tag{113}$$

The gravitational force on the tightly bound $e-p$ system is (given the large mass ratio in favour of the proton!)

$$\frac{GMm_p}{R^2} \tag{114}$$

From their equality we obtain

$$L = \frac{4\pi G m_p M}{\sigma_T} = 10^{38}\frac{M}{M_\odot}{\rm erg/s}\ \ [{\rm Eddington\ \ luminosity}] \tag{115}$$

(Note that $L_\odot = 3.83\times 10^{33}$ erg/s, so our Sun is safely below that limit!)

B. The Klein-Nishina regime

Actually, one can look at the same process from a quantum point of view, as a two-body collision between a photon and an electron. The more general formula for differential cross-section derived by Klein & Nishina in 1929, based on QED (see the diagrams in Fig. 11):

$$\frac{d\sigma}{d\Omega} = \frac{3}{16\pi}\sigma_T\left(\frac{\epsilon_f}{\epsilon_i}\right)^2\left(\frac{\epsilon_i}{\epsilon_f}+\frac{\epsilon_f}{\epsilon_i}-\sin^2\theta\right)\,, \tag{116}$$

$$\sigma = 2\pi\int_0^\pi \frac{d\sigma}{d\Omega}\sin\theta d\theta = \frac{3}{4}\sigma_T\left[\frac{1+x}{x^3}\left(\frac{2x(1+x)}{1+2x}-\ln(1+2x)\right)+\frac{1}{2x}\ln(1+2x)-\frac{1+3x}{(1+2x)^2}\right]\,, \tag{117}$$

where $x\equiv \epsilon_i/m_e$, so that

$$\begin{aligned} \sigma(x) &\simeq \sigma_T(1-2x+\ldots) \quad \text{for} \quad x\ll 1 \text{ (Thomson)} \\ \sigma(x) &\simeq \frac{3}{8}\sigma_T\frac{1}{x}\left(\ln 2x+\frac{1}{2}\right) \quad \text{for} \quad x\gg 1 \quad \text{(extreme KN)}\ . \end{aligned} \tag{118}$$

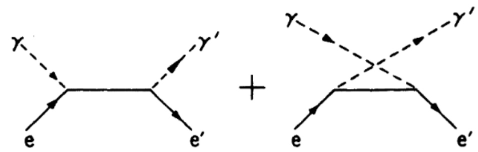

FIG. 11: Diagrams describing the Klein-Nishina process (tree-level electron photon scattering in QED).

1. Compton kinematics

Let us focus on the kinematics of the quantum process. Impose energy momentum balance in a photon (momentum k)-electron (momentum p) scattering:

$$k_i^\mu + p_i^\mu = k_f^\mu + p_f^\mu \,, \tag{119}$$

and the on-shell condition for the electron mass, $m_e^2 = p_i^2 = p_f^2$, and that $k_{i,f}^2 = 0$:

$$m_e^2 = (p_i + k_i - k_f)_\mu (p_i + k_i - k_f)^\mu \Longrightarrow m_e^2 = m_e^2 + 2(p_i k_i - p_i k_f - k_i k_f)\,. \tag{120}$$

In the frame where the electron is at rest, $p_i = (m_e, 0, 0, 0)$. We choose the x-axis as the direction of the photon, and the outgoing direction generates with that the x-y plane. Then $k_i^\mu = \epsilon_i(1, 1, 0, 0)$, while $k_f^\mu = \epsilon_f(1, \cos\theta, \sin\theta, 0)$, so that

$$m_e(\epsilon_i - \epsilon_f) = \epsilon_i \epsilon_f (1 - \cos\theta) \Longrightarrow \epsilon_f = \frac{\epsilon_i}{1 + \frac{\epsilon_i}{m_e}(1 - \cos\theta)}\,. \tag{121}$$

Note that unless the energy of the photons is of the order of the mass of the electron or larger in the electron rest-frame, the energy of the photons is only slightly altered (the ratio ϵ_i/m_e controls the energy change).

What happens if the electron is not at rest in the frame of interest ("lab"), rather has a velocity β? Well, the above relation (121) is valid in a frame (let us denote it via a prime) boosted via $\gamma(\beta)$ with respect to the lab:

$$\epsilon_f' = \frac{\epsilon_i'}{1 + \frac{\epsilon_i'}{m_e}(1 - \cos\theta')}\,, \tag{122}$$

where $\epsilon_i' = \epsilon_i \gamma(1 - \beta\cos\alpha)$, where α is the angle between photon and electron in the Lab frame (θ', as above, is the angle between outgoing and incoming photon directions). If we want to express ϵ_f' in the Lab frame, we need to perform the "reverse boost", now accounting for the emission angle α' in the comoving frame, such that

$$\epsilon_f = \gamma(1 + \beta\cos\alpha') \frac{\epsilon_i'}{1 + \frac{\epsilon_i'}{m_e}(1 - \cos\theta')} = \gamma^2 \epsilon_i \frac{(1 + \beta\cos\alpha')(1 - \beta\cos\alpha)}{1 + \frac{\epsilon_i'}{m_e}(1 - \cos\theta')}\,. \tag{123}$$

In the limit where $\epsilon_i'/m_e \ll 1$ (which requires $\epsilon_i \gamma \ll m_e$, i.e. $\epsilon_i E_e \ll m_e^2$ in the Lab frame), one has

$$\epsilon_f \simeq \gamma^2 \epsilon_i (1 + \beta\cos\alpha')(1 - \beta\cos\alpha)\,. \tag{124}$$

Also, for isotropically impinging radiation, the second $(\ldots)$ is averaging to 1 (and, as a consequence, also the first one, since the outgoing direction will be essentially isotropic in the electron comoving frame). Hence, a good estimate is

$$\boxed{\epsilon_f \simeq \gamma^2 \epsilon_i = 30 \left(\frac{\epsilon_i}{\mathrm{eV}}\right) \left(\frac{E_e}{\mathrm{GeV}}\right)^2 \mathrm{MeV}} \tag{125}$$

For this approximation to be correct, one requires $\epsilon_i E_e \ll m_e^2$ which implies $\epsilon_f < E_e$ as well. Note that while the scattering angle is arbitrary in the comoving frame, in the lab frame the outgoing radiation is beamed in the forward direction with an angle $1/\gamma$ (now you should be familiar with that!)

Exercise What is the maximum final energy of a photon in the Thomson regime ($\epsilon \ll m_e$), see Eq. (123)?

Exercise Have a look at Eq. (123): What is the typical energy in the KN regime? (put the angles to zero and consider $\epsilon \gg m_e$, should get $\sim \gamma m_e = E_e$; hence, in the KN regime the scattering becomes less likely, but when happening the scattered photons carries a sizable fraction of the energy of the incoming electron.)

C. Notions on SSC

Imagine that the synchrotron photons produced by an energetic electron populations also constitute the main target for further upscatter via inverse Compton. The resulting spectrum corresponds is dubbed Synchrotron-Self-Compton. We know the link between electron power-law index ($-\alpha$) and Syn. spectral index ($(1-\alpha)/2$), which is also the same for the IC (check that you can repeat the same argument, at least in the Thomson regime!)

This means that, at least in the Thomson regime, SSC sources are expected to show a power law between $\gamma_{\rm min}^2\nu_g$ and $\gamma_{\rm max}^2\nu_g$, and another power-law parallel to the previous one from $\gamma_{\rm min}^4\nu_g$ and $\gamma_{\rm max}^4\nu_g$, where the $\gamma_i's$ refer to the parent electron spectrum. Also note that the Syn. peak is proportional to the number of high-energy electrons, while the SSC peak, is quadratic in it (since also the target photon number is proportional to the high-energy electrons). I won't cover the physics of the sources, but if we are confident that SSC is realized, a lot of information can be gathered from the spectrum.

Exercise The Crab nebula, associated to the explosion of a SN in AD 1054, presents a roughly broken power-law spectrum, with a steepening around 5 eV, interpreted as synchrotron radiation. If we attribute this phenomenon to a "cooling break" (i.e. the electrons producing the photons below the break have a cooling lifetime longer than its age, and the ones producing photons above it have a shorter cooling timescale), determine the magnetic field in the source, and the energy of the electrons associated to this "break point". What is the energy of IC photons produced when those electrons hit the very synchrotron photons at the break point?

VI. LEPTONIC INTERACTIONS WITH MATTER

A. Ionization and Coulomb interactions

When crossing a medium with density of electrons n_e, on dimensional grounds an electron undergoes an energy loss per unit length of the order of $\sigma_T\, n_e\, m_e\, f(E/m_e)$, with the dimensionless function $f(x)$ further depending on atomic or plasma properties of the medium (such as the electron binding energy or the plasma frequency); A simplified estimate comforting the above expression can be obtained as follows in a classical limit: consider a non-relativistic electron crossing a medium containing electrons ("almost at rest") with density n_e. If b is the impact parameter at which the energetic electron passes, and v its velocity, the target electron feels an impulsive force due to the CR electron electric field $F = e^2/b^2$ over a time $\delta t\sim b/v$ (factors of order unity are omitted), associated to a momentum gain $\Delta p = F\,\Delta t\sim e^2/(b\,v)$, or equivalently a gain of kinetic energy $\Delta E\sim(\Delta p)^2/m_e\sim e^4/(b^2\,v^2\,m_e)$. The energy-loss rate per unit length for the cosmic ray electron can be obtained by integrating over the individual losses,

$$-\frac{{\rm d}E}{{\rm d}x}=\int 2\pi\, b\, n_e\,\Delta E{\rm d}b\simeq 2\pi\, n_e\frac{e^4}{v^2\,m_e}\ln\frac{b_{\rm max}}{b_{\rm min}}\simeq\frac{\sigma_T\,n_e\,m_e}{v^2}\ln\frac{b_{\rm max}}{b_{\rm min}} \tag{126}$$

where, apart from factors of order unity omitted above, the formula also depends on the so-called Coulomb logarithm: Qualitatively, $b_{\rm min}$ is determined by the distance at which the collision leads to a deflection angle of order unity (a "hard" collision); $b_{\rm max}$ is clearly dependent on the type of medium, related to bound electron orbital characteristics, or the effective plasma screening length. Typically, $\ln b_{\rm max}/b_{\rm min}\sim\mathcal{O}(10)$. Also note that the derivation requires the energy transfer to be non-relativistic, not the initial particle to be. In fact the result for a relativistic impinging particle is the same, since even if there is a time contraction $\Delta t\to\Delta t/\gamma$, the electric force increases by the same factor $F\to\gamma F$ (relativistic transformation of the electric field).

A more correct formula was computed by Bethe within quantum mechanics, and should be familiar from elementary physics courses,

$$-\frac{{\rm d}E}{{\rm d}x}=\frac{3}{2}\frac{\sigma_T\,m_e\,n_e}{v^2}\left[\log\left(\frac{E}{m_e}\right)+\kappa-\beta^2\right] \tag{127}$$

where the dimensionless constant κ depends on the material crossed, but typically dominates the E-dependence. A number of corrections are needed at both low and high velocities, which however are not particularly relevant for us since at higher energy the electromagnetic energy transfer described here is hardly the dominant one, and at low-energy one exits the realm of interest for most cosmic ray phenomenology. It suffices to notice that in the non-relativistic regime the E-loss becomes less and less efficient with higher velocity, until it saturates (actually, grows logarithmically) at relativistic energies. But, by the time a particle is relativistic, other E-loss mechanism dominate.

Exercise Remember Eq. (111). Estimate the range (and timescale) over which an electron with a kinetic energy of 60 keV stops in a ISM with density of 1 cm^{-3}.

The above considerations also apply to charged CRs, of course: We computed the energy transferred via interactions with the electrons in the medium via the impulsive electric field. So, the same considerations apply, but for the change in

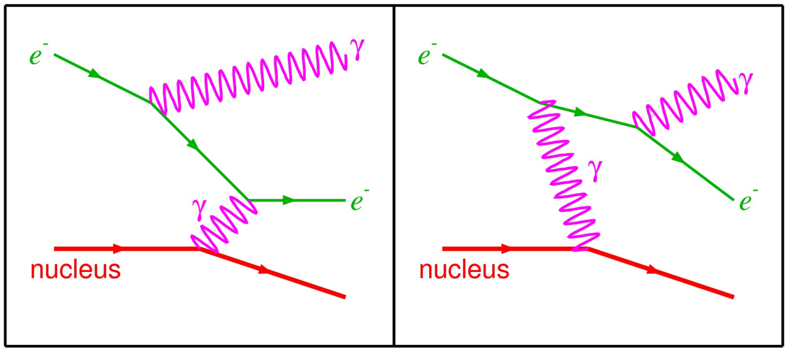

FIG. 12: Diagrams for Bremsstrahlung process.

the force (now with factors e^2 replaced by $Z\,e^2$, so that the previous formulae acquire a factor Z^2 more. Also, since m_e sets the energy scale, these energy losses are *in percentage* way less efficient that for leptons *of the same velocity*.

Exercise Estimate the range (and timescale) over which a proton with a kinetic energy of 60 keV or 100 MeV stops in a ISM with density of 1 cm^{-3}.

B. Bremsstrahlung

The previously described process is an (quasi)elastic scattering between a charged energetic particle and electrons in the medium. But the energetic particle can also *radiate* a photon under the braking action of the nuclear electric fields. Its calculation requires a spectral decomposition of the Larmor formula, plus kinematical considerations similar to the ones previously developed, and will not be reported here. It suffices to know that this is the dominant X-ray emission process from clusters of galaxies.

A few heuristic considerations suffice: Since there is an extra photon leg attached (see Fig. 12) compared to the previous process, we can quickly estimate that it should be suppressed by a factor α, i.e. its cross section to scale as

$$\sigma_{\rm Brems} \sim \alpha Z^2 \sigma_T \,, \tag{128}$$

For relativistic electrons, it can also be shown that the energy loss rate scales linearly with energy, i.e.

$$-\frac{{\rm d}E}{{\rm d}t} \simeq \sigma_{\rm Brems}\, n\, E\,. \tag{129}$$

VII. HADRONIC INTERACTIONS

For a particle of mass $m = A\,m_p$ and charge $Z\,e$, the Thomson cross-section σ_T scales as $\sigma_{\rm e.m.}(m,Z) = Z^4 A^{-2}(m_e/m_p)^2\sigma_T \simeq 3\times 10^{-7}(Z^4/A^2)\sigma_T$, which immediately shows how much more inefficient the direct scattering photon-nucleus is compared to the photon-electron.

However, nucleons and nuclei are subject to the strong nuclear force, and the associated inelastic processes are much more important for energy losses, in collisions with matter but also with radiation (if above appropriate thresholds). One useful way to think intuitively about nuclear interactions is the good old Yukawa's idea, of an exchange of massive mediators (the pions), responsible for the relatively short-range of the interaction. Note that the numerous particle physics discoveries in the cosmic rays between the thirties and the fifties of the XX century (the muon, the pion, the strange hadrons. . .) are due to hadronic interactions of primary protons or nuclei in the atmosphere.

A. Generalities and spallation

The key property of reactions involving nucleon-nucleon, or more in general nucleus-nucleus, is that they are strong, but short range. Also, nucleons and nuclei, differently from leptons and photons, are not 'pointlike', but extended objects of radius approximately given by

$$R \simeq 1.2\, A^{1/3}\,{\rm fm}\,. \tag{130}$$

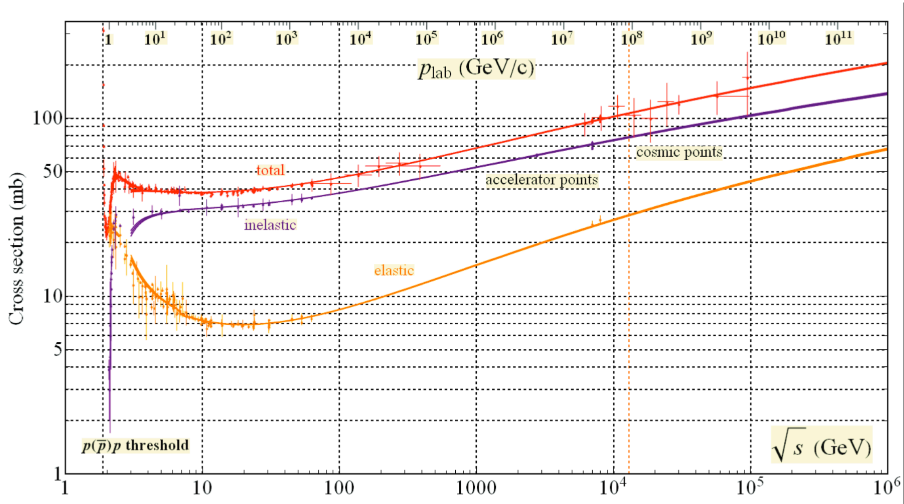

FIG. 13: Elastic and inelastic cross-section for pp scattering, from the PDG.

Together, these properties suggest a total cross-section value of the order of

$$\sigma_{\rm tot} \sim \pi R^2 \sim 45\, A^{2/3}{\rm mb}\,, \tag{131}$$

which is a good rule of thumb (see Fig. 13). At a more microscopic level, this can be thought to arise from (possibly multiple) exchanges of pions with an interactions coupling with "order 1" coupling, so that the nucleon nucleon cross-section is also of the order of $1/m_\pi^2$. At nucleon-nucleon level, inelastic processes are typically characterized by the emission of one or more pions (mostly in the forward direction). In a nuclear collision at GeV energies or above, however, the de Broglie wavelength $1/p \simeq 1/(\gamma m)$ is well below the nucleon size, so the nucleus is resolved in its constituents, and the nucleon typically strikes only one or a few nucleons (for multiple scatterings, more frequent for heavier nuclei). They are kicked out of the nucleus, in a process known as "spallation", with cross-sections which are only weakly dependent on energy, and that are empirically measured and tabulated (see Fig. 14.) In this "superposition model", most of the target nucleus remains essentially unaffected. This is confirmed by the fact that the *energy per nucleon* is approximately conserved, in such processes (this also explains the usefulness of plotting nuclear fluxes in terms of this variable E/A, notably at low-energies.) Note how partial x-sec are a fraction of the estimate of Eq. (131), and typically the byproducts differing from the parent by a single nucleon or two are the preferred final states (e.g. ^{11}C and ^{11}B from ^{12}C, ^{14}N, ^{15}N from ^{16}O, ...) These are the most relevant catastrophic interactions for nuclei propagating in a "matter-rich" environment (as the Galactic disk, as opposed to the extragalactic space).

It is worth mentioning that quite often the projectile or target nucleus can end up in an unstable state and de-excite e.g. by gamma-ray emission, at typical energies of few MeV for nuclei at rest. Although this has long been recognized as a potential exquisite diagnostic tool for the study of energetic phenomena in the interstellar medium, the observational challenges in MeV gamma-ray astronomy make this field still in its infancy.

B. Adiabatic energy losses

An ultrarelativistic particle propagating over distances undergoing the "cosmological" stretching (Hubble expansion wih rate $H(z)$) sees its wavelength stretched similarly. Hence

$$-\frac{1}{E}\frac{dE}{dt} = H(z) = H_0\sqrt{\Omega_\Lambda + \Omega_m(1+z)^3}\,. \tag{132}$$

Product nucleus	Z	A	Parent nucleus ^{11}B	^{12}C	^{14}N	^{16}O	^{20}Ne	^{24}Mg	^{28}Si	^{56}Fe
Lithium	3	6	12.9	12.6	12.6	12.6	12.6	12.6	12.6	17.4
		7	17.6	11.4	11.4	11.4	11.4	11.4	11.4	17.8
Beryllium	4	7	6.4	9.7	9.7	9.7	9.7	9.7	9.7	8.4
		9	7.1	4.3	4.3	4.3	4.3	4.3	4.3	5.8
		10	15.8	2.9	1.9	1.9	1.9	1.9	1.9	4.1
Boron	5	10	26.6	17.3	16.0	8.3	7.1	6.2	5.3	5.3
		11	—	31.5	15.0	13.9	12.0	10.4	9.0	8.1
Carbon	6	10	—	3.9	3.3	2.9	2.1	1.6	1.2	0.5
		11	0.6	26.9	12.4	10.6	7.9	5.9	4.5	1.3
		12	—	—	38.1	32.7	13.5	10.1	7.6	4.7
		13	—	—	10.5	14.4	10.7	8.0	6.0	3.7
		14	—	—	—	2.3	3.9	3.0	2.2	2.1
Nitrogen	7	13	—	—	10.7	3.6	2.7	2.0	1.5	0.5
		14	—	—	—	26.3	10.9	8.1	6.1	2.9
		15	—	—	—	31.5	10.0	7.5	5.7	4.3
		16	—	—	—	—	3.4	2.6	1.9	1.6
Oxygen	8	14	—	—	—	3.4	2.5	1.9	1.4	0.3
		15	—	—	—	27.8	11.8	8.9	6.7	1.0
		16	—	—	—	—	27.0	13.5	10.2	3.9
		17	—	—	—	—	15.5	11.6	8.7	4.1
		18	—	—	—	—	4.5	4.7	3.5	2.6

FIG. 14: Spallation cross-sections, in mb, From [1].

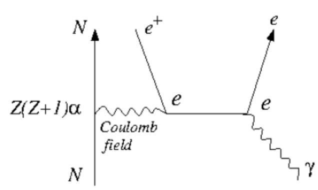

FIG. 15: Feynman diagram for the Bethe-Heitler process.

C. Bethe-Heitler process

It is the process $p + \gamma_{\rm CMB} \to p + e^+e^-$ (and, by extension, its analogous for nuclei). Its threshold is

$$m_p^2 + 2\epsilon_\gamma E_p(1-\cos\theta) > (m_p + 2m_e)^2 \Longrightarrow E_p \gtrsim \frac{m_e m_p}{\epsilon_\gamma} \simeq 2\times 10^{18}\,{\rm eV}\,. \tag{133}$$

Its inelasticity is however rather low, of the order of $2m_e/m_p \simeq 10^{-3}$ at threshold (where the energy repartition basically follows the relative mass carried by the product, with the ensemble considered as a "decaying" particle). In its Feynman diagram, shown in Fig. 15, there are three interactions vertices, so not surprising its cross-section is of the order α^3 (or, if you wish, parametrically suppressed by a factor α compared to the Thomson cross-section). A rough estimate of the cross-section is

$$\sigma_{\rm BH} \simeq \frac{\alpha^3}{m_e^2} Z^2 f(E,Z)\,, \tag{134}$$

with $f(E,Z)$ of order one. Some more detailed parameterization can be found e.g. in [15]. For particles propagating over cosmological distances, if this channel is open onto the CMB photons it is important in limiting the range below the one due to Eq. (132). At low redshift, the associated energy-loss length is shown in green Fig. 16, which we see becomes more

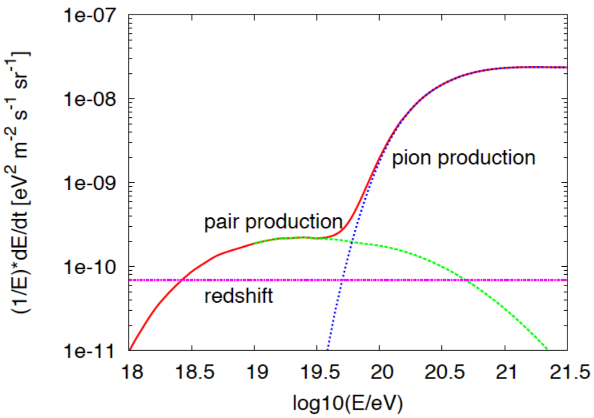

FIG. 16: Attenuation lengths at $z = 0$ for the main energy loss mechanisms for UHECR protons, from [16].

important than Eq. (132) above about $E \gtrsim 10^{18.5}$ eV. This mechanism also affects nuclei, for which it is comparatively more prominent since depending upon the nuclear charge as Z^2.

D. Nuclear photodisintegration

For nuclei propagating over extragalactic distances, the photodisintegration process $A+\gamma \to (A-1)+N+\gamma$ in the EBL (first) and CMB (at higher energies) is kinematically open. For typical nuclear binding energies of $\mathcal{O}(10)$ MeV, it is easy to check that the threshold lies at $E > 10^{19}$eV (just think that the threshold should be one order of magnitude higher than for the $e^{\pm}$ pair production for the Bethe-Heitler.) with details depending on the nucleus: It typically damps the propagation of light nuclei before heavier one, with the Fe case whose flux is only affected by this process closer to 10^{20} eV.

E. Inelastic pp collisions

This is the main energy-loss phenomenon affecting protons of Galactic Cosmic rays. If we specialize Eq. (81) to the case $p+p \to p+p(n)+\pi^0(\pi^+)$, neglecting the neutron-proton mass difference we get

$$2m_p^2 + 2E_p m_p > (2m_p + m_\pi)^2 \Longrightarrow 2m_p E_p > 2m_p^2 + m_\pi^2 + 4\,m_p m_\pi \Longrightarrow E_p > m_p + 2m_\pi + \frac{m_\pi^2}{2\,m_p} \simeq 1.2\,\mathrm{GeV}\,. \quad (135)$$

Exercise: Without neglecting proton neutron mass difference, estimate the minimum energy needed to produce: i) at least one positive pion; b) at least one negative pion in a pp reaction. *Hint :* impose baryon number, lepton number and electric charge conservation, and remember that free nucleons are not always the ground state of a multi-nucleon configuration!

F. $p\gamma$ collisions

If we specialize Eq. (81) to the case $p+\gamma \to p+\pi$, we get

$$m_p^2 + 2E_\gamma m_p > (m_p + m_\pi)^2 \Longrightarrow E_\gamma > m_\pi + \frac{m_\pi^2}{2\,m_p} \simeq 145\,\mathrm{MeV}\,, \quad (136)$$

where the numerical estimate is for π^0. If the proton is instead relativistic, at threshold (heads-on)

$$m_p^2 + 4\epsilon_\gamma E_p > (m_p + m_\pi)^2 \Longrightarrow E_p > \frac{2m_p m_\pi + m_\pi^2}{\epsilon_\gamma} \simeq 4\times 10^{19}\,\mathrm{eV}\,, \quad (137)$$

for CMB photons. The inelasticity of the order of $m_\pi/m_p \sim 15\%$ (actually a bit higher). The cross-section for this process, shown in Fig. 17, is dominated by the resonant production of a spin-3/2 and isospin 3/2 Δ^+ particle (of mass 1.232 MeV) just above threshold. Multi-particle processes dominate at much higher energy.

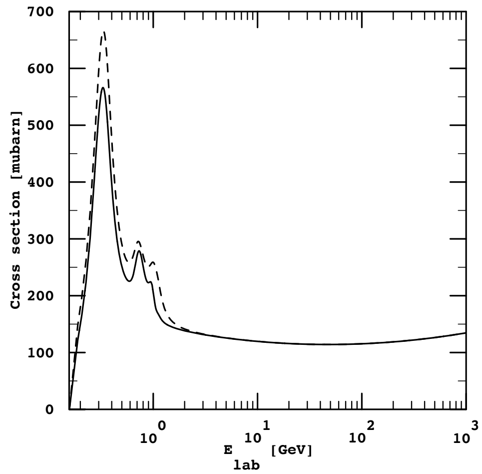

FIG. 17: The total photo-pion production cross section for protons (solid line) and neutrons (dashed line) as a function of the photon energy in the nucleon rest frame, $E_{\rm lab}$.

This process onto CMB photons is the most important process limiting the propagation of extragalactic protons, so dramatic to be known as *Greisen-Zatsepin-Kuzmin limit or cutoff* [17, 18] .

Exercise: Based on the cross-section value of Fig. 17, compute the mean free path of a proton propagating in the CMB, and compare with Fig. 16. Differently from the BH process, here the inelasticity is quite high, hence losses are "catastrophic" (even more so, given the steep CR spectrum).

VIII. SPECTRA OF PION DECAY BYPRODUCTS

A. Gamma spectra emitted in neutral pion decays

Let us consider the neutral pion decay process, $\pi^0\to\gamma\gamma$: The photons are back-to-back in the π^0 frame (momentum conservation), each carrying $E_\gamma=m_\pi/2\simeq 65.7\,$MeV (energy conservation). If the pion moves at β, in the Lab frame the energy of the photon emitted is $m_\pi\gamma(1+\beta\cos\theta)/2$, θ being the angle of the emitted photons with respect to the the direction of flight of the pion. Hence, the maximal and minimal energy of the photons is

$$E^{\max}_{\min}=\frac{m_\pi}{2}\gamma(1\pm\beta)\,, \tag{138}$$

i.e. from 0 to E_π in the ultra-relativistic limit. Also note that there is a one-to-one correspondence between photon energy and emission angle, whose law is

$$\mathrm{d}E=\frac{m_\pi}{2}\gamma\beta\mathrm{d}\cos\theta\,. \tag{139}$$

Since the pion is a scalar particle, its decay products are emitted isotropically, i.e. (just normalizing to one)

$$\frac{\mathrm{d}N}{\mathrm{d}\Omega}=\frac{1}{4\pi}\Longrightarrow \mathrm{d}N=\frac{1}{2}\mathrm{d}\cos\theta\,, \tag{140}$$

hence the energy distribution of the photons is

$$\frac{\mathrm{d}N}{\mathrm{d}E}=\frac{1}{m_\pi\gamma\beta}=\frac{1}{E_\pi\beta}=\frac{1}{\sqrt{E_\pi^2-m_\pi^2}}\,, \tag{141}$$

i.e. flat in energy space, between $E_{\rm min}\simeq 0$ and $E^{\rm max}\simeq E_\pi$ in the relativistic limit. Basically, one expects a box spectrum, symmetric around $m_\pi/2$. In log-Energy space,

$$\frac{1}{2}(\log E^{\min}+\log E^{\max})=\log\sqrt{E^{\min}E^{\max}}=\log\left(\frac{m_\pi}{2}\sqrt{\gamma^2(1-\beta^2)}\right)=\log\left(\frac{m_\pi}{2}\right)\,, \tag{142}$$

the center of the interval is half the pion mass, independently of the pion energy distribution, hence of the parent nucleon distribution. This is dubbed "pion bump" and considered in principle the cleanest signature of hadronic origin of a gamma-ray spectrum. (In practice, very often one has only access to the high-energy tail of the spectrum!)

A useful approximation for the spectrum of pions from hadronic interaction is the delta approximation, where a fixed inelasticity is considered,

$$E_\pi \simeq \kappa_\pi E_p \,. \tag{143}$$

Then (assuming for simplicity a pure hydrogen target and relativistic projectiles)

$$q_\pi(E_\pi) = n_H \int \mathrm{d}E_p \delta(E_\pi - \kappa_\pi E_p)\sigma_{pp\to\pi}(E_p)\Phi(E_p) = \frac{n_H}{\kappa_\pi}\,\sigma_{pp\to\pi}\left(\frac{E_\pi}{\kappa_\pi}\right)\Phi\left(\frac{E_\pi}{\kappa_\pi}\right)\,, \tag{144}$$

where $\sigma_{pp\to\pi}(E_p)$ is the pion-production cross section, rising quickly above threshold to $\mathcal{O}(100)$ mb and then growing only slowly with energy (logarithmically), not unlikely the overall inelastic cross-section ($\delta(\ldots)\sigma_{pp\to\pi}$ is a "simple" way to write the differential cross section that a proton of energy E_π produces a pion of energy E_π. For more advanced formulae, see e.g. [19] or [20])

To a leading order, the pion source spectrum at E_π is thus proportional to the proton flux at an energy $E_p = E_\pi/\kappa_\pi$ (with sizable corrections due to the x-sec energy dependence relevant close to threshold). The photon spectrum is then trivially obtained as

$$q_\gamma(E_\gamma) = 2\int_{E_\pi^{\rm min}(E_\gamma)}^{\infty} \mathrm{d}E_\pi \frac{dN}{dE} q(E_\pi) = 2\int_{E_\gamma+\frac{m_\pi^2}{4E_\gamma}}^{\infty} \mathrm{d}E_\pi \frac{q(E_\pi)}{\sqrt{E_\pi^2 - m_\pi^2}}\,, \tag{145}$$

where the minimum energy of the pion to produce a photon of energy E_γ, $E_\pi^{\rm min}(E_\gamma)$, is given by the relation linking the maximal energy of a photon produced by a pion of E_π. Note that the last step follows from the properties

$$E_\gamma^{\rm max}E_\gamma^{\rm min} = \frac{m_\pi^2}{4}\,,\quad E_\gamma^{\rm max} + E_\gamma^{\rm min} = E_\pi\,. \tag{146}$$

For a given energy E_π, the maximum energy of the photons is

$$E_\gamma^{\rm max} = E_\pi - E_\gamma^{\rm min} = E_\pi - \frac{m_\pi^2}{4E_\gamma^{\rm max}}\,, \tag{147}$$

hence solving for E_π by inversion it follows

$$E_\pi^{\rm min}(E_\gamma) = E_\gamma + \frac{m_\pi^2}{4E_\gamma}\,. \tag{148}$$

Optional Exercise: Compute the *shape* of $q_\gamma(E_\gamma)$ for some test pion source term, such as a power-law, a gaussian, etc. For a more advanced/realistic application, you can also use the fitting formula of Eq. (1) in [21] and the approximation $q_\pi(E_\pi) \propto \Phi_p\left(\frac{E_\pi}{\kappa_\pi}\right)$. Plot $q_\gamma(E_\gamma)$ in linear scale and $E_\gamma^2 q_\gamma(E_\gamma)$ in log-log one. Compare with [21].

B. Neutrino spectra emitted in charged pion decays

For the $\pi \to \mu + \nu$, the spectrum is monochromatic in the pion rest frame, at an energy satisfying (neutrino assumed massless)

$$(p_\pi - p_\nu)^2 = p_\mu^2 \Longrightarrow E_\nu^* = \frac{m_\pi^2 - m_\mu^2}{2\,m_\pi} \simeq 29.8\,\mathrm{MeV}\,, \tag{149}$$

If the pion moves at β, in the Lab frame the energy of the neutrino emitted is $E_\nu = E_\nu^*\gamma(1+\beta\cos\theta)$, θ being the angle of the emitted neutrino with respect to the the direction of flight of the pion. Hence, the maximal energy of the neutrinos is

$$E_\nu^{\rm max} = \frac{m_\pi^2 - m_\mu^2}{2\,m_\pi}\gamma(1+\beta) \simeq \frac{m_\pi^2 - m_\mu^2}{2\,m_\pi}2\gamma = \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)E_\pi \equiv \lambda E_\pi \simeq 0.427 E_\pi\,, \tag{150}$$

where the last equalities hold for very relativistic pions, $\beta \simeq 1$. Similarly to what seen for photons, inverting the latter given the minimum pion energy yielding a given neutrino energy, hence

$$q_\nu(E_\nu) = \int_{E_\pi^{\min}(E_\nu)}^{\infty} \mathrm{d}E_\pi \frac{dN}{dE} q(E_\pi) = \int_{E_\nu/\lambda}^{\infty} \mathrm{d}E_\pi \frac{q(E_\pi)}{\lambda E_\pi}\,. \tag{151}$$

Note that Eq. (141) holds unchanged for neutrinos and, given that $\mathrm{d}E_\nu = E_\nu^* \gamma\beta \mathrm{d}\cos\theta$, the energy distribution of the neutrinos is

$$\frac{\mathrm{d}N}{\mathrm{d}E} = \frac{1}{2E_\nu^*\gamma\beta} \simeq \frac{1}{\lambda E_\pi}\,. \tag{152}$$

Note that for a power-law form of $q_\pi(E_\pi)$, the above expression indicates that the neutrino from pion decay has the same power-law.

Similar considerations (but a bit more involved) can be applied to the muon spectrum from the pion decay and, in turn, to its three body decay $\mu \to e+\nu\bar{\nu}$ to yield the complete neutrino spectrum. Note that the muons from charged pion decays are also the main parents of the so-called *secondary* electrons and positrons in CRs. In particular, until a few years ago, it was believed that most CR positrons in the GeV-TeV range should originate via this mechanism (ultimately coming from $pp \to \pi^+$ reactions and analogous nuclear processes), while now a sizable contribution of primary sources seems favoured.

It is worth noting that each process producing a π^0 is associated to a process producing a π^+ (that is the case of $p\gamma \to p\pi^0$, associated to $p\gamma \to n\pi^+$) or to both π^+ and π^-: For the pp process one is practically always well above threshold, multipion production is frequent, and isospin symmetry yields equal numbers of π^0, π^+, and π^-. For the pp process, one is often close to threshold and the number of π^- is much lower than the one of π^0 and π^+, which are instead comparable. Similarly, the energies of the two types of particles are similar, within a factor of 2: neutrinos carry a fraction between $1/4$ and $1/2$ of the parent pion, while photons exactly a fraction $1/2$ of it. All in all, the gamma-ray spectrum and the spectrum of neutrinos of supposedly hadronic origin turn out to be comparable *at the source*. More often, one has access to a "degraded" gamma spectrum, since photons do experience interactions. Even so, in terms of bolometric energy flux, they should be comparable. Although we shall not develop them in detail, these are the type of considerations that were used to estimate the size of neutrino telescopes needed for a first detection of high-energy neutrino sources (a km^3 of water or Ice). IceCube has proven these estimates to be roughly correct!

IX. THE DIFFUSION-*LOSS* EQUATION: INCLUDING COLLISIONAL EFFECTS

In the light of what we have seen, the previously derived Eq. (73) requires some final generalizations concerning loss terms (besides a source term Q, to be provided by some theory of acceleration and injection). The two types of terms missing are "continuous energy losses" (such as Bremsstrahlung on the gas) and catastrophic ones, such as spallations and pp inelastic processes.

A. Catastrophic losses

These are treated as "source and sink terms" at the RHS of the propagation equation, since the species "changes nature" (disappears and/or evolves discontinuously in energy space). In particular, you are familiar with the fact that a radioactive species suffers of a term $-n/\tau$ (with n phase space density) at the RHS, where τ is its decay lifetime. Such a term is obviously present for unstable nuclei, apart for the generalization that in the lab frame, the decay time is boosted by the gamma factor of the nucleus: If τ_0 is the proper decay time of the species there is a $-f/(\gamma(p)\tau_0)$ at the RHS. For a species subject to collisional *interactions* that make it disappear, $1/(\gamma\tau_0) \to \Gamma$, with the interaction rate defined already in Eq. (1), with n the gas target and the σ the cross-section for the specific process. Note how, differently from a decay, this term is in general space-dependent, and $\Gamma \to 0$ when $n \to 0$. On the other hand, these interaction processes do produce secondary particles, so that they are also associated to *source* terms for other species. We can symbolically write this "secondary" source term for a species α as $\sum_\beta \Gamma_{\beta\to\alpha} n_\beta$. For nuclei sourced via spallation, if the transport equation is written in terms of Energy/nucleon, then this expression is almost exact, and not only symbolic. However, one should be aware that the true expression requires a convolution over energies (and the differential cross sections) since the secondaries have a degraded energy distribution with respect to the primaries.

B. Continuous energy losses

Under this category fall ionisation and Coulomb losses (however only important at energies below ∼GeV). In addition, electrons and positrons interact with the ISM emitting bremsstrahlung (again, only important at $\sim$ few GeV energies), but also synchrotron radiation on the galactic magnetic fields and inverse Compton scattering on interstellar radiation fields, which are instead very important at tens of GeV or above. To deduce the form of such a term, let us consider the problem of particle distributions only subject to injection and energy-losses. Since particles do not disappear once injected, a *continuity equation in E space holds* (just compare with the continuity equation in fluid dynamics $\partial\rho/\partial t + \nabla(v\rho) = Q$)

$$\frac{\partial \mathcal{N}}{\partial t} + \frac{\partial}{\partial E}(\dot{E}\mathcal{N}) = Q(E)\,, \tag{153}$$

where $Q(E)$ is the injection term. If written in terms of phase space density $n \propto \mathcal{N}(E)/p^2$, it can be shown that the term analogous to $\partial(\dot{E}\mathcal{N})/\partial E$ writes

$$-\frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(\frac{dp}{dt}\right)_\ell n\right]. \tag{154}$$

C. The "complete" diffusion-loss equation and some benchmark solutions

For a coupled set of species α, we can thus write the master equations

$$\frac{\partial n_\alpha}{\partial t} - \frac{\partial}{\partial x_i}K_{ij}\frac{\partial n_\alpha}{\partial x_j} + u_i\frac{\partial n_\alpha}{\partial x_i} - \frac{1}{3}\frac{\partial u_i}{\partial x_i}\left(p\frac{\partial n_\alpha}{\partial p}\right) + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(\frac{dp}{dt}\right)_\ell n_\alpha\right] - \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 K_{pp}\frac{\partial n_\alpha}{\partial p}\right) = Q - \Gamma\, n + \sum_\beta n_\beta \Gamma_{\beta\to\alpha}\,, \tag{155}$$

where we have identified in color the terms introduced previously in this section. Note that each term in the above equation has the dimension of n/time. One can thus define "characteristic timescales" (analogous to Eq. (111)) which allow for a quick parametric assessment of the importance of each term. Public available codes exist that allow for a numerical (such as GALPROP, DRAGON) or semi-analytical (USINE) solution of the problem. It is important however to grasp the key features of these terms via some analytical, limiting solution, notably of the steady state problem.

Actually, the flux resulting from the solution of the previous equation cannot be directly compared to what measured at the Earth, since the ISM flux is further subject to the effect of the solar wind (*solar modulation*). This is a quite complex phenomenon that goes well beyond the topic of these lectures. For practical purposes, the simplest (and largely) successful model is the so-called *force-field* approximation, where the effects is parameterized in terms of an effective and universal (electric) potential $\Phi \sim \mathcal{O}(0.5)$ GV[22]. The flux at the top of the atmosphere (TOA)—now expressed as differential with respect to Energy, $\mathcal{N}_{\mathsf{TOA}}(E)$, such that $n(p)p^2dp \propto \mathcal{N}(E)dE$—is suppressed with respect to the interstellar flux $\mathcal{N}_{\mathsf{IS}}(E)$ by

$$\mathcal{N}_{\mathsf{TOA}}(E) = \frac{E^2 - m^2}{(E + |Z|e\Phi)^2 - m^2}\,\mathcal{N}_{\mathsf{IS}}(E + |Z|e\Phi)\,, \tag{156}$$

where $(Z\,e)$ is the charge and m the mass of the CR species.

1. *E-loss dominated propagation*

If continuous energy loss timescales are the shortest ones (or the only one of relevance, in "quasi-homogeneous" problems), the steady state equation approximates to

$$-\frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(\frac{dp}{dt}\right)_\ell n_\alpha\right] = Q \implies n(p) \propto -\frac{1}{p^2(dp/dt)_\ell}\int^p \mathrm{d}p' Q(p')\, p'^2\,, \tag{157}$$

which, for $Q \propto p^{-s}$ and $(dp/dt)_\ell \propto -p^\ell$, leads to

$$n(p) \propto p^{-s-\ell+1}\,. \tag{158}$$

Namely, the resulting spectrum is $\ell - 1$ softer than the injected one. It turns out that for CR leptons, the above situation is close to truth, with ionization and Coulomb Energy losses ($\ell \simeq 0$) dominating at low-energies (spectrum is harder than the source), eventually overcome by bremsstrahlung energy losses ($\ell \simeq 1$, spectrum matching the source one) and Compton-Synchrotron energy losses ($\ell \simeq 2$ steeper spectrum). Note also that the diffusion-continuous loss problem (at least for space-independent loss term) is reduced to the diffusion problem (and thus to Eq. (51)) via a variable transformation into a "pseudo-time", see for instance Sec. II in [23], where the adaptation of the Green's function method to a geometry with boundaries is also shown.

2. *The 2D and 1D approximation*

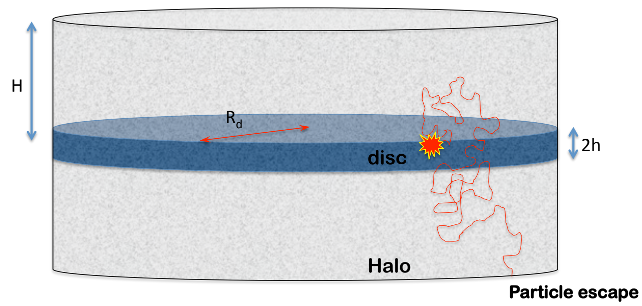

FIG. 18: Propagation setup. The thin disk of sources and interstellar gas of half-thickness h is contained within a cylindrical cosmic ray halo of half-height H and radius R_d, with a hierarchy of scales $R_g > H \gg h$. Within the halo cosmic rays diffuse in coordinate and momentum space, get convected, spallate and lose energy, depending on the details of the propagation model.

Usually, GCR propagation is considered to be limited to a cylindrical volume of radius R_d and half-height H, where the Galactic thin disk of matter of half-thickness h is contained in, see Fig. 18. Given the rough hierarchy $h : H : R_d \sim 0.1 : 5 : 20$ between the three scales, the gaseous disk of the Galaxy containing the most plausible sources is often considered infinitesimal, and sometimes a further simplification to an effective 1-D models and to a piece-wise constant u (or even a vanishing one) is adopted, allowing one to obtain quite transparent analytical solutions. In the 1D case, the Galaxy is considered as a (radially) "infinite" gas thin disk of uniform surface density, sandwiched in a thicker diffusive halo, only the vertical coordinate is relevant:

$$\frac{\partial n}{\partial t} - \frac{\partial}{\partial z}\left(K\frac{\partial n}{\partial z}\right) + u\frac{\partial n}{\partial z} - \frac{1}{3}\frac{\mathrm{d}u}{\mathrm{d}z}p\frac{\partial n}{\partial p} = Q - \Gamma n \,. \tag{159}$$

3. *"Leaky box" from slab model*

Let us assume that $u = 0$ and sources as well as catastrophic losses are confined to an infinitesimal disk. The steady state transport equation simplifies into

$$-\frac{\partial}{\partial z}\left(K\frac{\partial n}{\partial z}\right) = 2\, q_0(p)\, h\delta(z) - 2h\, \Gamma_\sigma\, n\, \delta(z) \;, \tag{160}$$

where $\Gamma_\sigma \equiv \sigma\, v\, n_{\rm ISM}$. Eq. (160) at $z \neq 0$ reduces to $\frac{\partial^2 n}{\partial z^2} = 0$, whose solution is $n(z,p) = a(p) + b(p)|z|$. If we denote with n_0 the solution in the plane $n_0 = n(0,p)$, the vanishing at the boundary $|z| = H$ gives

$$n(z,p) = n_0(p)(1 - |z|/H) \,. \tag{161}$$

An equation for $n_0(p)$ can be found by integrating Eq. (160) over a small interval around $z = 0$. One thus obtains

$$-2K(p)\left.\frac{\partial n}{\partial z}\right|_0 + 2\, h\, n_0 \Gamma_\sigma = 2\, q_0(p)\, h \,. \tag{162}$$

Using Eq. (161) one finds

$$n_0(p) = q_0(p)\tau_{\rm eff}(p)\,, \quad \text{where} \quad \tau_{\rm eff}^{-1}(p) = \tau_d^{-1}(p) + \Gamma_\sigma(p)\,. \tag{163}$$

and

$$\tau_d(p) \equiv \frac{H\,h}{K(p)} \approx 10^7\,\mathrm{yr}\frac{H}{3\,\mathrm{kpc}}\frac{h}{100\,\mathrm{pc}}\frac{10^{28}\,\mathrm{cm}^2\mathrm{s}^{-1}}{K}\,, \quad \tau_\sigma(p) \approx 10^7\,\mathrm{yr}\left(\frac{1\,\mathrm{cm}^{-3}}{n_{\mathrm{ISM}}}\right)\left(\frac{100\,\mathrm{mb}}{\sigma}\right). \tag{164}$$

The above solution *in the plane* is equivalent to the so-called "leaky box" model solution: The result of this model in the plane are equivalent to a homogeneous model where the diffusion operator is replaced by an effective confinement time, see Eq. (163). We also note that, as long as $K(p)$ is a growing function of p, the diffusive timescale τ_d dominates over collisional losses at sufficiently high energies. This remains true in more general models.

Exercise: Generalize the above derivation to solve Eq. (159) for the stationary solution in the case $u = \pm const. \neq 0$ ($+|u|$ above and $-|u|$ below the plane, respectively). You should find a closed-form for the altered z-profile, and a first order differential equation replacing the algebraic Eq. (163) to determine $n_0(p)$.

4. *Secondaries over primaries*

Let us apply the previous equation to the case of secondaries, i.e. nuclei only produced by spallation during propagation, such as the above-mentioned Boron, Lithium, and Beryllium. The distribution of secondaries in the plane, $n_S(p)$ is sourced by the injected nuclides per unit time, i.e. $q_0(p) \to n_P\Gamma_{P\to S}$, n_P being the primary population. Hence we obtain the solution for the ratio of primary to secondary distribution, assuming that the effective propagation time is species-independent

$$\frac{n_S(p)}{n_P(p)} \simeq \Gamma_{P\to S}\tau_{\mathrm{eff},P} \simeq \Gamma_{P\to S}\frac{H\,h}{K(p)}. \tag{165}$$

where the second relation holds if collisions are subdominant with respect to diffusion (which is not true at low energies!). Since, at least in principle, In principle, $\Gamma_{P\to S}\,h$ can be inferred by independent means, from this ratio (for instance, Boron-to-Carbon ratio in CRs) one can gauge the value of the "diffusive" ratio H/K, as well as of its energy dependence.

5. *Proxy for a dark matter production of a charged species, like antiprotons*

For a DM origin, the production (e.g. of antiprotons) is of a *primary* origin, and way more uniform, notably vertically, than the simple model for astrophysical secondaries considered above. A more appropriate (although geometrically not quantitatively correct!) toy model is

$$-\frac{\partial}{\partial z}\left(K\frac{\partial n}{\partial z}\right) = q_{\mathrm{DM}}(p) - 2h\,\Gamma_\sigma\, n\,\delta(z)\,. \tag{166}$$

I invite you to show as a simple **Exercise** that it follows

$$\frac{1}{\tau_{\mathrm{eff}}(p)}n_0 = \frac{H}{h}\,q(p)\,, \tag{167}$$

where $\tau_{\mathrm{eff}}(p)$ is the same defined in Eq. (163). Note that now the expected antiproton signal from DM, say, depends on one extra parameter, that we can take as H (This scaling is physical, because it's linear in the size of the volume from which one collects injected particles.) This means that antimatter signal at the Earth is much more uncertain!

This simple example suggests the following: imagine that the actual flux at the Earth is made of two contributions: astrophysical one, plus a DM one. Even if the respective source terms were exactly known (not true, think for instance of cross-section uncertainties), you have to account for propagation effects. And even if the "astrophysical" ones were under control (e.g. B/C ratio, etc.), the possibility to determine e.g. the ratio K/H from secondary to primary ratio observations as shown above (and there are errors associated to that!), the DM contribution is subject to further *astrophysical/propagation* uncertainties, which are more difficult to reduce, since they depend differently on the astrophysical parameters. Although one has some handles (such as radioactive CRs for the above example of H determination), this "propagation problem" is one of the main difficulties to keep in mind associated to these types of searches.

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