Cooperative effects and photon localization in atomic gases :

Phase transition in non Hermitian random matrices

Eric Akkermans
Technion

PRL **101**, 103602 (2008),
EPL **101,** (2013)
PR **A88**, (2013),
PR A **90**, 063822 (2014)

Benefitted from discussions and collaborations with:

Ari Gero, Technion
Robin Kaiser INLN, Nice
Louis Bellando, INLN, Nice

What is it about ?

- Coherent multiple scattering of photons/waves
- Anderson photon localization : phase transition and scaling

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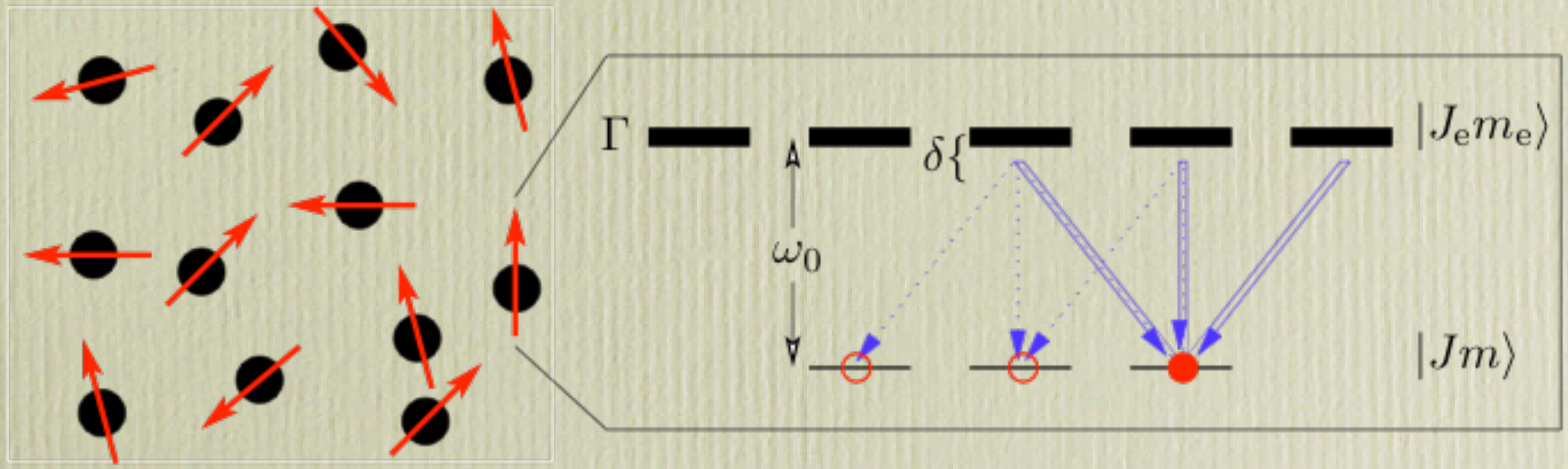
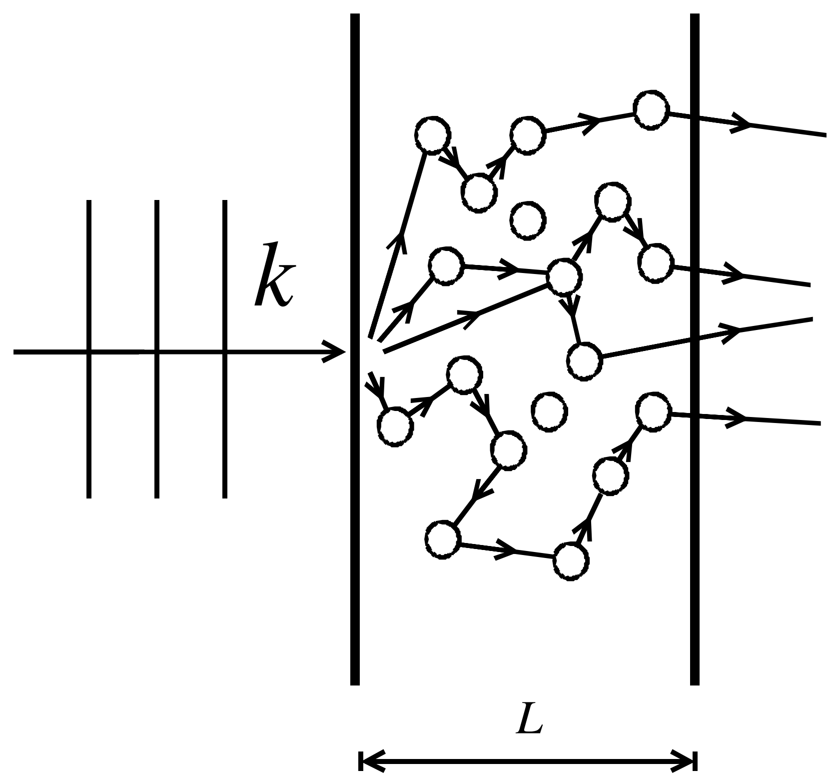
- Coherent multiple scattering of photons/waves
- Anderson photon localization : phase transition and scaling
- Cooperative effects and Dicke superradiance
- Photon escape rates : *Competition between Anderson and Dicke mechanisms*

Framework

Multiple scattering of photons/waves by a cold atomic gas.



Multiple scattering



2 characteristic lengths:

Wavelength: λ_0

Elastic mean free path: $l = \dfrac{1}{n_i \sigma} \gg n_i^{-1/d}$

density of scatterers

scattering cross section $\sigma \propto \lambda^2$

Disorder strength :

$$W = \frac{1}{k_0 l}$$

$$\lambda = {}^{2\pi}\!/_{k_0}$$

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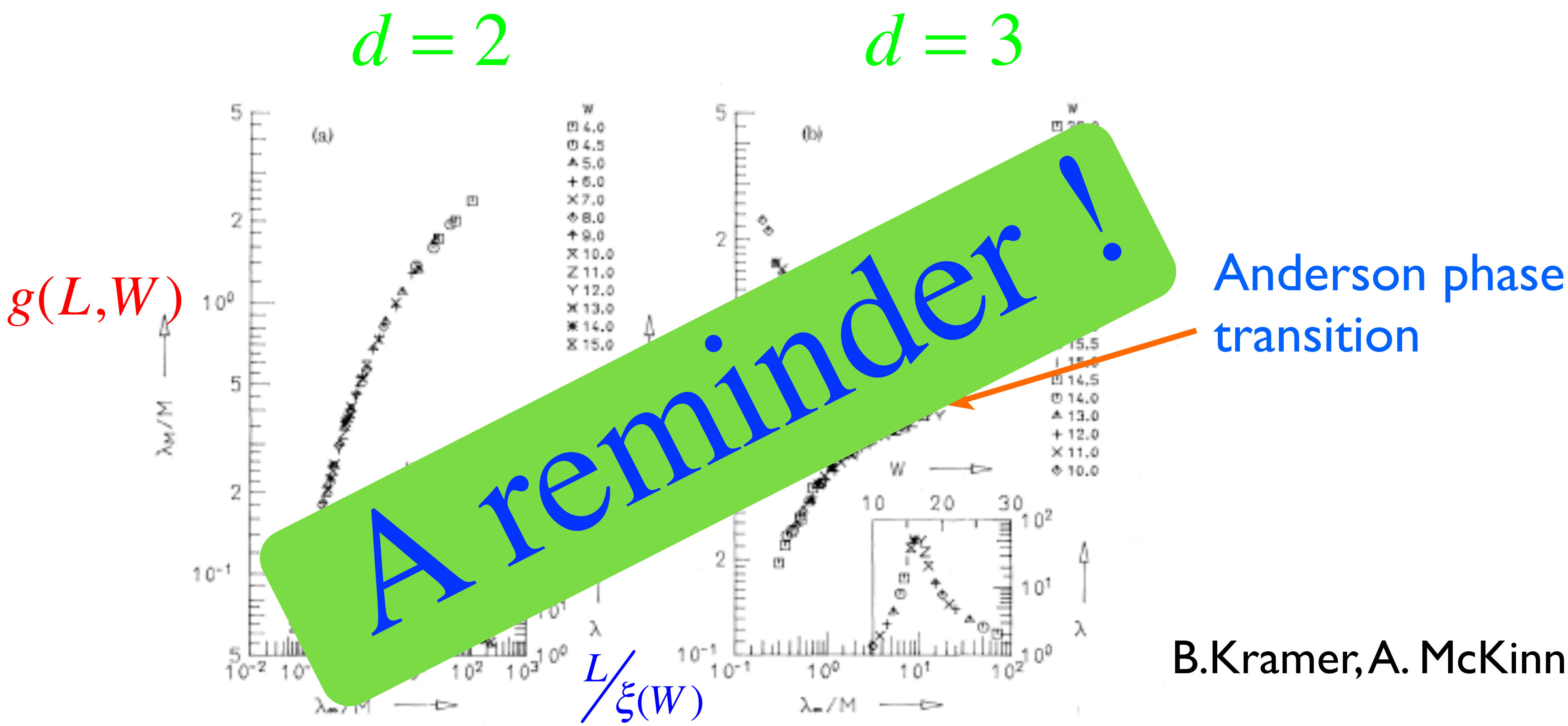
Weak disorder limit $W \ll 1$

Numerical calculations on the Anderson Hamiltonian



B.Kramer, A. McKinnon, 1981

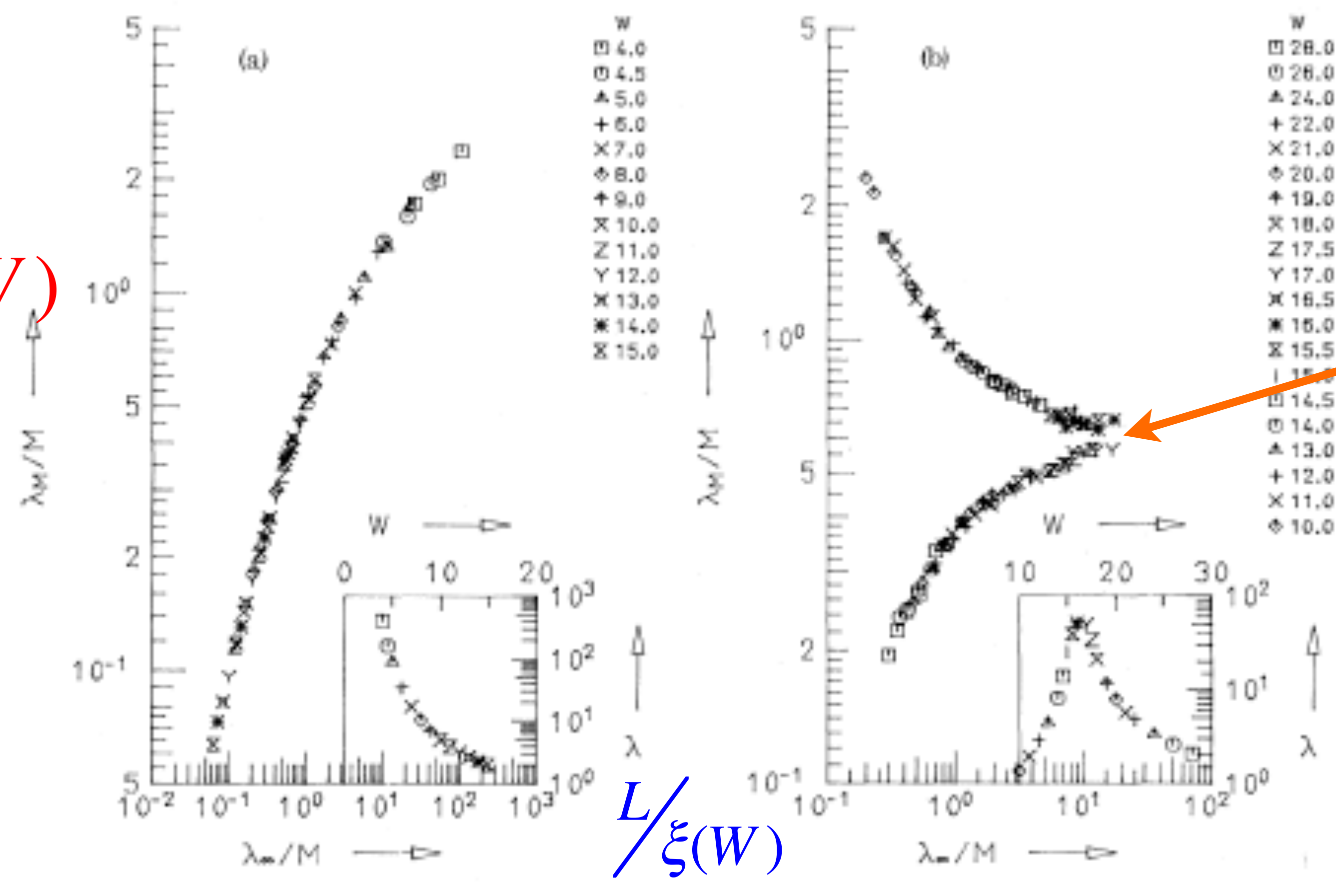
FIG. 1. Scaling function λ_M/M vs λ_∞/M for the localization length λ_M of a system of thickness M for (a) $d=2$ ($M \geq 4$) and (b) $d=3$ ($M \geq 3$). Insets show the scaling parameter λ_∞ as a function of the disorder W.

Anderson localisation phase transition occurs in $d > 2$

Numerical calculations on the Anderson Hamiltonian

$d = 2$ $\qquad$ $d = 3$

$g(L,W)$



$L/\xi(W)$

Anderson phase transition

B.Kramer, A. McKinnon, 1981

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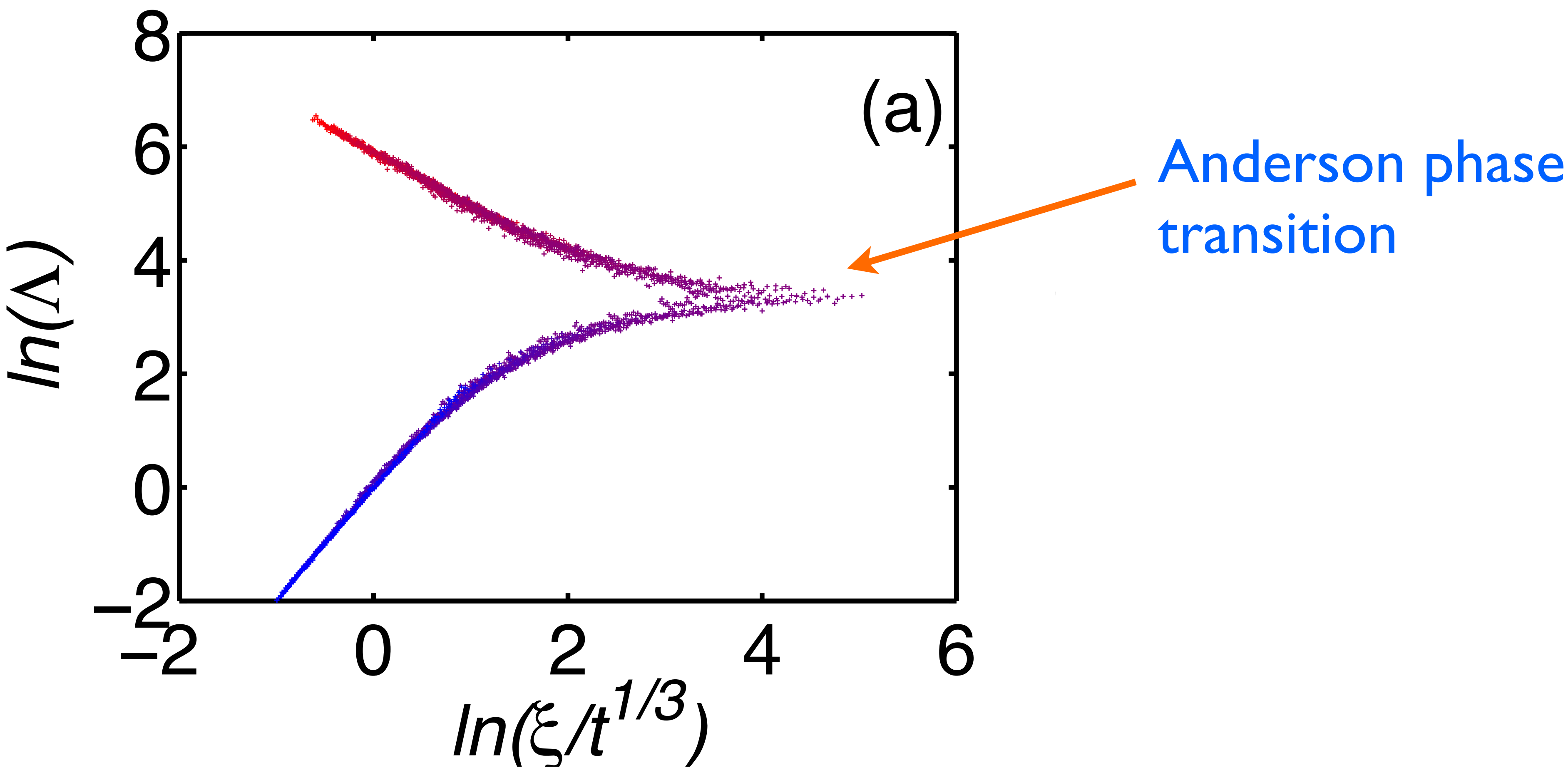
Realisations of the Anderson Hamiltonian

Quantum evolution of the atomic kicked rotor

(localisation of the momentum in phase space (d=3))

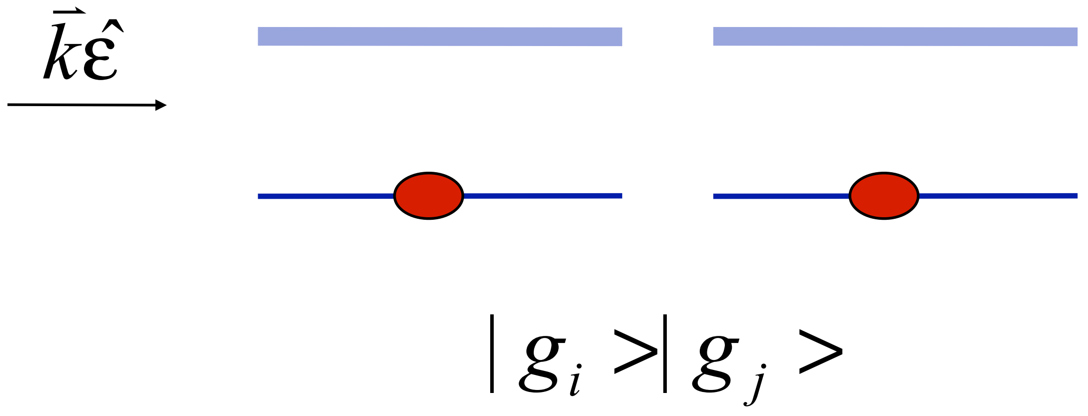


(P. Szriftgiser et al. 2010, for the experiment, theory : Casati, Chirikov, ('79) Fishman, Grempel, Prange, ('84), Guarneri et al. ('89),

Cooperative effects (superradiance-subradiance)

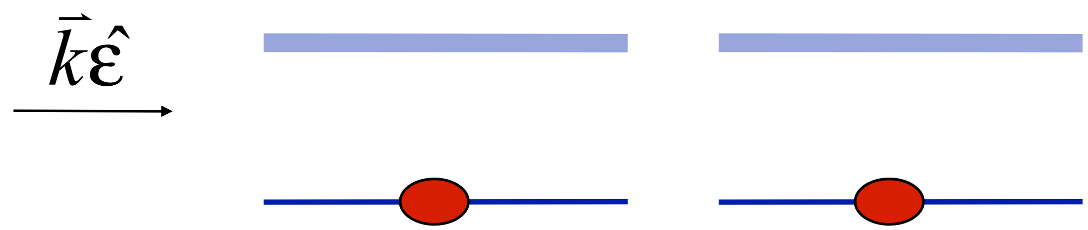
Cooperative spontaneous radiation (Superradiance) results from *quantum phase correlations* induced between atoms by dipole-dipole interactions.

Cooperative effects

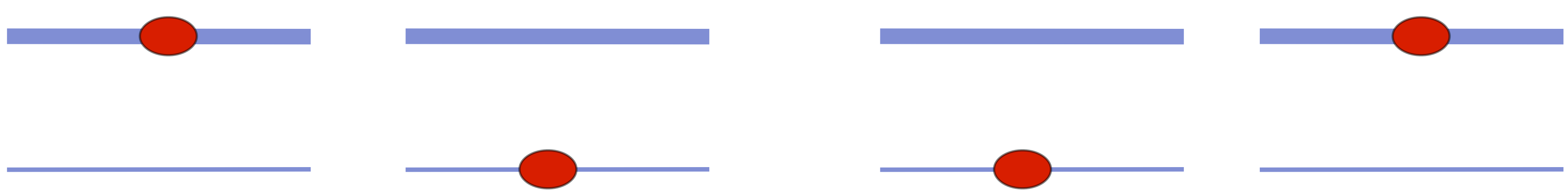


Cooperative effects

Dicke states



$$|g_i>|g_j>$$



$$|\pm>\equiv\frac{1}{\sqrt{2}}\left(|e_i>|g_j>\pm|g_i>|e_j>\right)$$

R. H. Dicke, Phys. Rev. **93**, 99 (1954)

Second order in perturbation theory in the coupling to photons

$$\varepsilon V_e(r) = -\varepsilon \frac{\hbar\Gamma}{2}\frac{\cos k_0 r}{k_0 r}$$

$$\Gamma^{(\varepsilon)} = \Gamma\left(1+\varepsilon\frac{\sin k_0 r}{k_0 r}\right)$$

Superradiance

$$V_e(r) \propto -\frac{1}{r}$$

(attractive at short distance)

Superradiant state $\varepsilon = +1$

$$\frac{1}{\sqrt{2}}\left[|e_1 g_2\rangle + |g_1 e_2\rangle\right]$$

$\Gamma^{(+1)} = 2\Gamma$ (for $r = 0$)

Characteristics of superradiance

Subradiant state $\varepsilon = -1$

$$\frac{1}{\sqrt{2}}\left[|e_1 g_2\rangle - |g_1 e_2\rangle\right]$$

$\Gamma^{(-1)} = 0$

Photon is trapped by the two atoms

A.Gero, E.A, Phys. Rev. Lett. **96**, 093601 (2006)

A.Gero, E.A. , Phys. Rev. A **75**, 053413 (2007)

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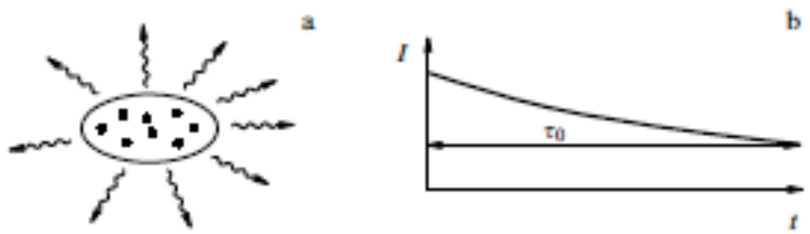

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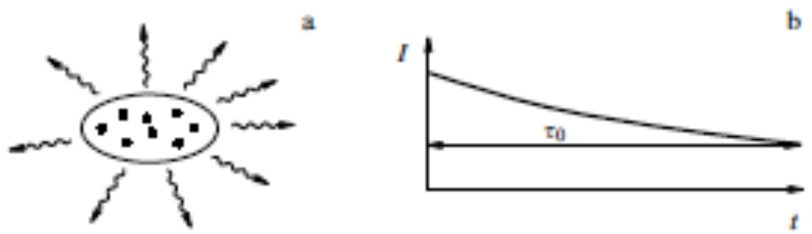

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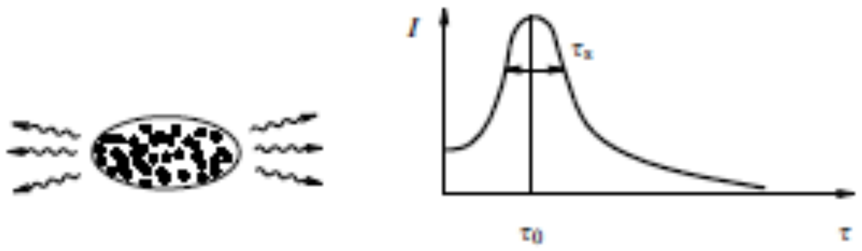

$$I \approx N^2$$

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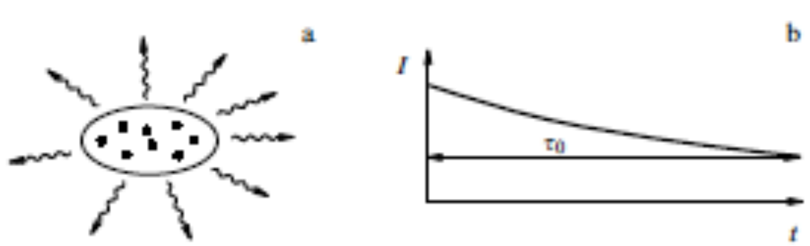

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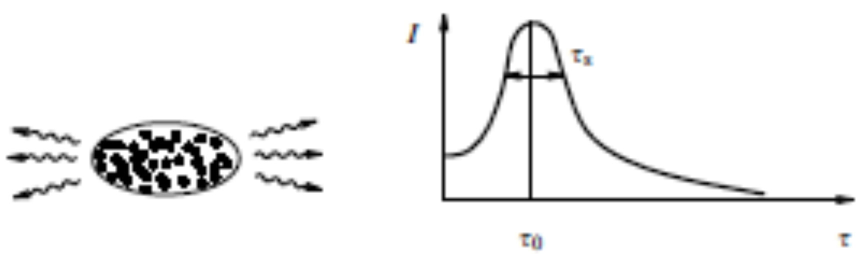

$$I \approx N^2$$

But the dependence $I \approx N^2$ *does not constitute the main distinguishing feature of superradiance*.

It is rather the mechanism leading to *coherent phasing of atoms*.

Superradiant emission : all atoms must see (in phase) the same

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small volumes (*Dicke limit*)

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- small volumes (*Dicke limit*)
- large systems : *Anderson localization* may play a role :

photon modes are spatially localized in volumes ξ^d

only a fraction $N\left(\xi/L\right)^d$ of atoms are coherent so that the emission time τ_S becomes large:

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$$\tau_s \approx \left(\frac{L}{\xi}\right)^d \frac{\ln N}{N} \gg \frac{\ln N}{N}$$

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Anderson localization and cooperative spontaneous emission are competing effects.

Model

N identical two-level atoms located at random positions $\vec{r}_i$ (uniform distribution) with electric dipole moments $\vec{d}_i$ in the quantum radiation field $\vec{E}$

- Total Hamiltonian

$$H = H_0 + U$$

- Non-interacting Hamiltonian

$$H_0 = \hbar\omega_0 \sum_{i=1}^{N} |e_i\rangle\langle e_i| + \sum_{\vec{k}\varepsilon} \hbar\omega_k a^{+}_{\vec{k}\varepsilon} a_{\vec{k}\varepsilon}$$

- Electric dipole representation of the interaction

$$U = -\sum_{i=1}^{N} \vec{d}_i \cdot \vec{E}(\vec{r}_i)$$

Model

- Effective Hamiltonian
  - Tracing over the EM field degrees of freedom

$$H_e = \left(\hbar\omega_0 - i\frac{\hbar\Gamma_0}{2}\right)\sum_{i=1}^{N} \left|e_i\right\rangle\left\langle e_i\right| + \frac{\hbar\Gamma_0}{2}\sum_{i\neq j} V_{ij}\Delta_i^+\Delta_j^-$$

  - Atomic raising and lowering operators

$$\Delta_i^+ = \left|e_i\right\rangle\left\langle g_i\right| \qquad \Delta_j^- = \left|g_j\right\rangle\left\langle e_j\right|$$

Model

- Effective Hamiltonian
  - Tracing over the EM field degrees of freedom

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  - Atomic raising and lowering operators

$V_{ij} = \beta_{ij} - i\gamma_{ij}$ is random and complex valued

- Real part : interaction potential

$$\beta_{ij} = \frac{3}{2}\left[-p\ \frac{\cos k_0 r_{ij}}{k_0 r_{ij}} + q\left(\frac{\cos k_0 r_{ij}}{(k_0 r_{ij})^3} + \frac{\sin k_0 r_{ij}}{(k_0 r_{ij})^2}\right)\right]$$

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- Imaginary part : photon escape rate

$$\gamma_{ij} = \frac{3}{2}\left[p \ \frac{\sin k_0 r_{ij}}{k_0 r_{ij}} - q \left(\frac{\sin k_0 r_{ij}}{(k_0 r_{ij})^3} - \frac{\cos k_0 r_{ij}}{(k_0 r_{ij})^2} \right) \right]$$

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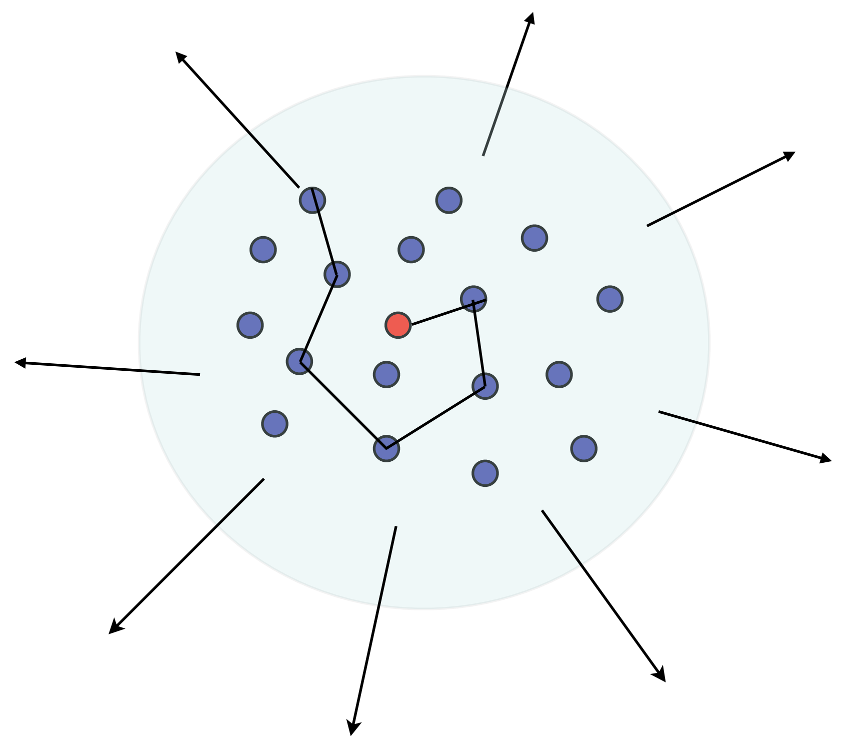
- For a scalar wave:

$$\beta_{ij} = -\frac{\cos\left(k_0 r_{ij}\right)}{k_0 r_{ij}}$$

$$\gamma_{ij} = \frac{\sin k_0 r_{ij}}{k_0 r_{ij}}$$

Which quantity to study ?

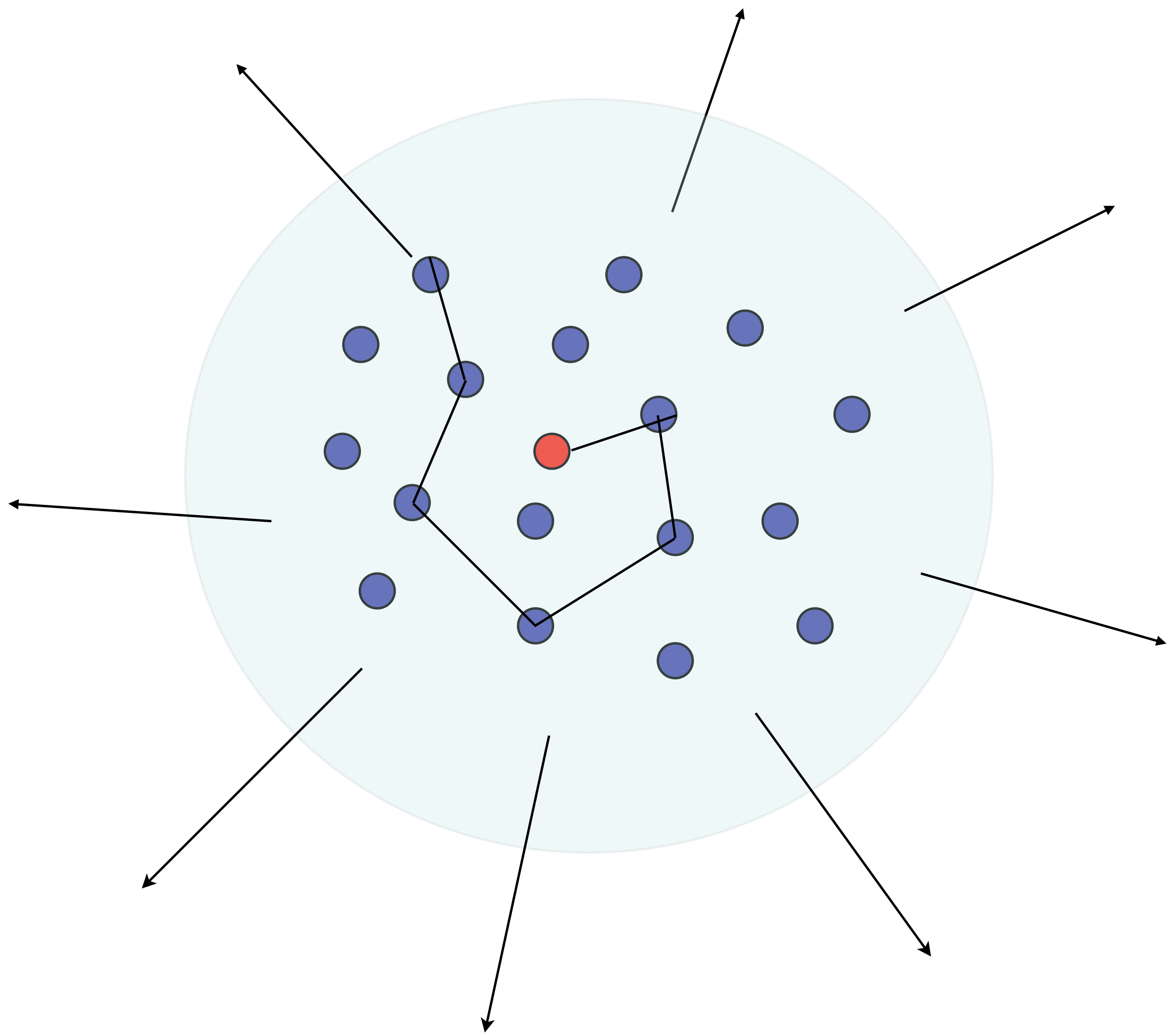
- The radiation pattern/intensity of the atomic cloud with a single excited atom



$$\Psi = \sum_{j=1}^{N} \beta_j(t)|b_1 b_2 \cdots a_j \cdots b_N\rangle|0\rangle + \sum_{\mathbf{k}} \gamma_{\mathbf{k}}(t)|b_1 b_2 \cdots b_N\rangle|1_{\mathbf{k}}\rangle.$$

Photon escape rates are a measure of localization and/or cooperative emission.

Escape rates are not a transport quantity.



More precisely : Photon escape rates

Evolution of the density matrix (Linblad form)

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}\left(H_e\rho - \rho H_e^\dagger\right) + \Gamma_0 \sum_{i \neq j} \gamma_{ij} \Delta_i^+ \rho \Delta_j^-$$

$$H_e = \left(\hbar\omega_0 - i\frac{\hbar\Gamma_0}{2}\right)\sum_{i=1}^{N} |e_i\rangle\langle e_i| + \frac{\hbar\Gamma_0}{2}\sum_{i \neq j} V_{ij}\Delta_i^+\Delta_j^-$$

$$V_{ij} = \beta_{ij} - i\gamma_{ij}$$

M. Stephen (1964), R.H. Lehmberg (1970), E. Ressayre and A.Tallet (1976), Ellinger, Cooper and P. Zoller (1994)

Photon escape rates from the atomic gas are obtained from the eigenvalues of the euclidean random matrix γ_{ij}

Eigenvalue density $P(\Gamma)$ of the $N \times N$ random matrix γ_{ij}

(Scalar case) $$\gamma_{ij} = \frac{\sin k_0 r_{ij}}{k_0 r_{ij}}$$ $$x_{ij} \equiv k_0 r_{ij}$$

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(Scalar case)

$$\gamma_{ij} = \frac{\sin k_0 r_{ij}}{k_0 r_{ij}} \qquad x_{ij} \equiv k_0 r_{ij}$$

Defining dimensionless quantities $L^d = (\lambda a)^d$ $\qquad \lambda = 2\pi / k_0$

$$\boxed{W = \frac{1}{k_0 l} = \frac{\pi}{2} \frac{\lambda}{L} \frac{N}{N_\perp}}$$

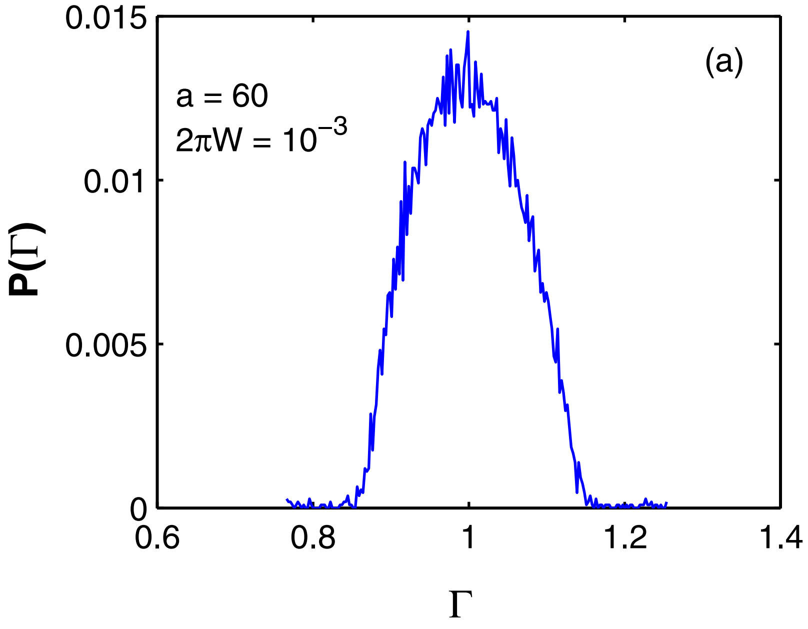
Elastic mean free path

$$l = \frac{1}{n\sigma} = \frac{L^3}{N\lambda^2}$$

Number of transverse channels $\qquad (d = 3)$

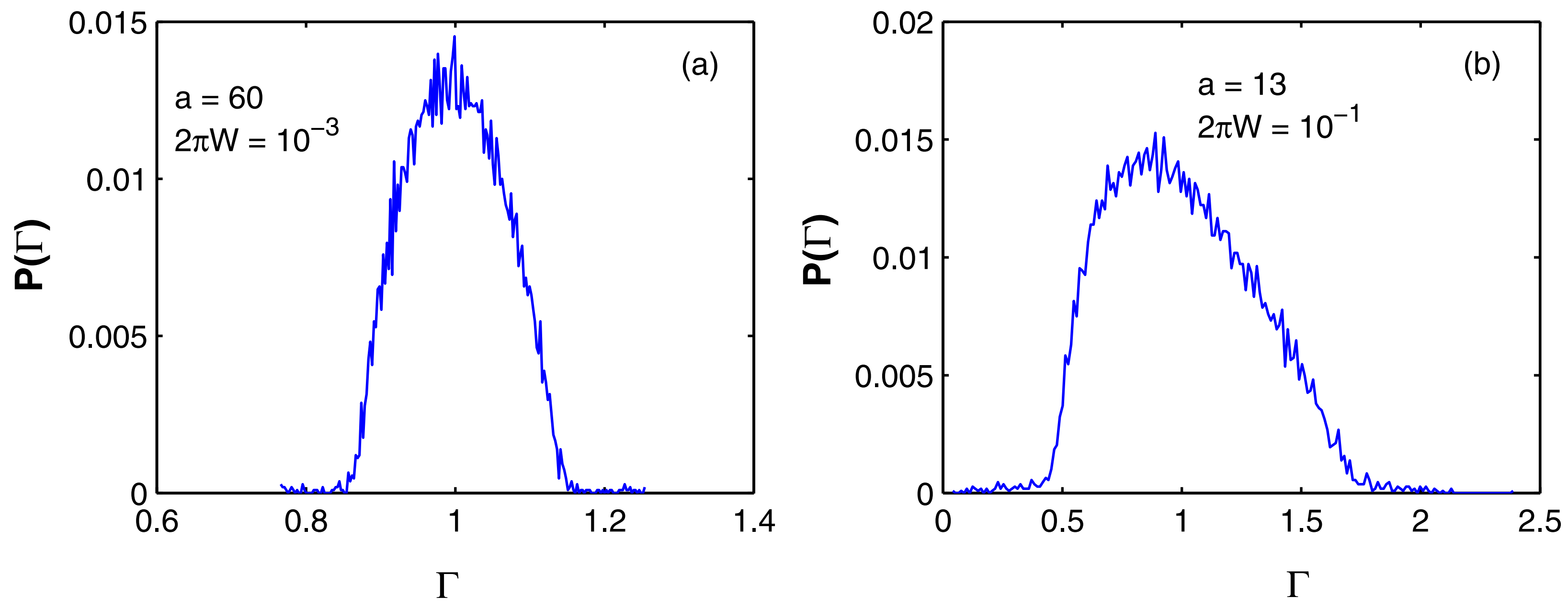
$$N_\perp = (k_0 L)^2 / 4$$

Eigenvalue density $P(\Gamma)$



Eigenvalue density $P(\Gamma)$

increasing disorder W



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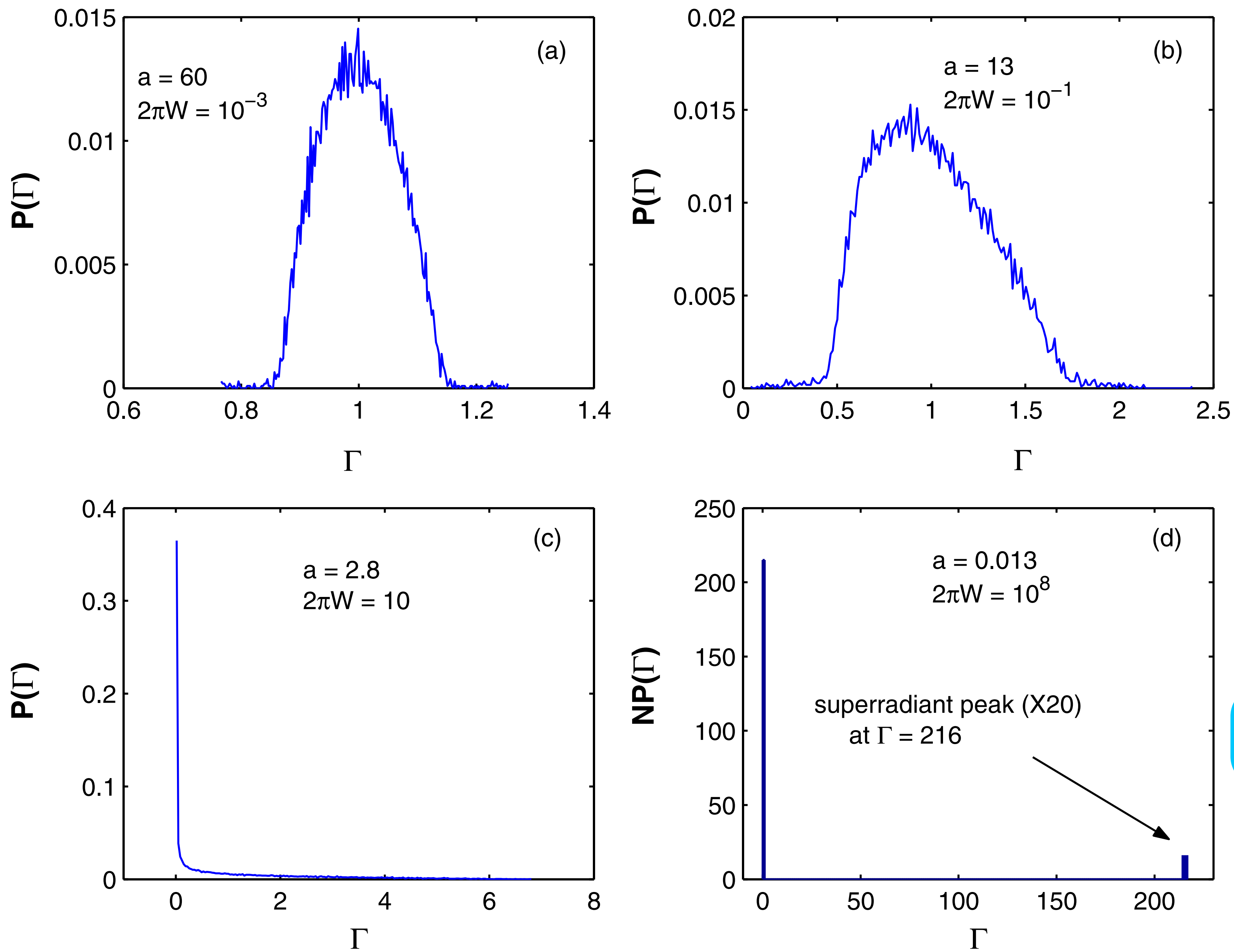

Dicke limit

localized photons

Numerical results $N=216$

A. Gero, R. Kaiser, E.A PRL **101**, 103602 (2008),

Scaling ?

To characterize $P(\Gamma)$ we look for a scaling function $C(a,W)$

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Relative number of localized states i.e. having a vanishing escape rate :

$$C(a,W) = 1 - 2\int_1^\infty d\Gamma \, P(\Gamma)$$

$C(a,W)$ is defined between 0 and 1.

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Relative number of localized states i.e. having a vanishing escape rate :

$$C(a,W) = 1 - 2\int_1^\infty d\Gamma\, P(\Gamma)$$

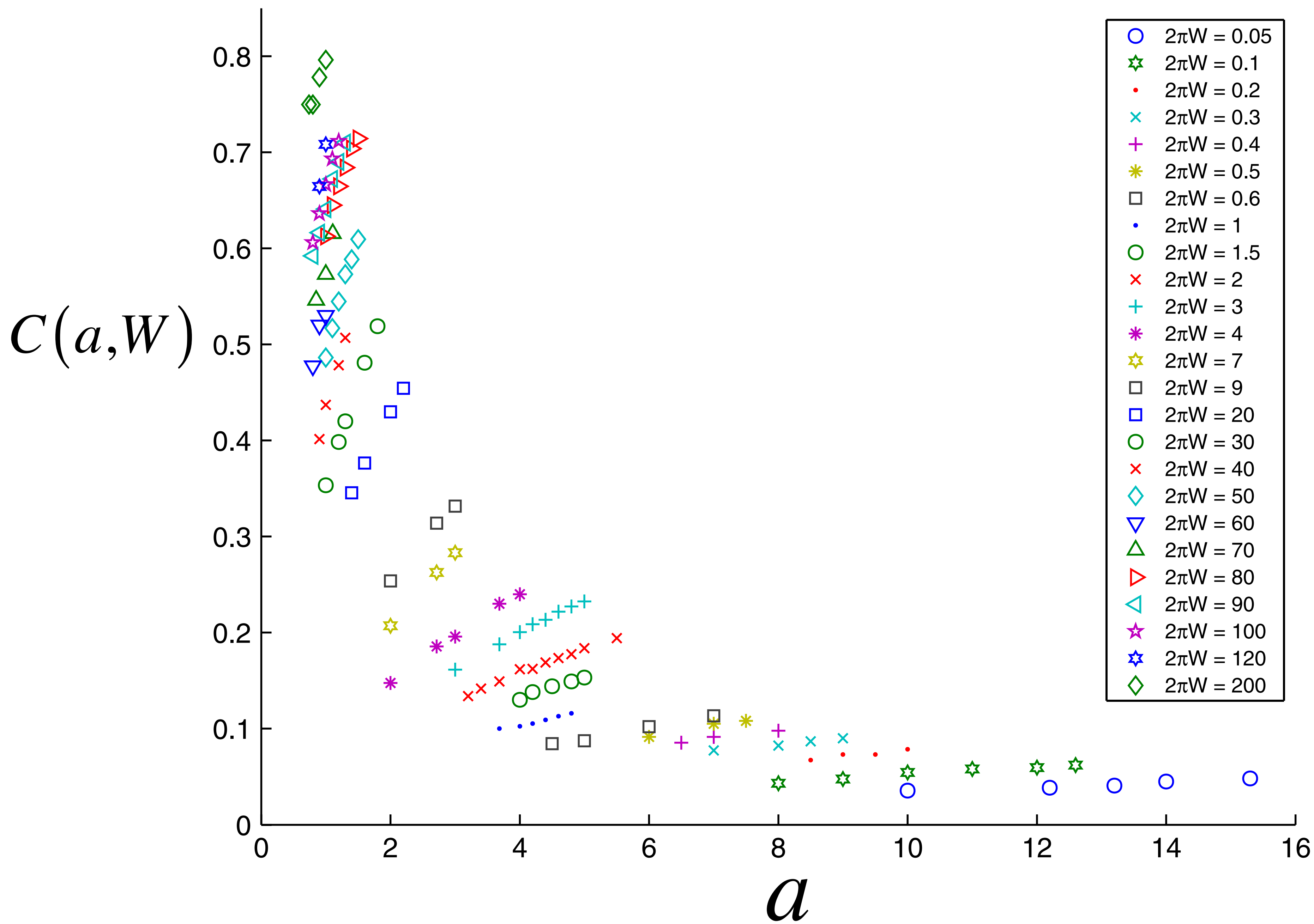
$C(a,W)$ is defined between 0 and 1. At finite size, we expect the scaling form:

$$C(a,W) = f\left(a / \xi(W) \right)$$

A. Gero, R. Kaiser, E.A PRL **101**, 103602 (2008),

Scaling behaviour

Large sample limit ($a \geq 1$)



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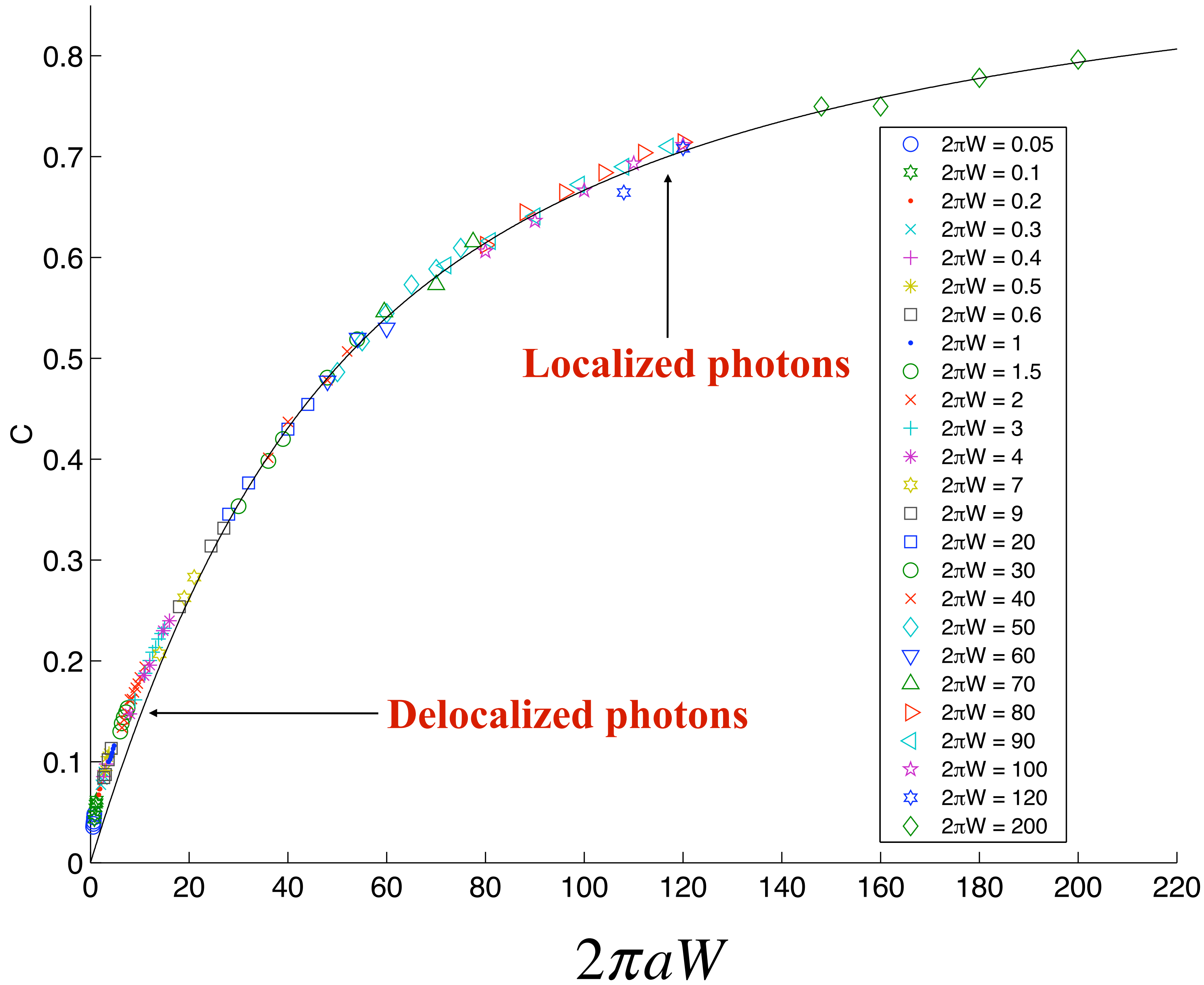
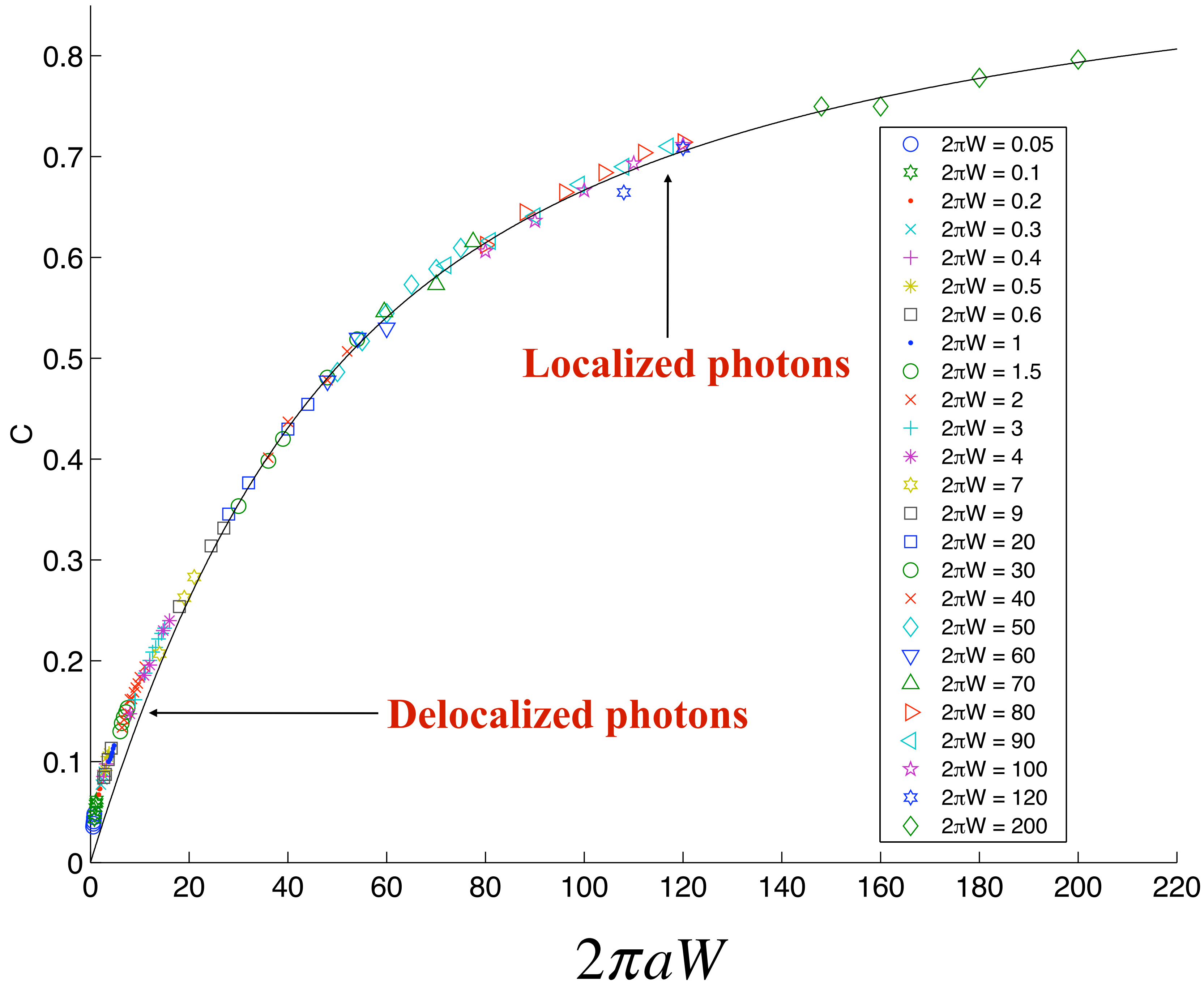
Scaling behavior *(large sample limit)*



$C(a,W)$ depends on $2\pi aW = \pi^2 N / N_\perp$

Scaling behavior *(large sample limit)*



Is there a localisation phase transition ?

$C(a,W)$ depends on $2\pi aW = \pi^2 N / N_\perp$

Is there a localisation phase transition ?

- Microscopic QED approach

  Large disorder limit $N \gg N_\perp$

- Phenomenological Markov process (Small world networks)

  For the whole range of disorder

Microscopic QED approach

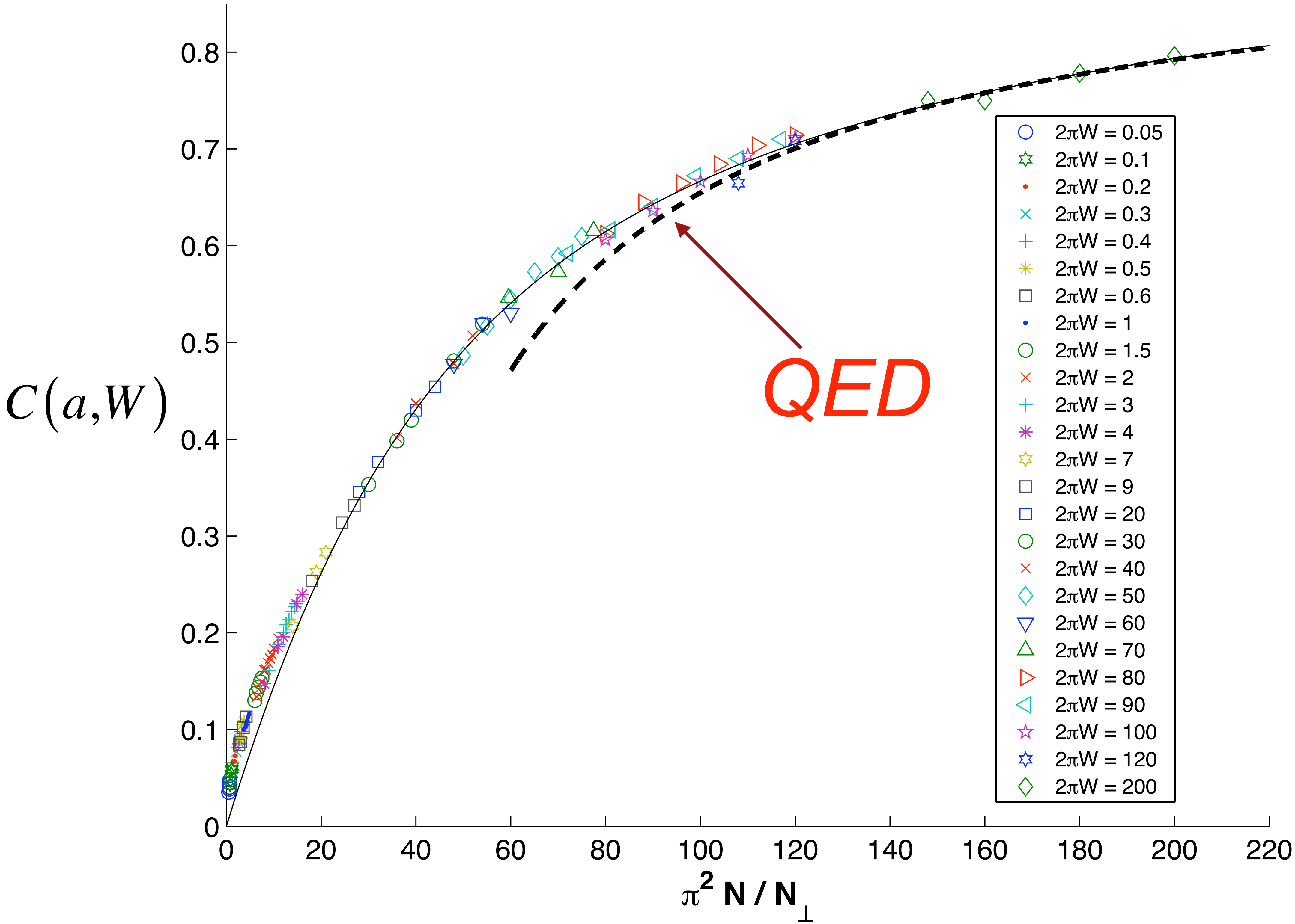
Large disorder limit $N \gg N_\perp$

Resummation of the cumulants of $P(\Gamma)$ leads to the asymptotic behavior

$$P(\Gamma) = \left(1 - \frac{3N_\perp}{2N}\right)\delta(\Gamma) + 3\Gamma\left(\frac{N_\perp}{N}\right)^3 \quad \textit{for} \;\; \Gamma \le {}^{N}\!/\!_{N_\perp}$$

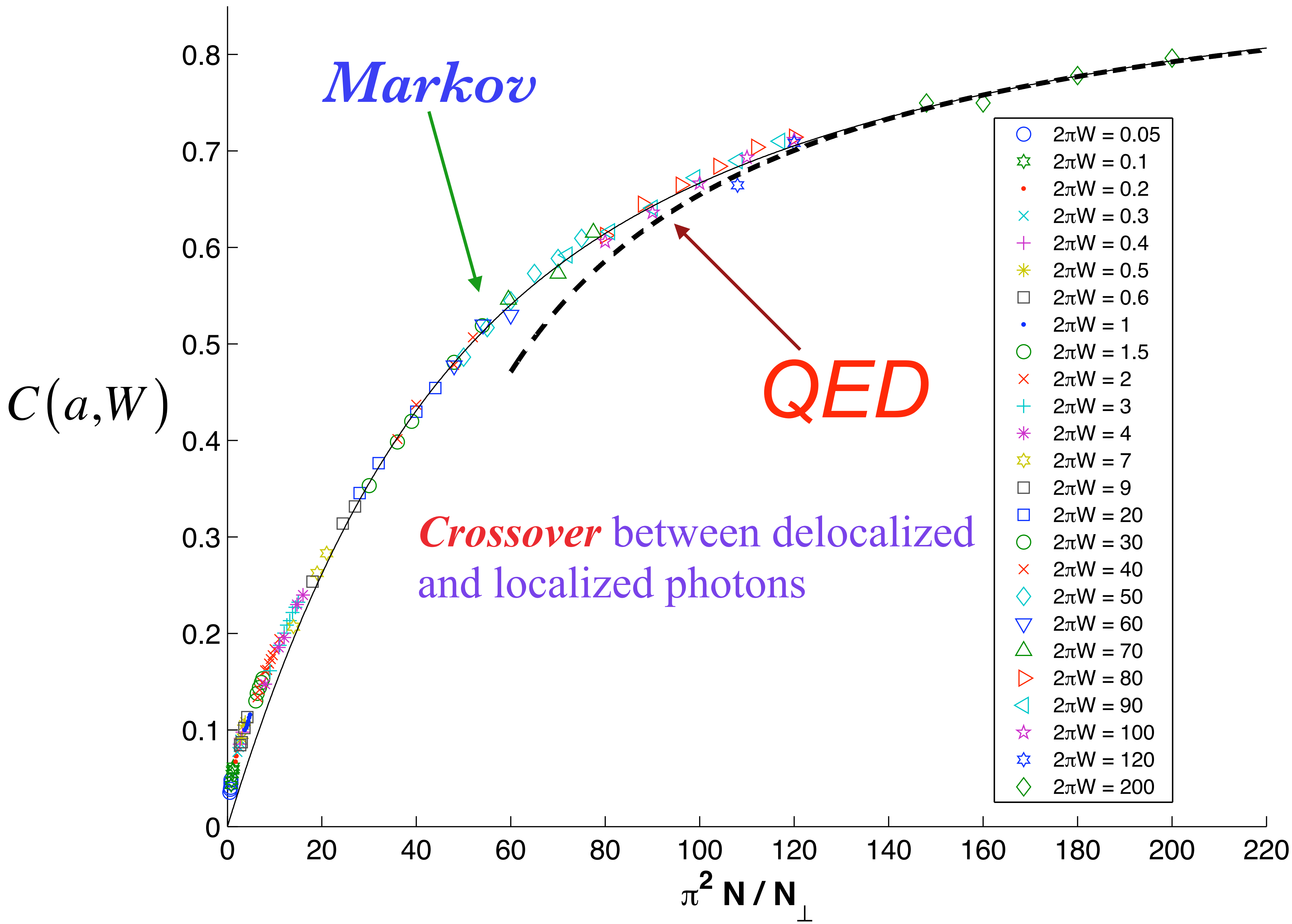
$$P(\Gamma) = 0 \quad \textit{otherwise}$$

so that $$\boxed{C\left({}^{N}\!/\!_{N_\perp}\right) = 1 - 3\frac{N_\perp}{N}}$$



C(a,W)
π² N / N⊥
QED
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
20
40
60
80
100
120
140
160
180
200
220
2πW = 0.05
2πW = 0.1
2πW = 0.2
2πW = 0.3
2πW = 0.4
2πW = 0.5
2πW = 0.6
2πW = 1
2πW = 1.5
2πW = 2
2πW = 3
2πW = 4
2πW = 7
2πW = 9
2πW = 20
2πW = 30
2πW = 40
2πW = 50
2πW = 60
2πW = 70
2πW = 80
2πW = 90
2πW = 100
2πW = 120
2πW = 200

Phenomenological Markov process
(Small world networks)



Markov
QED
Crossover between delocalized and localized photons
$C(a,W)$
$\pi^2 N / N_\perp$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
20
40
60
80
100
120
140
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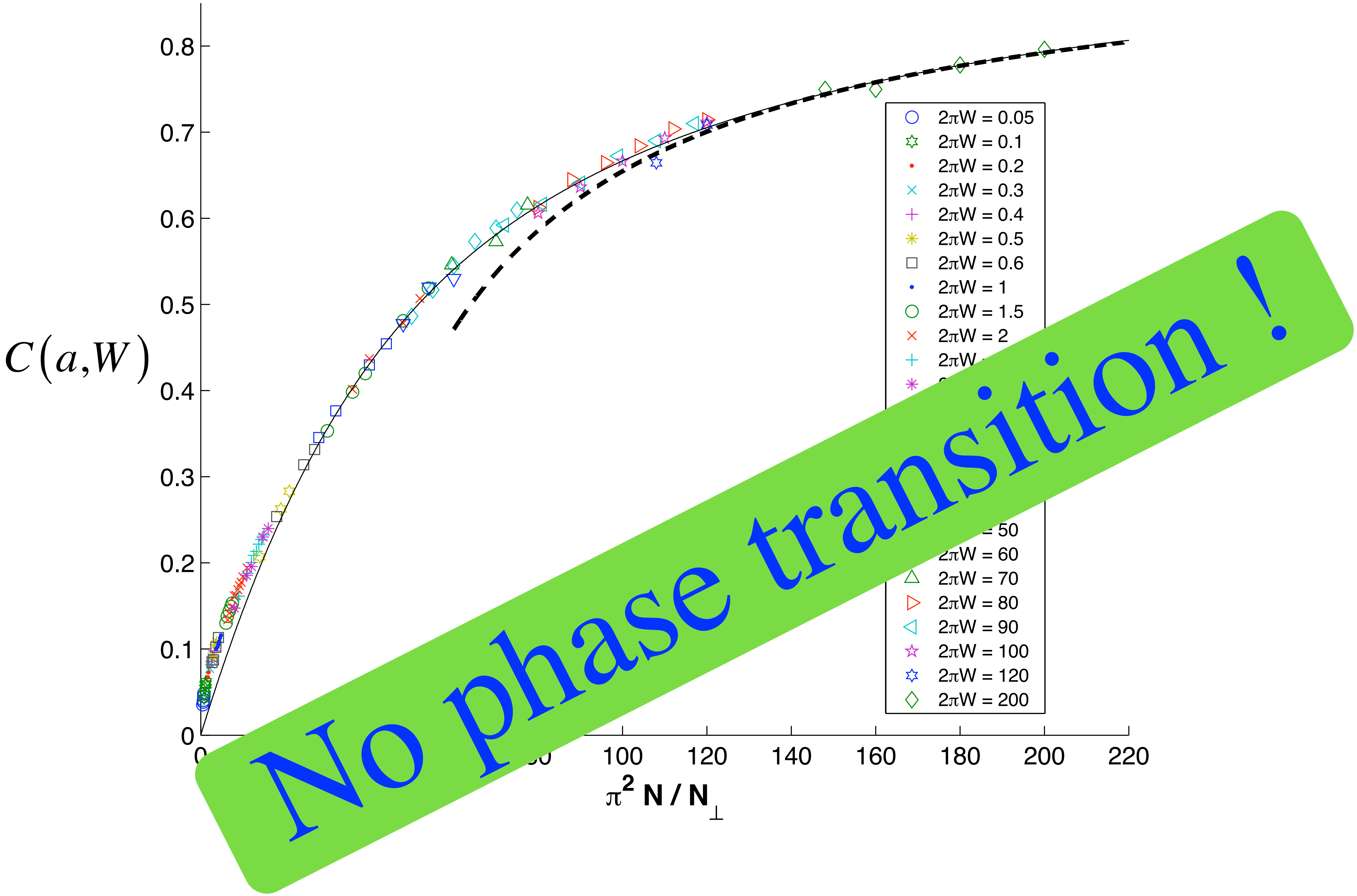

$C(a,W)$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
100
120
140
160
180
200
220
π^2 N / N$_\perp$
2πW = 0.05
2πW = 0.1
2πW = 0.2
2πW = 0.3
2πW = 0.4
2πW = 0.5
2πW = 0.6
2πW = 1
2πW = 1.5
2πW = 2
2πW = 60
2πW = 70
2πW = 80
2πW = 90
2πW = 100
2πW = 120
2πW = 200
No phase transition !

Dependence upon the space dimension ?

disorder driven localisation transition
(Anderson)

One-dimensional random atomic gas : Absence of single atom limit (Wigner-Weisskopf)

$d = 1$: Same expression of the effective atomic Hamiltonian H_e ,

$$H_e = \left(\hbar\omega_0 - i\frac{\hbar\Gamma_0}{2}\right)\sum_{i=1}^{N}|e_i\rangle\langle e_i| + \frac{\hbar\Gamma_0}{2}\sum_{i\neq j}V_{ij}\Delta_i^+\Delta_j^-$$

with $V_{ij} = \beta_{ij} - i\gamma_{ij}$ but $\gamma_{ij} = \cos k_0 r_{ij}$ instead of $\cancel{\gamma_{ij} = \frac{\sin k_0 r_{ij}}{k_0 r_{ij}}}$

A. Gero, E.A. ,EPL **101,** 2013

One-dimensional random atomic gas : Absence of single atom limit (Wigner-Weisskopf)

$d = 1$: Same expression of the effective atomic Hamiltonian H_e ,

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$a \ll 1$ Dicke limit (cooperative effects are expected) $\gamma_{ij} = \cos k_0 r_{ij} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$

so that $P(\Gamma) = \frac{1}{N}[(N-1)\delta(\Gamma)+\delta(\Gamma-N)]$

A. Gero, E.A. ,EPL **101**, 2013

Method : Decomposition into a product of matrices

$N \times N$ matrix $U_{ij} = \cos k_0 r_{ij}$ can be written $U = \dfrac{1}{2} A^{\dagger} A$

with A is the $2 \times N$ matrix defined by $A_{0j} = e^{ik_0 r_j}$ *and* $A_{1j} = e^{-ik_0 r_j}$

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$$U^\dagger = \frac{1}{2} \begin{pmatrix} N & M \\ M^* & N \end{pmatrix}.$$

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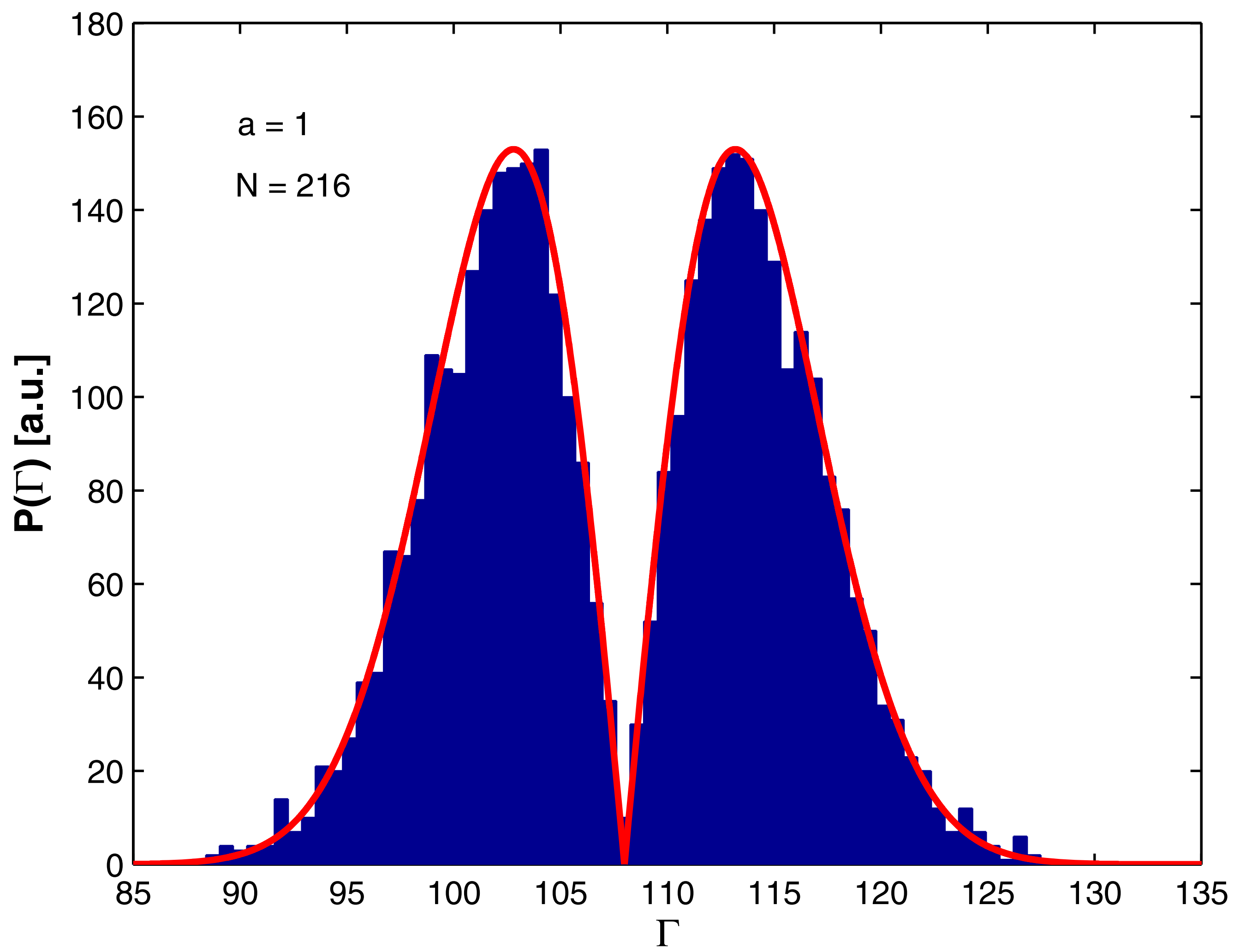
The two eigenvalues of $U^\dagger$ are $\lambda_\pm = \frac{N \pm |M|}{2}$,

and the spectrum of U is $$P(\Gamma) = \frac{1}{N}\left[(N-2)\delta(\Gamma) + \delta(\Gamma - \lambda_+) + \delta(\Gamma - \lambda_-)\right].$$

A. Gero, E.A. ,EPL **101**, 2013

One-dimensional random atomic gas : Absence of single atom limit



Subradiant mode is not represented

$$|M|^2 = N + \sum_{p \neq q} e^{2ik_0(r_p - r_q)}$$

Rayleigh distribution $P(|M|) = \frac{2|M|}{N} e^{-\frac{|M|^2}{N}}$

One-dimensional random atomic gas

- d=1 : no crossover between localised and delocalised photons.
- Single atom (Wigner-Weisskopf) limit is never reached.
- Results in d=1 are valid for both **ordered** and **disordered** media (M is not a random variable)
- Cooperative effects (not disorder) is the mechanism underlying photon localisation in d=1.

Two-dimensional random atomic gas : Marchenko-Pastur distribution

In $d=2$ the same expression of the effective atomic Hamiltonian H_e holds,

with $V_{ij}=\beta_{ij}-i\gamma_{ij}$ but $\gamma_{ij}=J_0\left(k_0 r_{ij}\right)$ instead of $\cancel{\gamma_{ij}=\frac{\sin k_0 r_{ij}}{k_0 r_{ij}}}$

A. Gero, E.A. ,PRA 88, 2013

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The $d=1$ trick does not work.

Instead we use the general decomposition: $U = H\,T\,H^{\dagger}$,

where $T(M\times M), H(N\times M)$

(S. Skipetrov and also "Free probability theory" (Voiculescu), "Wireless communications" (Debbah, Tulino, Verdu), "spin glasses" I. Kanter et al.

A. Gero, E.A. ,PRA 88, 2013

Two-dimensional random atomic gas : Marchenko-Pastur distribution

Spectrum of $\gamma_{ij} = J_0\left(k_0 r_{ij}\right)$

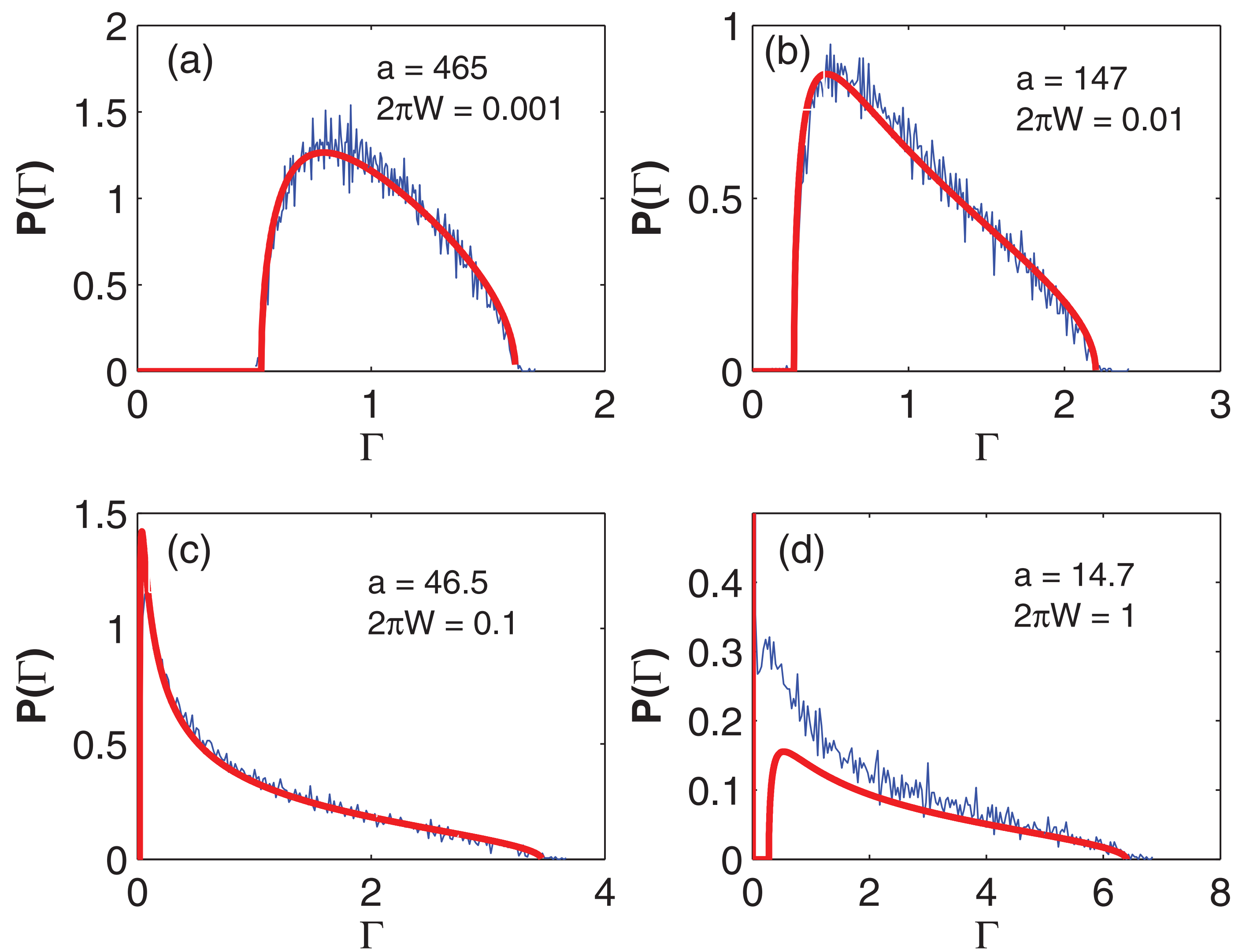
For $a \gg 1$, dilute limit,

$$P(\Gamma) \simeq \left(1 - \frac{M}{N}\right)^+ \delta(\Gamma) + \frac{\sqrt{(\Gamma - \Gamma_-)^+ (\Gamma_+ - \Gamma)^+}}{2\pi \frac{N}{M} \Gamma},$$

$$x^+ = \max(0, x)$$

$$\Gamma_\pm = (1 \pm \sqrt{N/M})^2$$

$$M = 2\pi a$$



Disorder parameter

$$L^d = (\lambda a)^d$$

$$W = N \big/ 2\pi a^2$$

Two-dimensional random atomic gas : Marchenko-Pastur distribution

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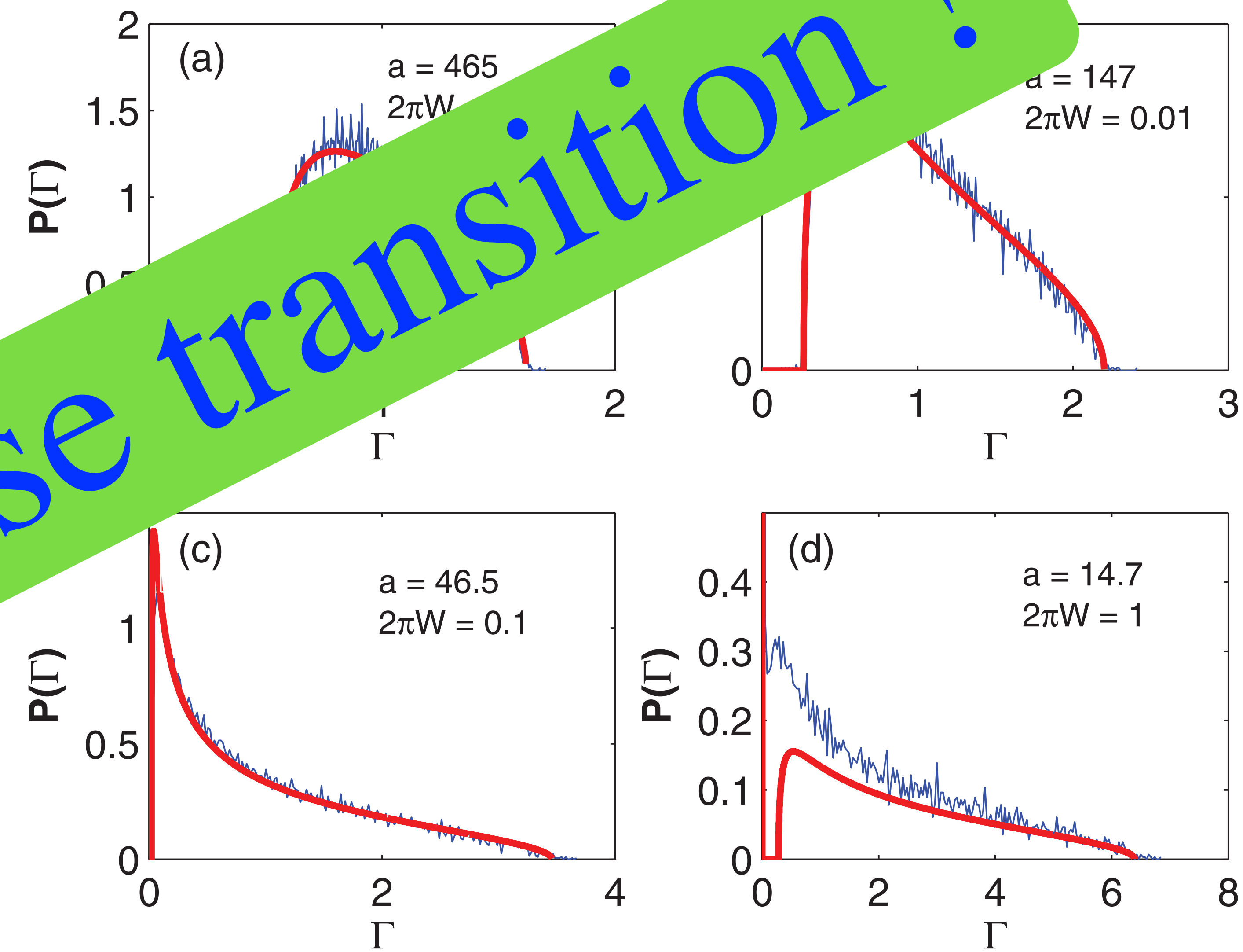
No phase transition !



Numerical results $N=216$

A. Gero, E.A. ,PRA 88, 2013

Eigenvalues of the non Hermitian random Hamiltonian

Time evolution of the ground state population is driven by the eigenvalues of the random matrix γ_{ij}

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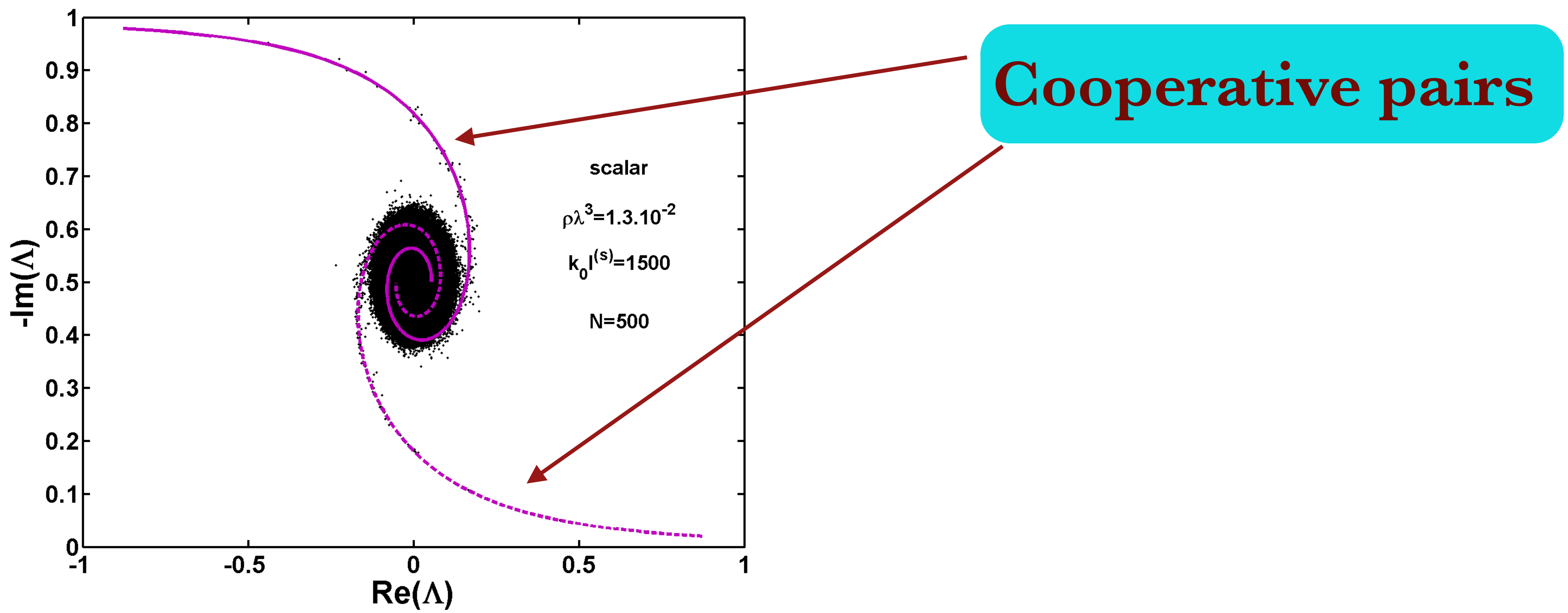
$$V_{ij} = \beta_{ij} - i\gamma_{ij}$$

Study the complex eigenvalues of H_e

$$\boxed{E_n - i\hbar\,{}^{\Gamma_n}\!/_{2} \equiv \hbar\omega_0 + \hbar\Gamma_0\Lambda_n}$$

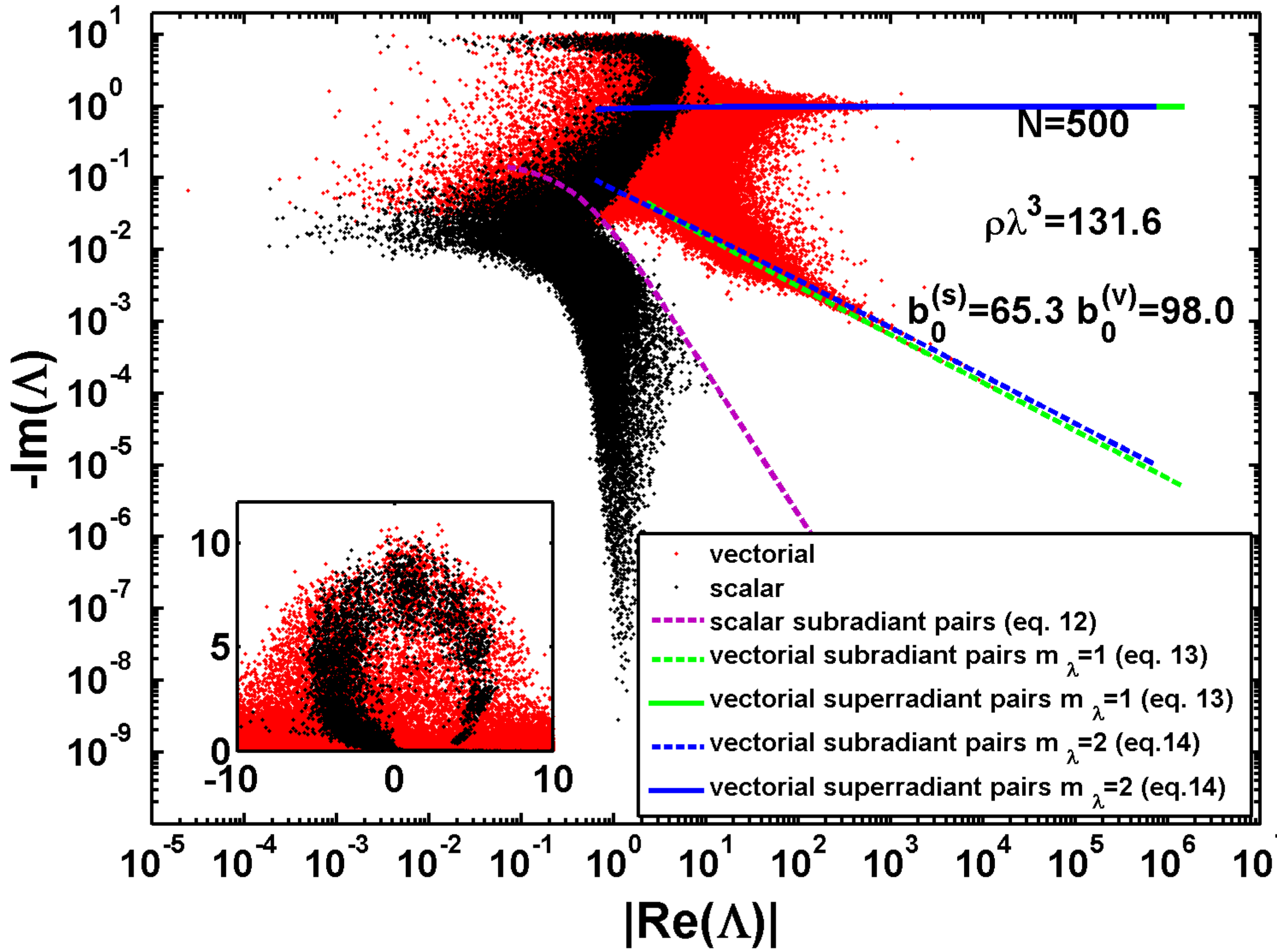
Eigenvalues of the non Hermitian random Hamiltonian

N=2 atoms case : The spectrum of H_e can be obtained explicitly (for both scalar and vectorial case)



$$E_n - i\hbar \Gamma_n/2 \equiv \hbar\omega_0 + \hbar\Gamma_0 \Lambda_n$$

For N large and in the dense limit



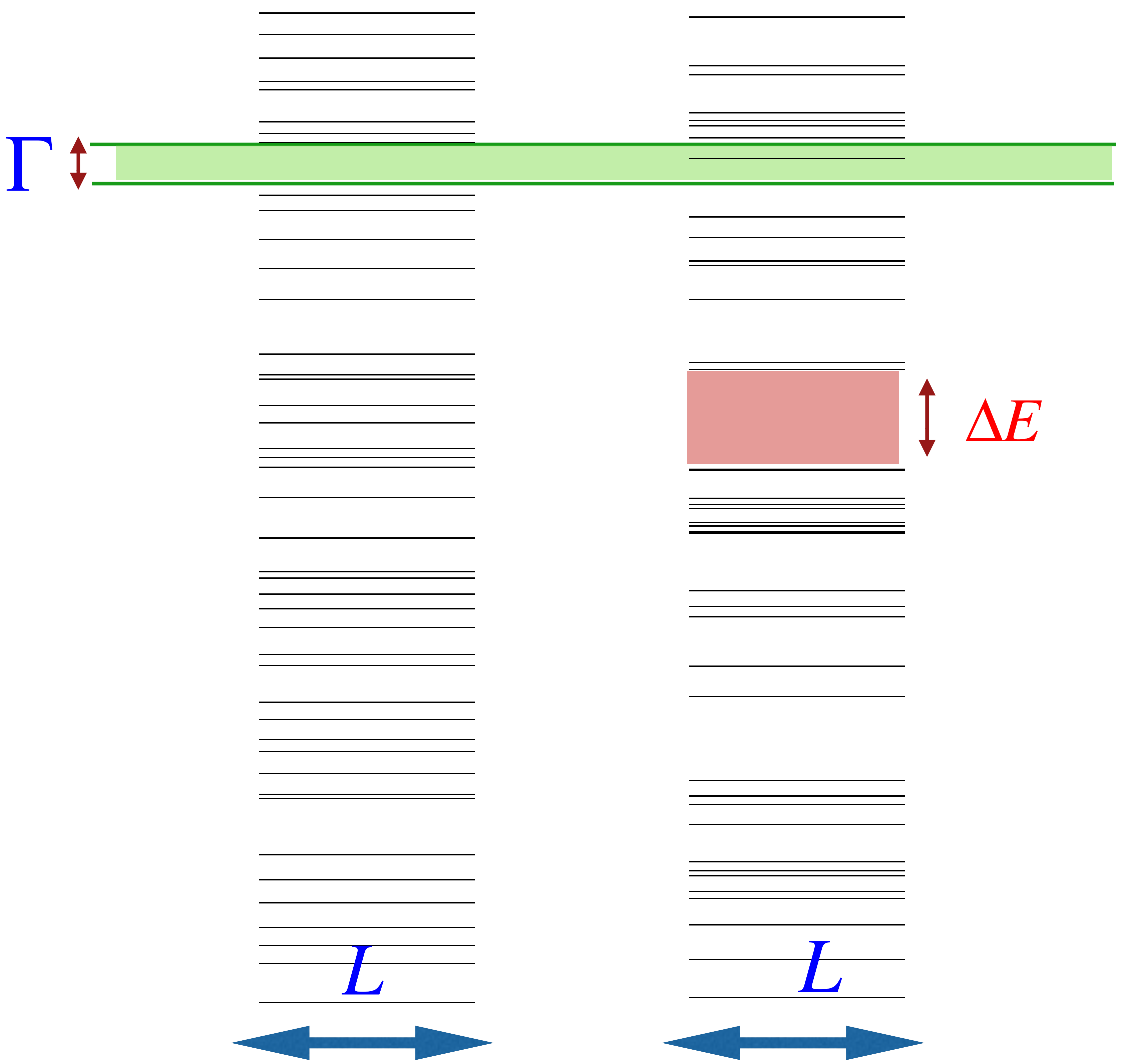
Thouless parameter : localisation phase transition

Edwards & Thouless ('72), Thouless ('77)

Coupling between open quantum systems Transport (conductance)

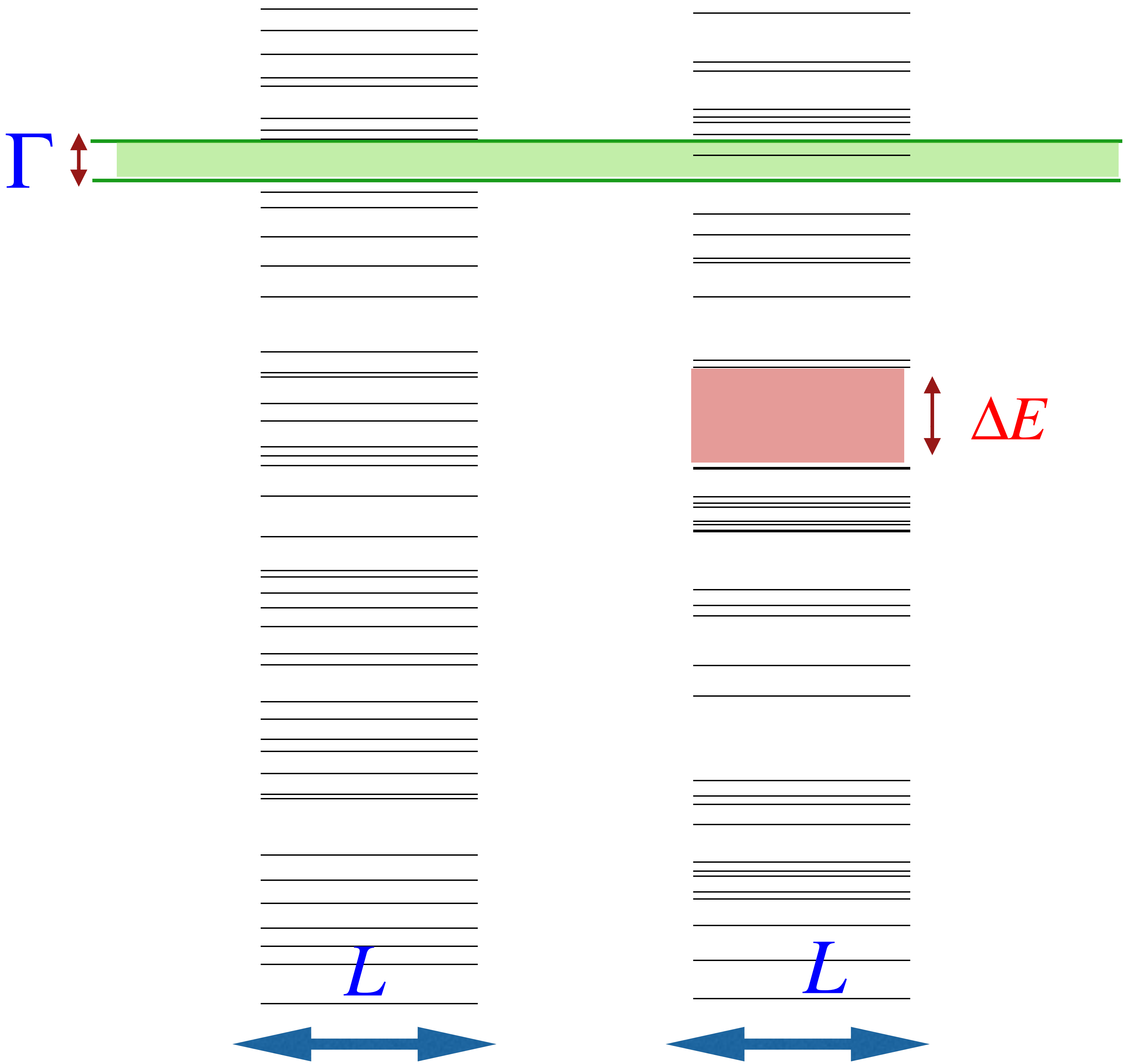
Using Random matrix theory :
G. Montambaux, E.A., (1992), I. Guarneri et al. (1994)

Thouless parameter - Resonance overlap



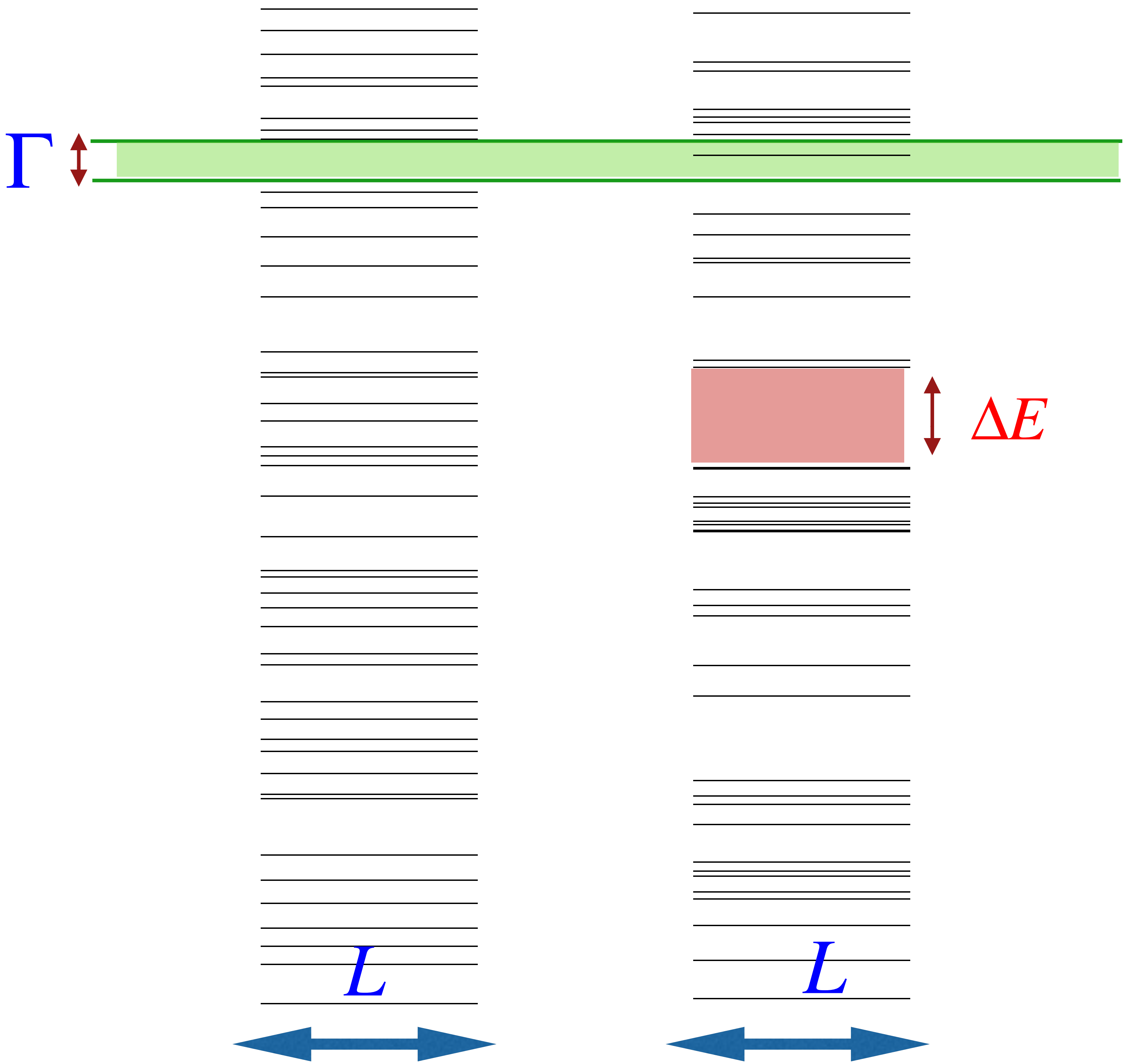
Energy spectrum of two random quantum systems

Thouless parameter - Resonance overlap



$$g(L) = \left\langle \frac{\langle \Gamma \rangle_i}{\langle \Delta E \rangle_i} \right\rangle$$

Thouless parameter - Resonance overlap



$$g(L) = \left\langle \frac{\langle \Gamma \rangle_i}{\langle \Delta E \rangle_i} \right\rangle$$

$$g(L \to \infty) \gg 1$$

large overlap : delocalised states - conductor

$$g \ll 1$$

Small overlap : localised states - insulator

Thouless parameter - localisation phase transition

Scaling and its meaning : (P.W. Anderson *et al*.,1979)

If we know $g(L)$, we know it at any scale :

$$g\big(L(1+\varepsilon)\big) = f\big(g(L),\varepsilon\big)$$

Scaling behavior :

$$\boxed{g(L,W) = f\left(L/\xi(W)\right)}$$

$\xi(W)$ is the localization length

Thouless parameter - localisation phase transition

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$$\beta(g)=\frac{d\ln g}{d\ln L}$$

is a function of g only.

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Thouless parameter - localisation phase transition

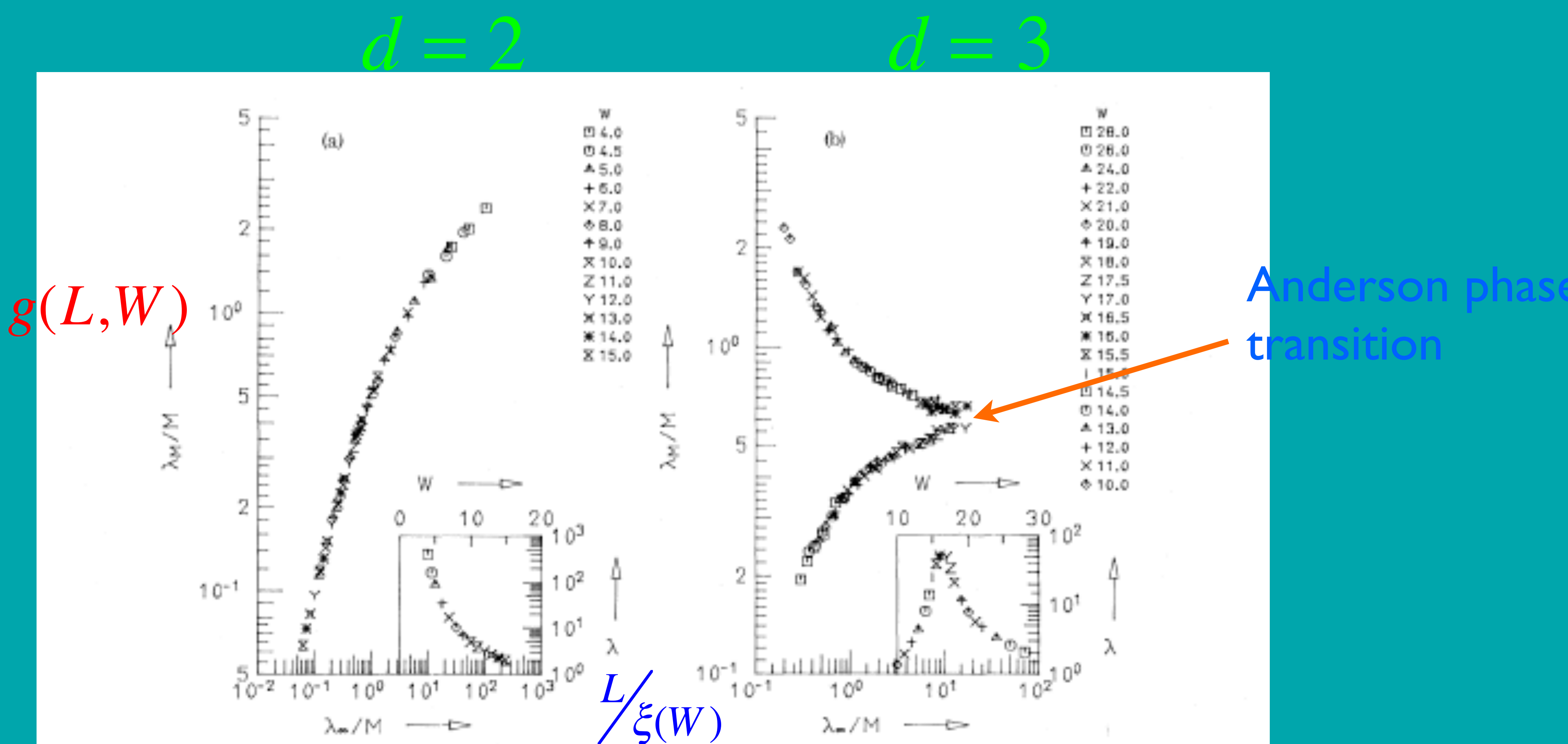

FIG. 1. Scaling function λ_M/M vs λ_∞/M for the localization length λ_M of a system of thickness M for (a) $d=2$ ($M \geq 4$) and (b) $d=3$ ($M \geq 3$). Insets show the scaling parameter λ_∞ as a function of the disorder W.

Scaling behaviour - phase transition

Thouless scaling parameter (conductance)

$$g(L) = \left\langle \frac{\langle \Gamma \rangle_i}{\langle \Delta E \rangle_i} \right\rangle$$

$$E_n - i\hbar \,\Gamma_n/2 \equiv \hbar\omega_0 + \hbar\Gamma_0 \Lambda_n$$

S. Skipetrov and Sokolov (2014), Bellando, Gero, Kaiser, E.A. , PRA, 2014

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because of the constraint : $\langle \Gamma \rangle_i = -2Tr(\Lambda)/N = 1$

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because of the constraint : $\langle \Gamma \rangle_i = -2Tr(\Lambda)/N = 1$

Instead we define :

$$\boxed{g \equiv \left\langle \frac{1}{\langle 1/\Gamma \rangle_i \langle \Delta E \rangle_i} \right\rangle}$$

S. Skipetrov and Sokolov (2014), Bellando, Gero, Kaiser, E.A. , PRA, 2014

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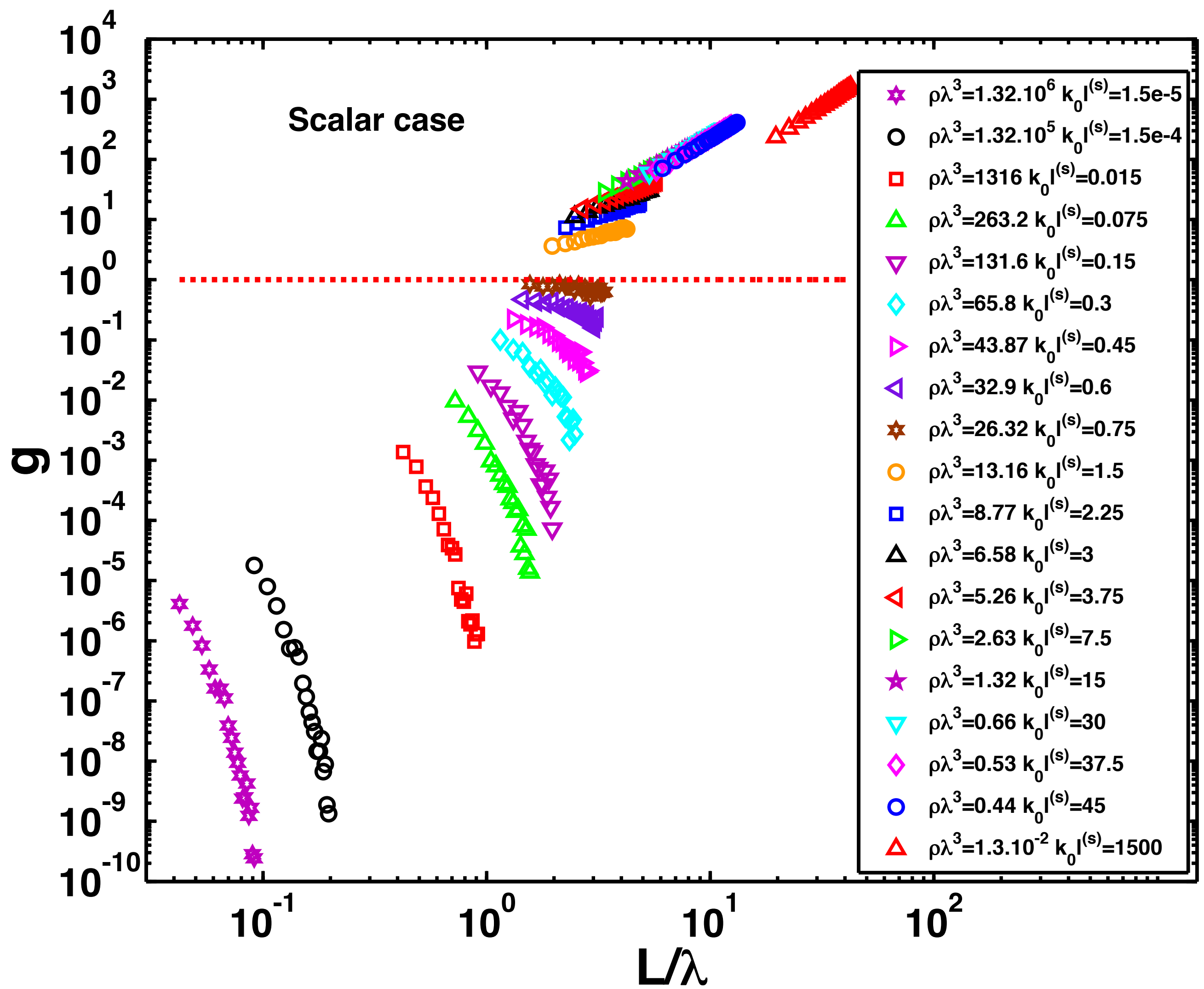
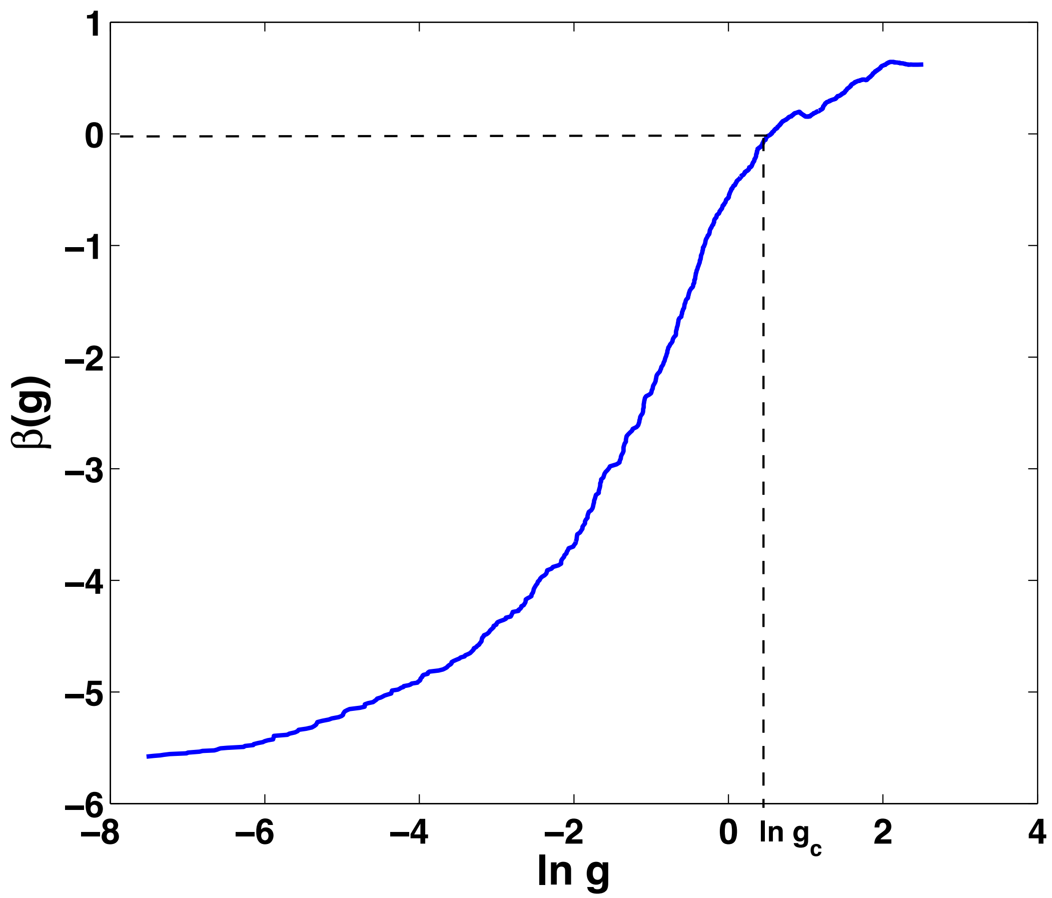
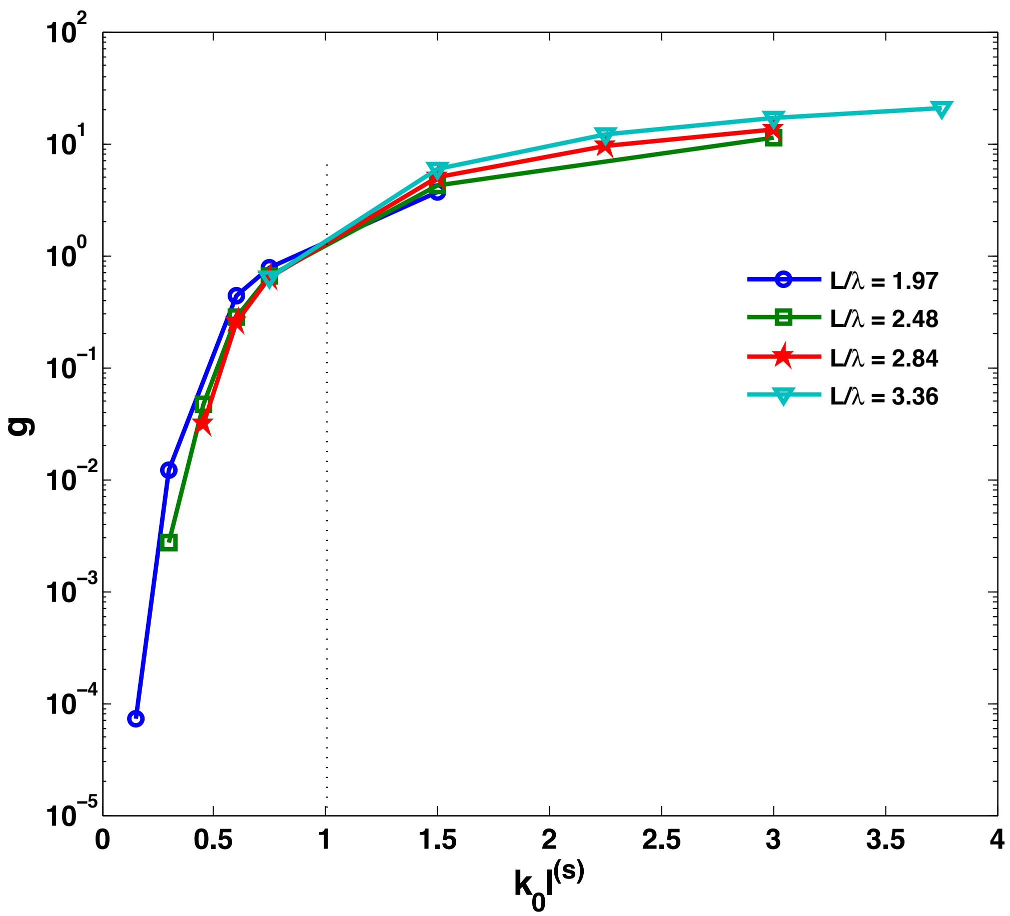


Critical point for a phase transition :

$g(L)$ becomes L- independent i.e. $\beta(g) = \frac{d \ln g}{d \ln L} = 0$





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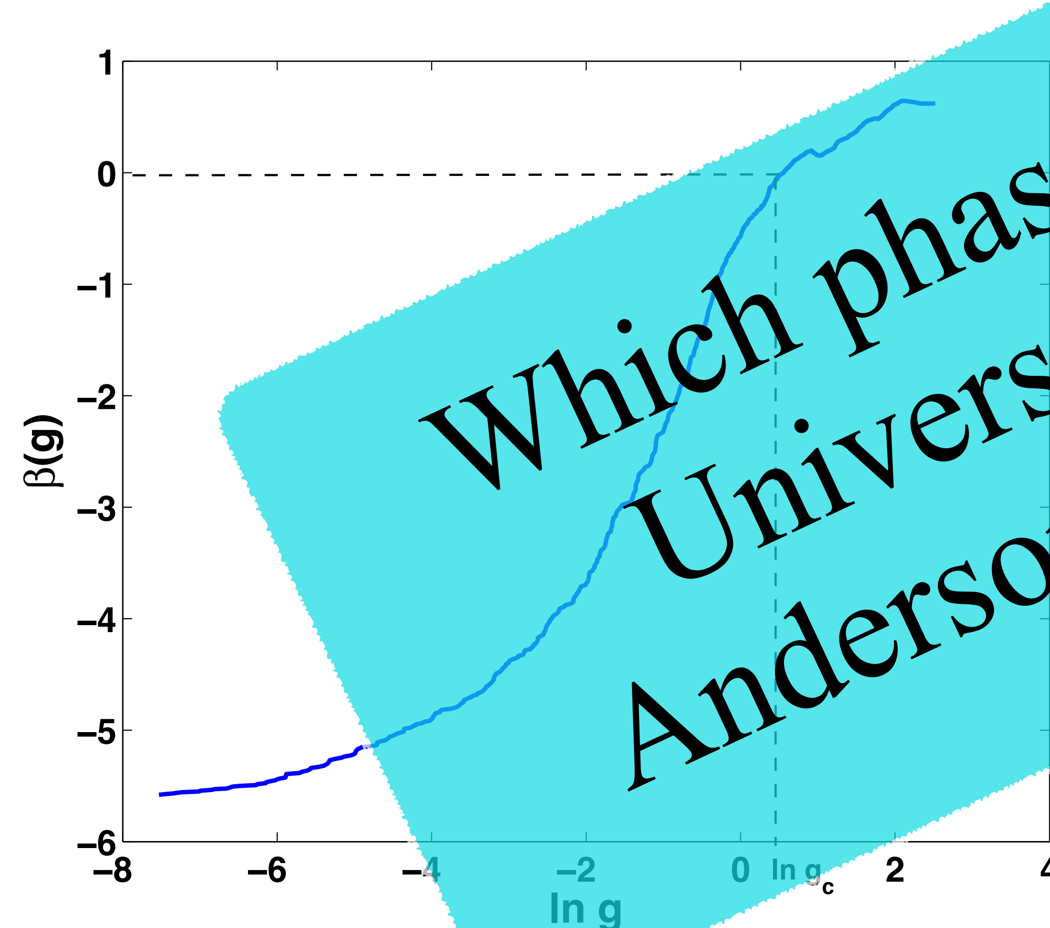
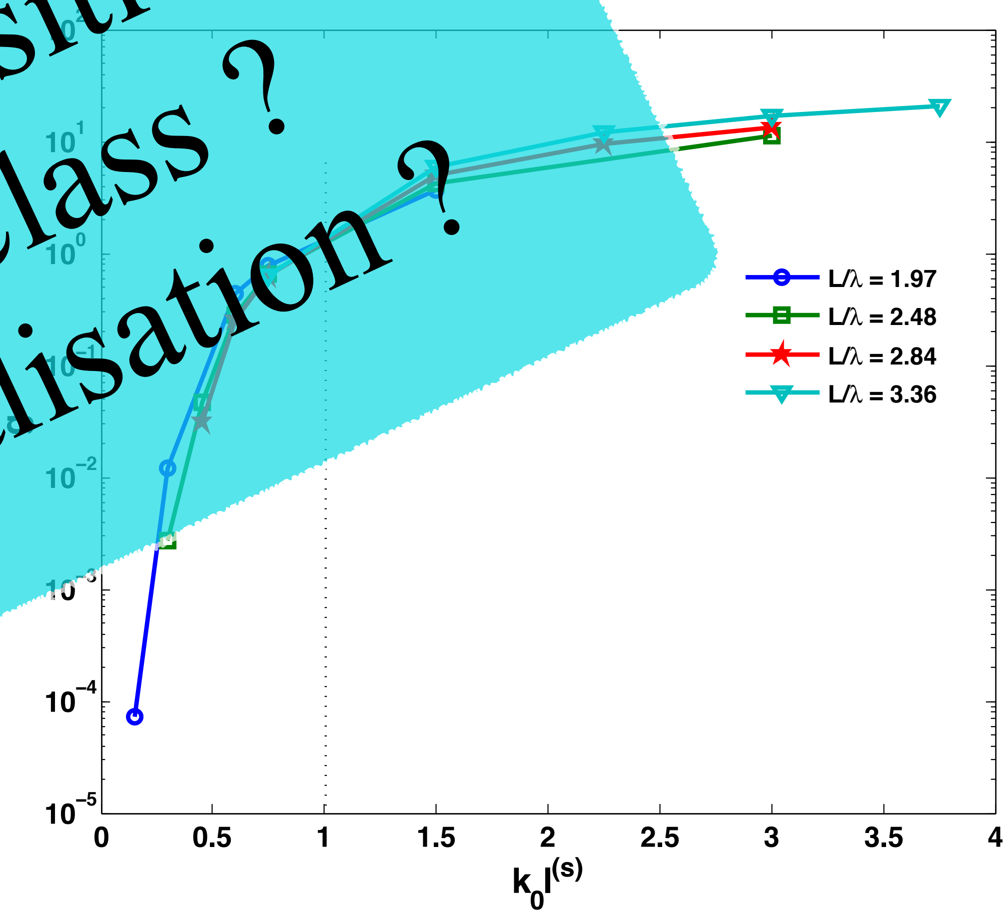
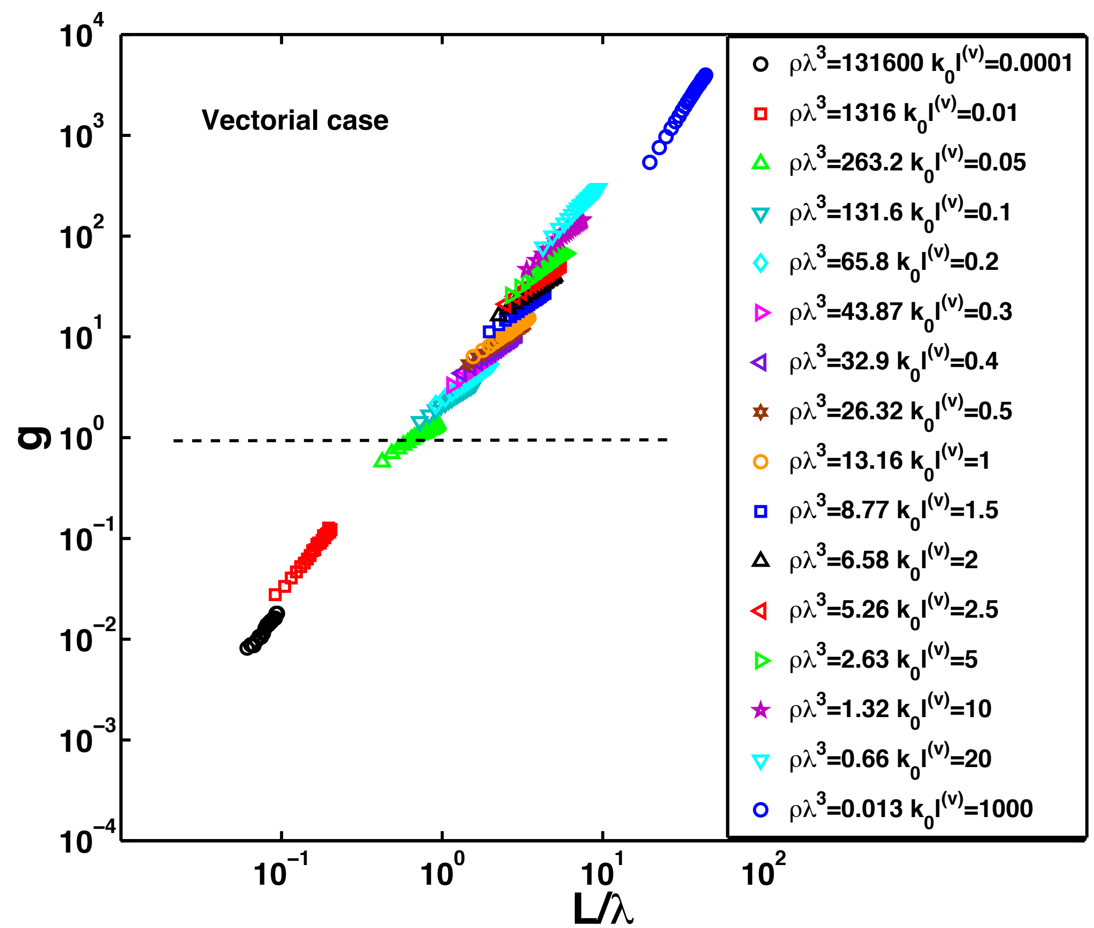
Which phase transition ?
Universality class ?
Anderson localisation ?

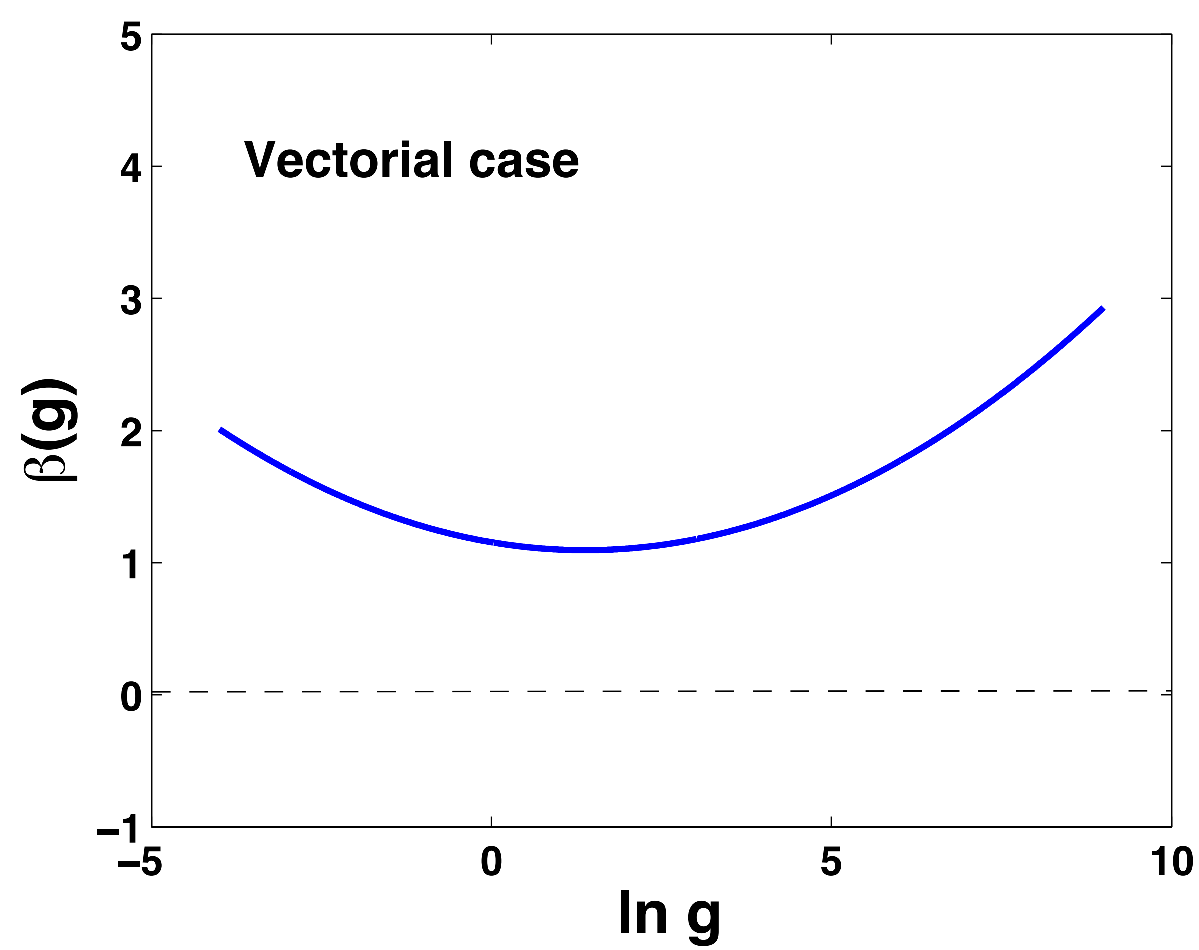




Scaling behaviour - phase transition
to make things more complicated ….

Vector case - polarised waves





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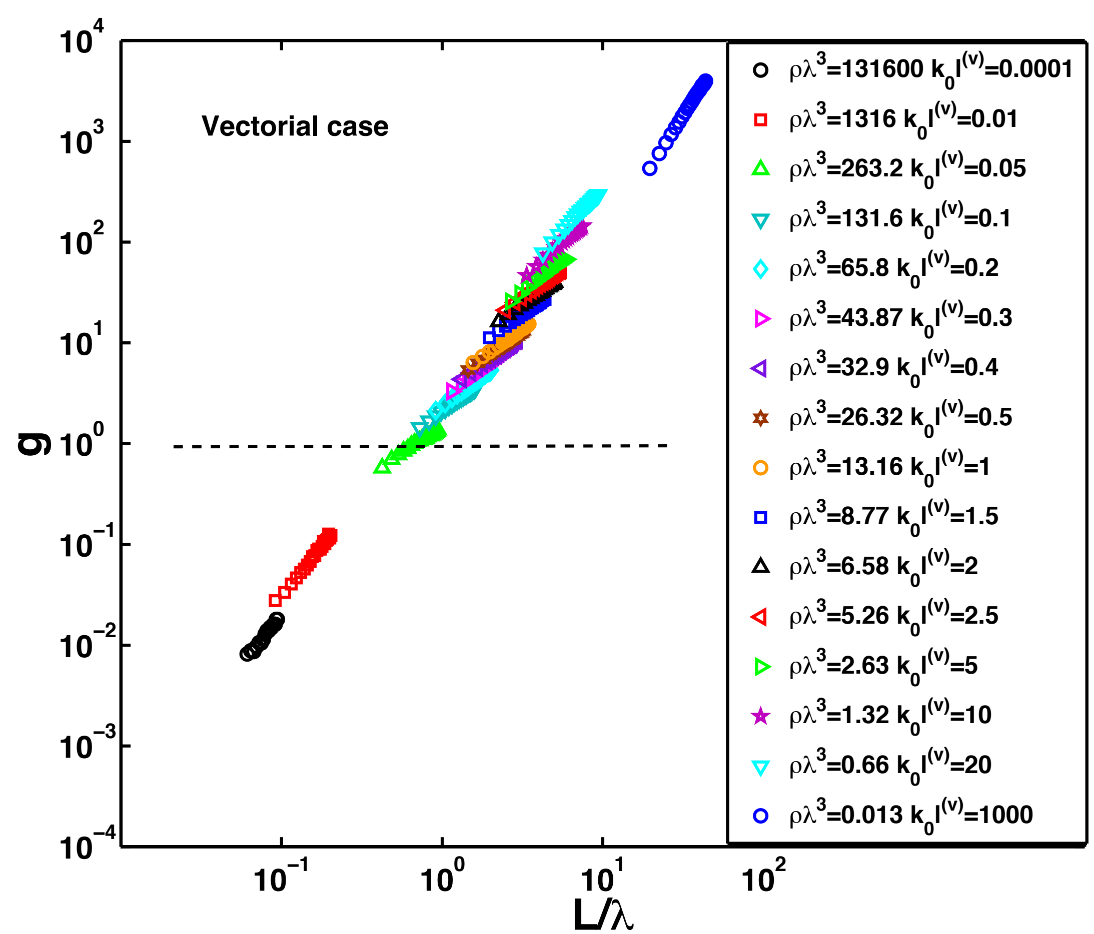

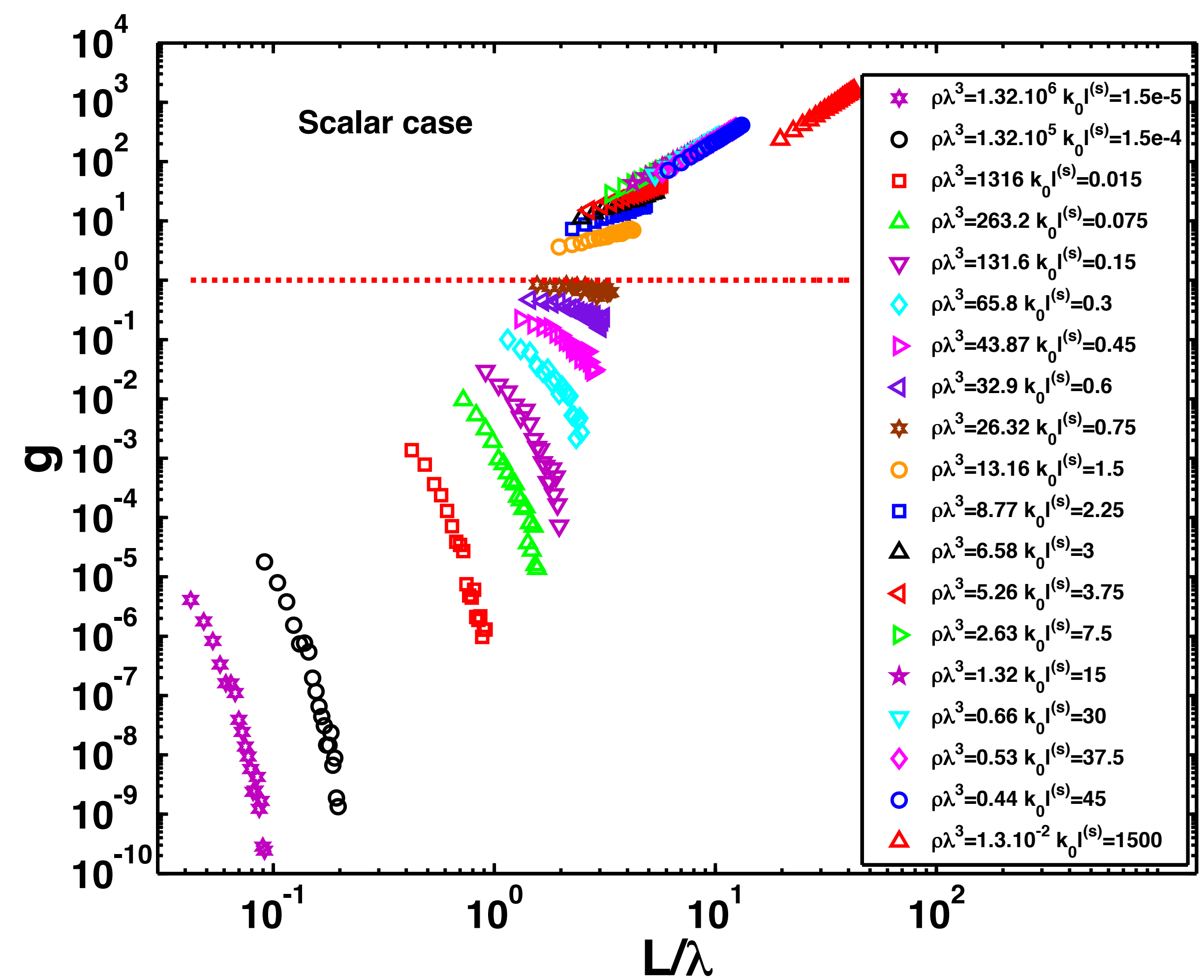

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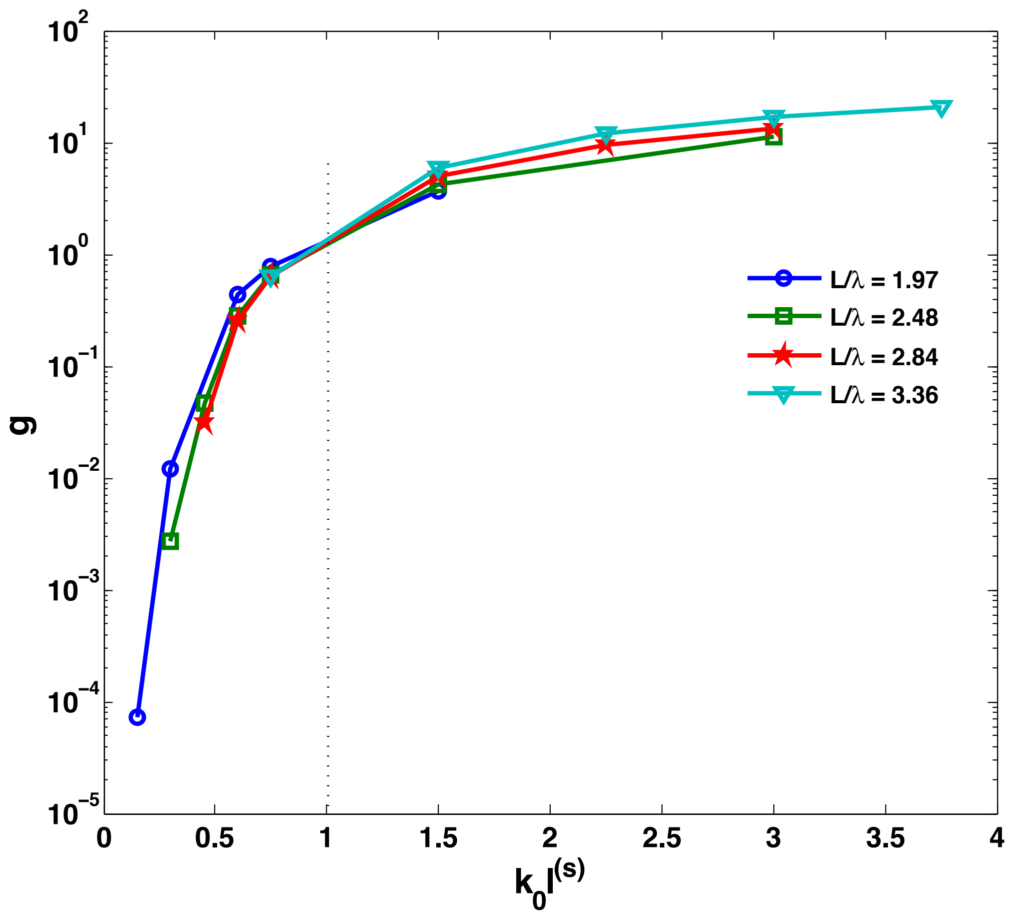

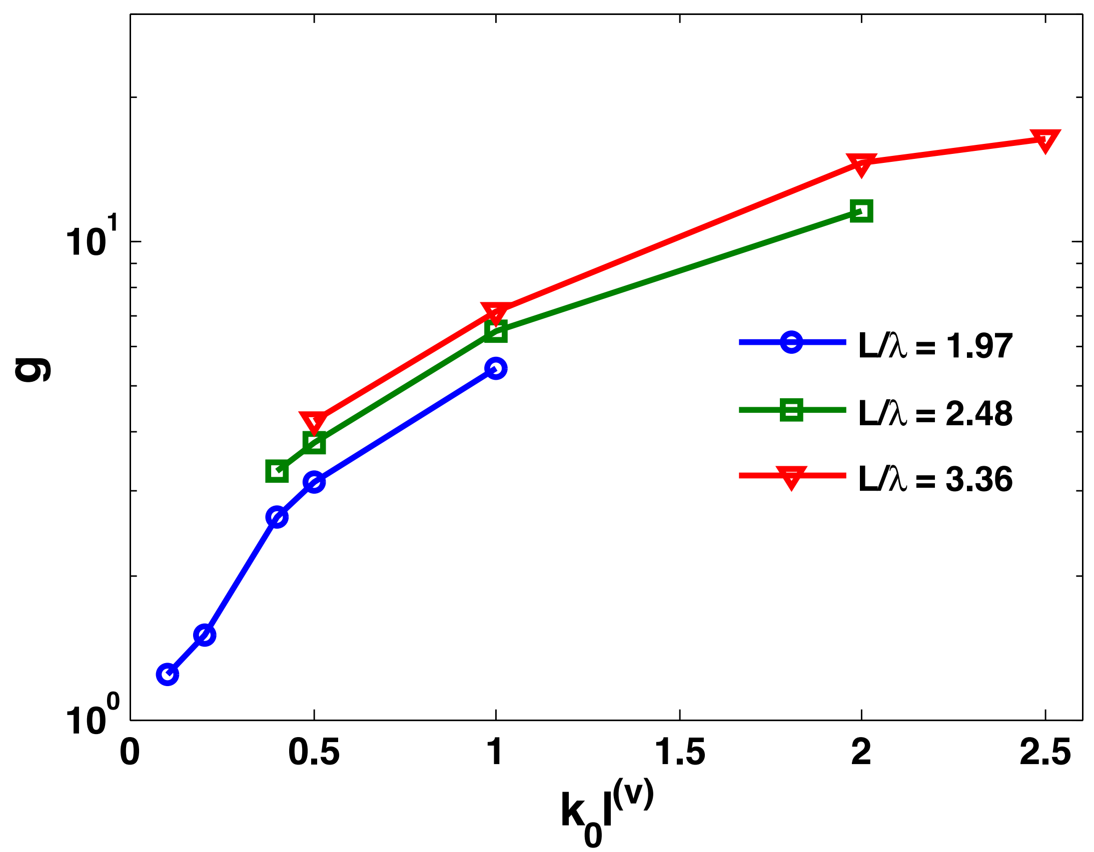

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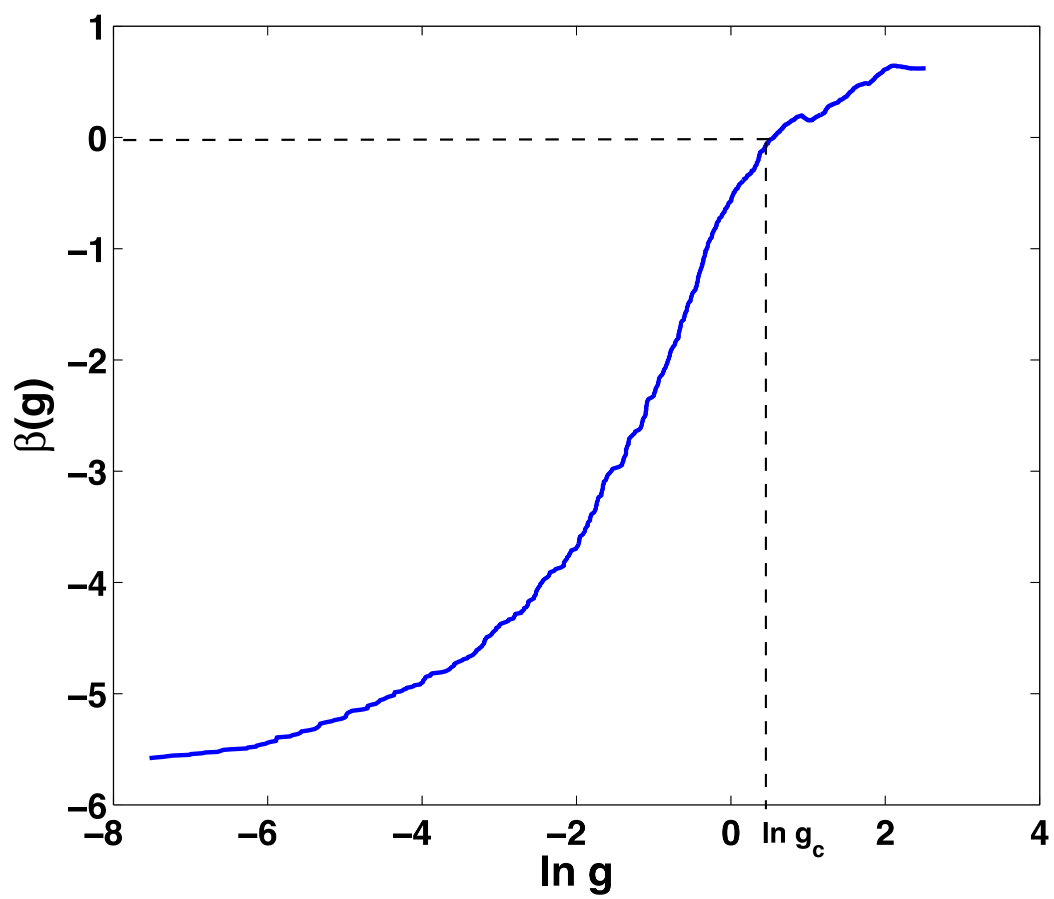

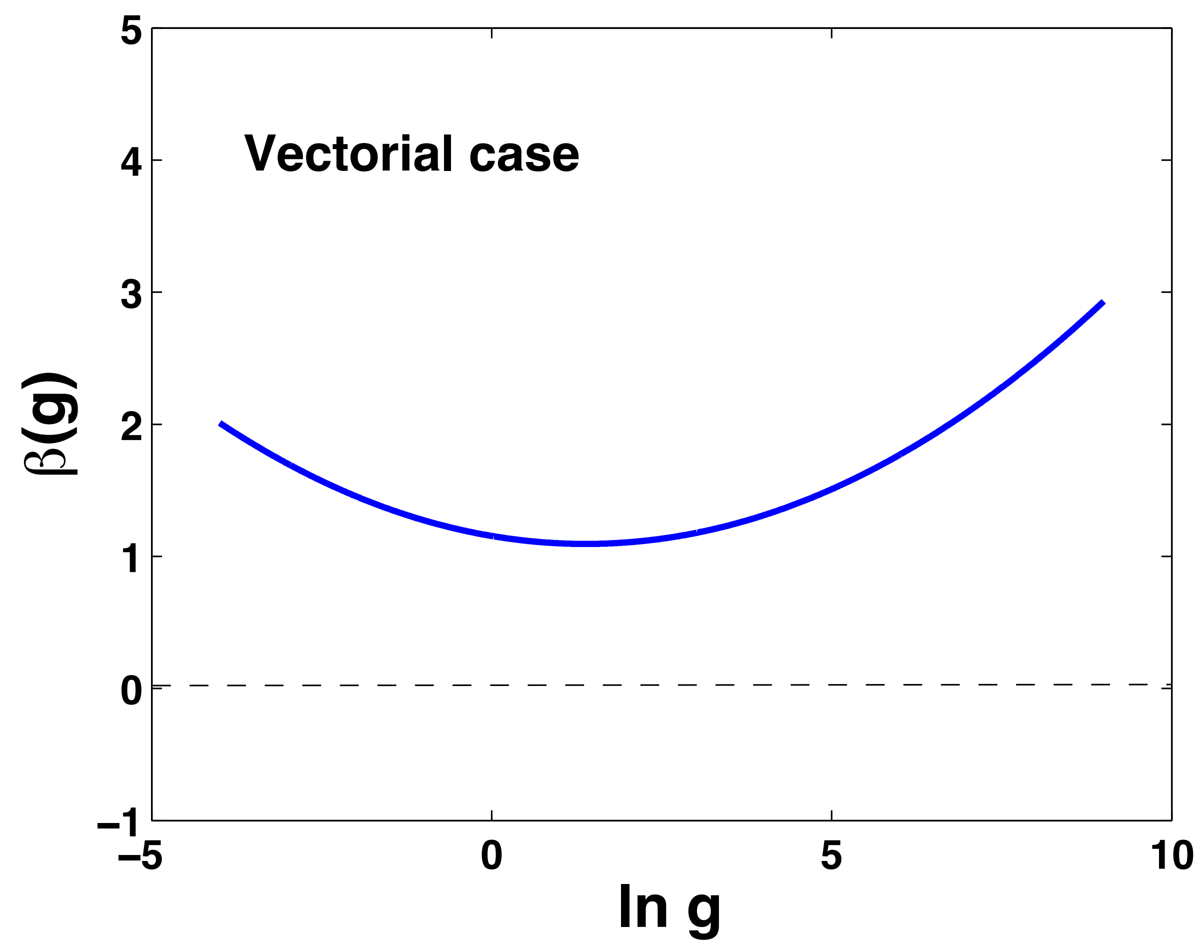

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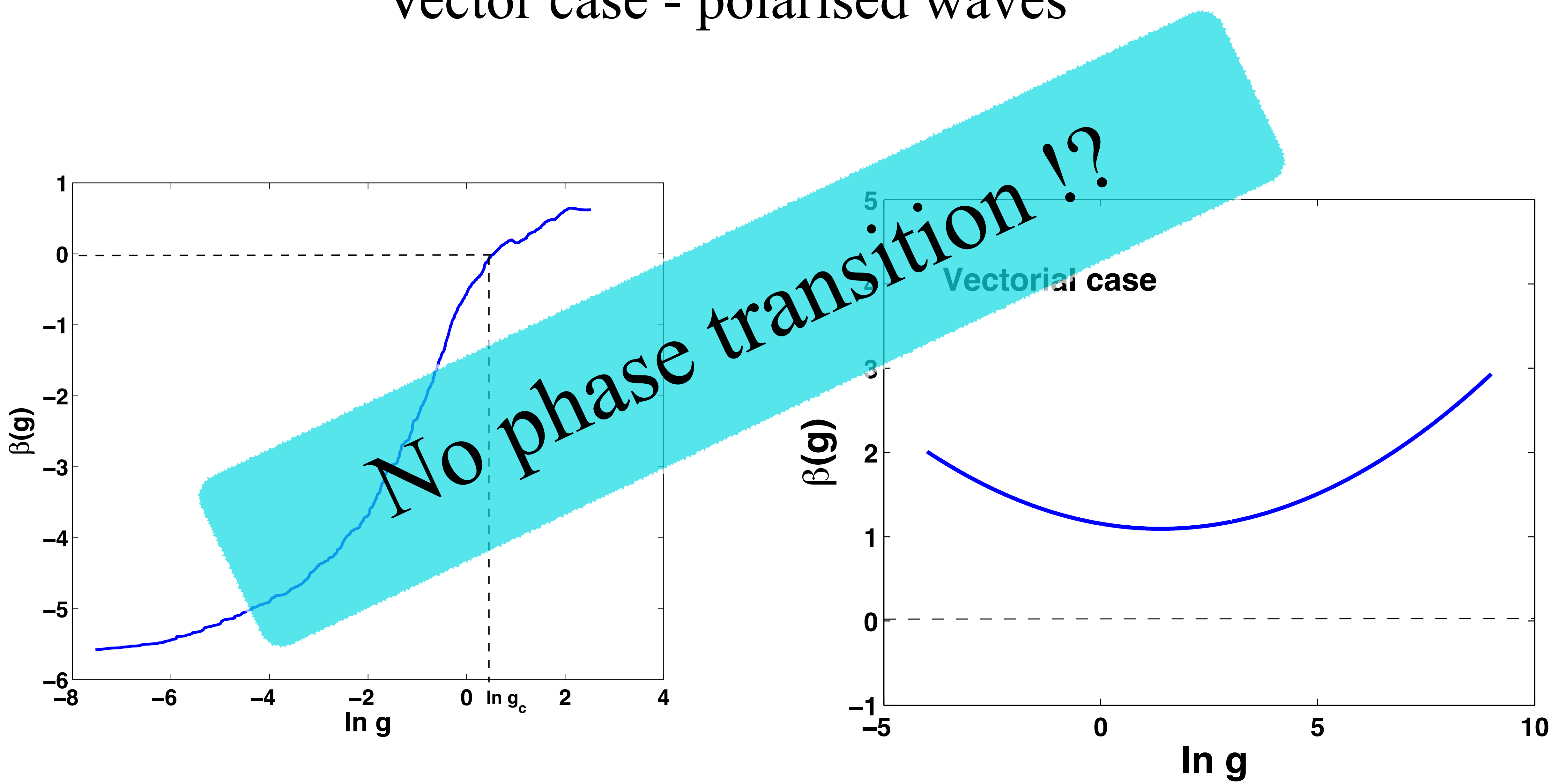

Conclusion - Summary

- Study of the scaling properties of the Non Hermitian Euclidean random Hamiltonian

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- H_e accounts for **cooperative** properties of the atomic gas (Super- and Sub-radiance). It also depends on the disorder.
- The radiation pattern is well accounted by the part γ_{ij} of the interaction.
- The distribution of eigenvalues of γ_{ij} exhibits scaling properties but there is ***no indication of the existence of a phase transition*** driven either by disorder or interactions.

- The interplay between disorder and cooperative effects depend upon the space dimensionality.
- For $d = 2,3$, there is a ***crossover*** between a delocalised (Wigner-Weisskopf) regime and a behaviour driven by cooperative effects (eventually Dicke regime)
- For $d = 1$, there is no single atom limit.

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- ***The critical behaviour disappears for vector waves.***
- The nature and universality of this transition is still unclear.
- Set of new experimental efforts to probe the interplay of disorder and cooperative effects (R. Kaiser, A. Browaeys, M. Havey,…)