Lectures on quantum gases

Lecture 2

Relative motion of interacting particles II

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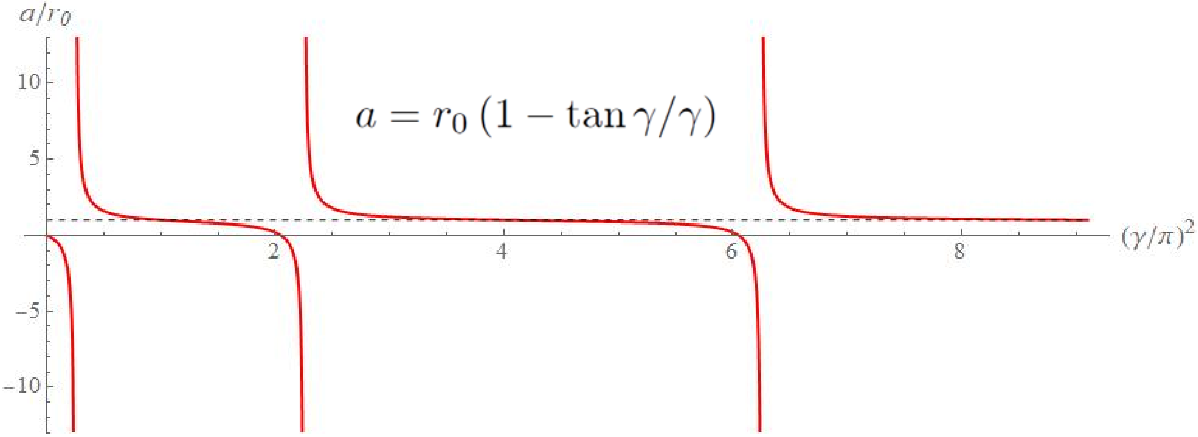
binding energy versus well parameter

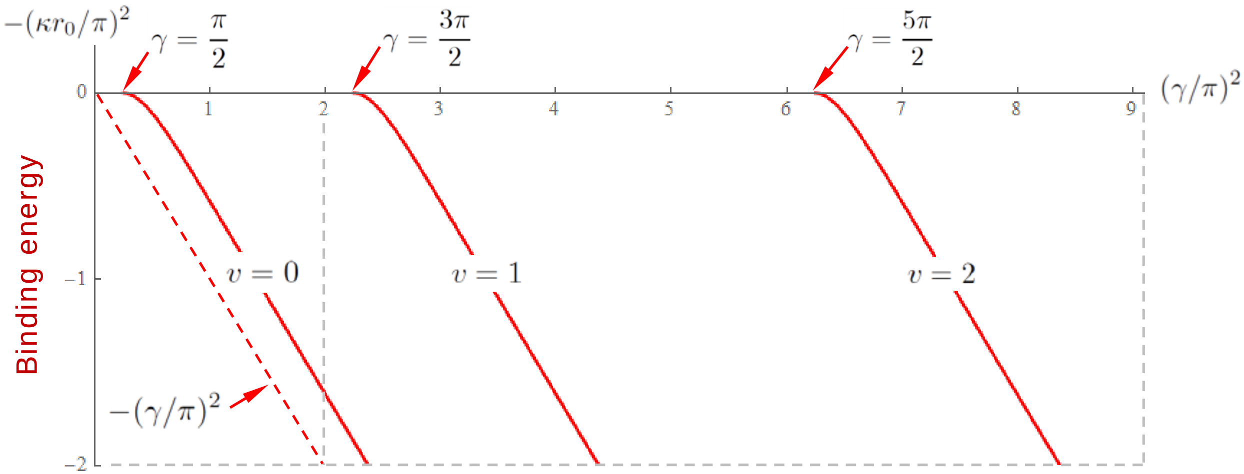
$$\chi_0'/\chi_0|_{r=r_0} = K_- \cot K_- r_0 = -\kappa$$

$$K_-^2 = -\kappa^2 + \kappa_0^2$$

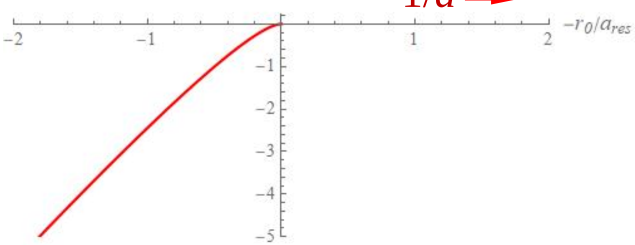
well parameter: $\gamma \equiv \kappa_0 r_0$



$$\boxed{E_b = -\frac{\hbar^2}{2\mu}\frac{1}{a^2}}$$

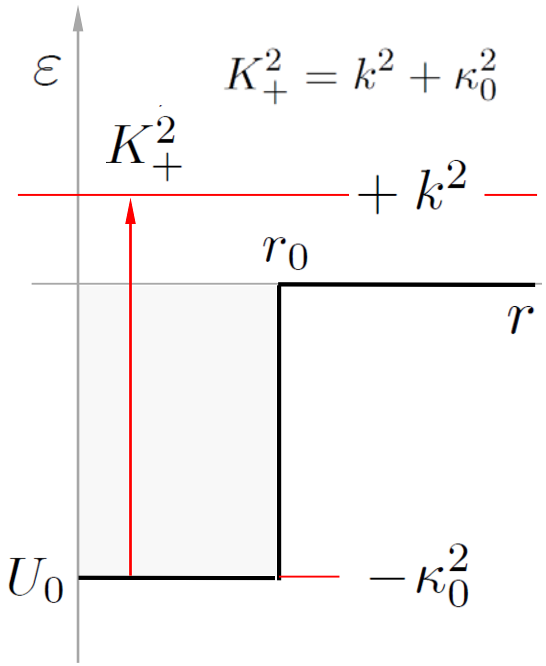


$-1/a$



Binding energy

s-wave scattering length a



$$\eta_0(k) = -kr_0 + \arctan \frac{k}{K_+ \cot K_+ r_0}$$

$$\eta_0(k) \equiv -ka(k)$$

$$\boxed{a(k) = r_0 - \frac{1}{k} \arctan \frac{k}{K_+ \cot K_+ r_0}}$$

Define scattering length: $a \equiv \lim_{k \to 0} a(k) = -\lim_{k \to 0} \eta_0(k)/k$

Define well parameter: $\gamma \equiv \kappa_0 r_0$ $\quad \boxed{a = r_0 \left(1 - \tan\gamma/\gamma\right)}$

flat-bottom potential

effective range - r_e

effective range r_e

We now analyze the *energy dependence* of the phase shift

$$\eta_0(k) = \underbrace{-kr_0}_{\substack{\alpha \\ \text{"regular"}}} + \underbrace{\arctan \frac{k}{K_+ \cot K_+ r_0}}_{\substack{\beta \\ \text{"resonant"}}}$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$\tan x = x + \tfrac{1}{3}x^3 + \cdots = x\left(1 + \tfrac{1}{3}x^2 + \cdots\right)$$

evaluate phase shift:

$$kr_0 \cot\eta_0 = \frac{K_+ r_0 \cot K_+ r_0 + k^2 r_0^2 + \cdots}{1 - \left(1 + \frac{1}{3}k^2 r_0^2 + \cdots\right) K_+ r_0 \cot K_+ r_0}$$

$$K_+^2 = k^2 + \kappa_0^2 \longrightarrow K_+ r_0 = \kappa_0 r_0 [1 + k^2/\kappa_0^2]^{1/2} = \gamma + \tfrac{1}{2}k^2 r_0^2/\gamma + \cdots$$

effective range r_e

$$k r_0 \cot \eta_0 = -\frac{1}{1-\tan\gamma/\gamma} + \tfrac{1}{2}k^2 r_0^2 \left(1 - \frac{3\,(1-\tan\gamma/\gamma)+\gamma^2}{3\gamma^2\,(1-\tan\gamma/\gamma)^2}\right) + \cdots$$

$$\boxed{a = r_0\,(1-\tan\gamma/\gamma)}$$

r_e : measure for energy dependence of the phase shift

Effective range expansion:

$$\boxed{k\cot\eta_0 = -\frac{1}{a} + \tfrac{1}{2}k^2 r_e + \cdots} \rightarrow \boxed{k\cot\eta_0 = -\frac{1}{a(k)}}$$

Effective range: $r_e = r_0\left(1 - \frac{3ar_0+\gamma^2 r_0^2}{3\gamma^2 a^2}\right)$

Two expansions to order k^2:

$$a \neq 0 \qquad k\cot\eta_0 = -\frac{1}{a} + \tfrac{1}{2}k^2 r_0\left(1 - \frac{3ar_0+\gamma^2 r_0^2}{3\gamma^2 a^2}\right) + \cdots$$

$$a \ll r_0 \qquad \frac{1}{k\cot\eta_0} = -a + \tfrac{1}{6}k^2 r_0^3\left[1 - 3\,(a/r_0)^2 + (3/\gamma^2)\,(a/r_0)\right] + \cdots$$

effective range r_e – special cases

$$a \neq 0 \qquad k\cot\eta_0 = -\frac{1}{a} + \underbrace{\tfrac{1}{2}k^2 r_0\left(1 - \frac{3ar_0 + \gamma^2 r_0^2}{3\gamma^2 a^2}\right)}_{r_e} + \cdots$$

positive

$a = \pm r_0$ (regular) $\qquad r_e = r_0\left(2/3 \mp 1/\gamma^2\right) \simeq \tfrac{2}{3}r_0$

$$k\cot\eta_0 = \mp\frac{1}{r_0} + \cancel{\tfrac{1}{3}k^2 r_0} + \cdots$$

$kr_0 \ll 1$ $\longrightarrow$ k dependence unimportant

positive

$|a| \gg r_0$ (anomalously large) $\qquad r_e \simeq r_0$

$$k\cot\eta_0 = -\cancel{\frac{1}{a}} + \tfrac{1}{2}k^2 r_0 + \cdots$$

$\longrightarrow$ strong k dependence of $a(k)$

effective range r_e – special cases

$a \ll r_0$

$$\frac{1}{k \cot \eta_0} = \cancel{-a} + \tfrac{1}{6} k^2 r_0^3 [1 - 3\,(a/r_0)^2 + (3/\gamma^2)\,(a/r_0)] + \cdots$$

$a = 0$ (anomalously small)

$$k \cot \eta_0 = -\frac{1}{a(k)} = \frac{6}{k^2 r_0^3} + \cdots \qquad a(k) = -\tfrac{1}{6} k^2 r_0^3$$

$k r_0 \ll 1$ $\longrightarrow$ k dependence large but a always small

Conclusions for 'open channel' potentials:

effective range always positive - important only for $|a| \gg r_0$

examples

ultracold atom collisions:

^{133}Cs |4,4>	$a = 2400\ a_0$	$r_0 = 100\ a_0$
^{85}Rb |3,3>	$a = -369\ a_0$	$r_0 = 83\ a_0$
^{88}Sr	$a = -2\ a_0$	
^{1}H |1,1>	$a = 1.22\ a_0$	$r_e = 348\ a_0$

2 hadron collisions at MeV energies:

s-wave regime ($kr_0 << 1$):

proton (uud) I=1/2

neutron (udd) I=1/2

r_0 is 6 orders of magnitude smaller

k can be 6 orders of magnitude larger

deuteron	I=1	bound state	$a > 0$	$a = 5.41$ fm $r_e = 1.75$ fm
	I=0	virtual state	$a < 0$	$a = -2.38$ fm $r_e = 2.67$ fm

flat-bottom potential

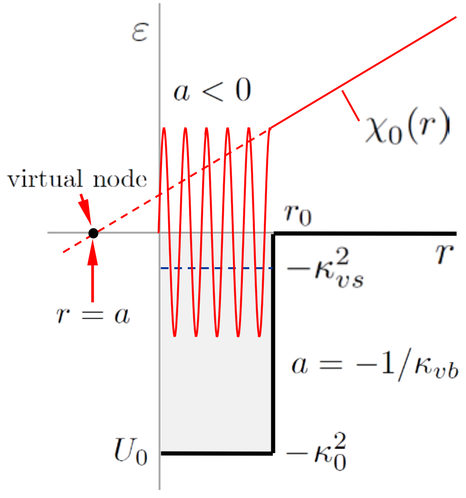
s-wave resonances

s-wave resonances

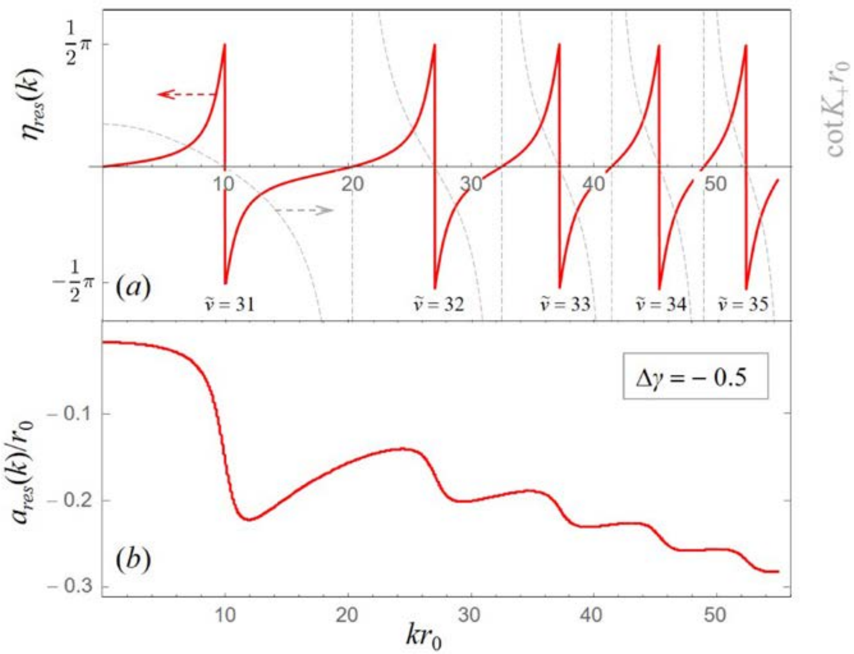
We return to the *energy dependence* of the phase shift

$$a(k) = \underbrace{r_0}_{\text{“background”}} \underbrace{- \frac{1}{k} \arctan \frac{k}{K_+ \cot K_+ r_0}}_{\text{“resonant”}}$$

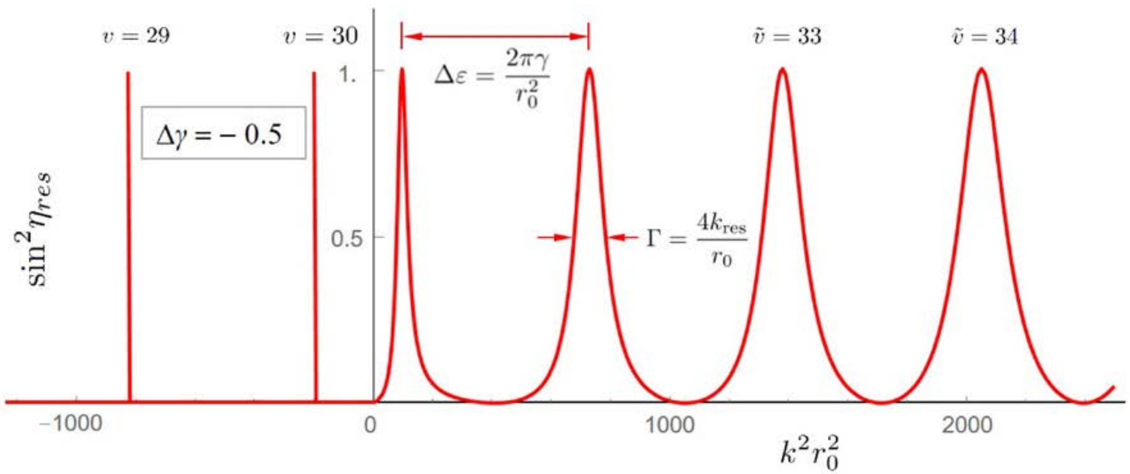
$$K_+^2 = k^2 + \kappa_0^2$$



virtual level



Breit-Wigner line shape



Expand about resonance ($k \simeq k_{\rm res}$)

$$\tan\eta_{\rm res} = \frac{k}{K_+ \cot K_+ r_0} \simeq -\frac{1}{\delta k\, r_0} = \frac{-(k+k_{\rm res})}{(k+k_{\rm res})(k-k_{\rm res})r_0} \simeq \frac{-2k_{\rm res}}{(k^2-k_{\rm res}^2)r_0} \equiv -\frac{\Gamma/2}{E-E_{res}}$$

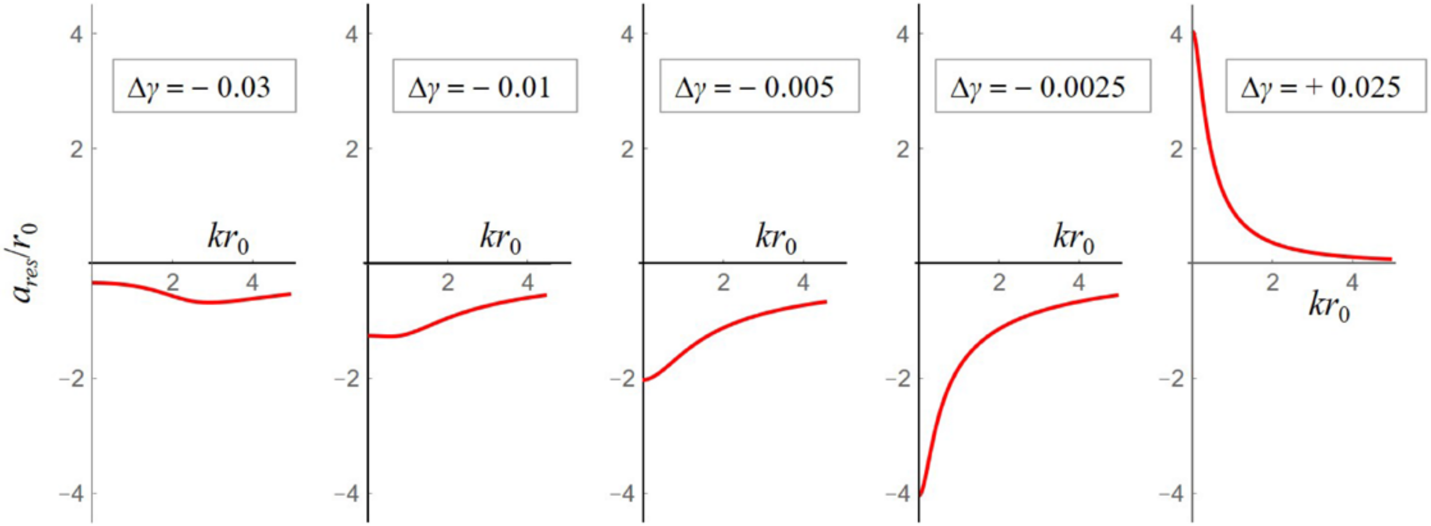
$$\boxed{\Gamma = (\hbar^2/m_r)(2k_{\rm res}/r_0)}$$

$$\boxed{\sin^2\eta_{\rm res} = \frac{(\Gamma/2)^2}{(E-E_{\rm res})^2 + (\Gamma/2)^2}}$$

s-wave resonance near threshold

$$\eta_{\rm res}(k) = -kr_0 + \arctan\frac{k}{K_+\cot K_+ r_0}$$

$$a(k) = r_0 - \frac{1}{k}\arctan\frac{k}{K_+\cot K_+ r_0}$$



$$\text{for } K_+ \simeq \kappa_0 \simeq K_- \longrightarrow \kappa_0 \cot \kappa_0 r_0 = -\kappa$$

$$a(k) \simeq r_0 + \frac{1}{k}\arctan\frac{k}{\kappa}$$

weakly-bound level

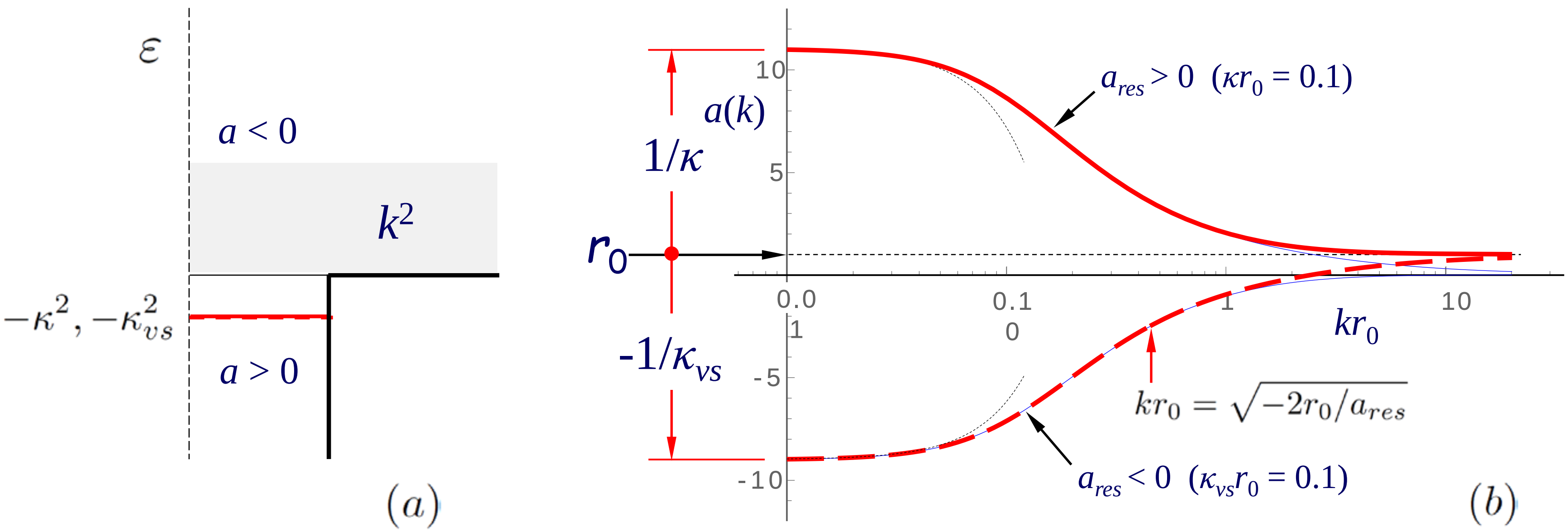
$$a(k) \simeq r_0 - \frac{1}{k}\arctan\frac{k}{\kappa_{vb}}$$

virtual level

s-wave resonance near threshold

$$\eta_{\mathrm{res}}(k) = -kr_0 + \arctan\frac{k}{K_+\cot K_+ r_0}$$

$$a(k) = r_0 - \frac{1}{k}\arctan\frac{k}{K_+\cot K_+ r_0}$$



Conclusion: scattering length approximation valid for $\boxed{\begin{array}{l} k \ll \kappa \\ k \ll \kappa_{vb} \end{array}}$

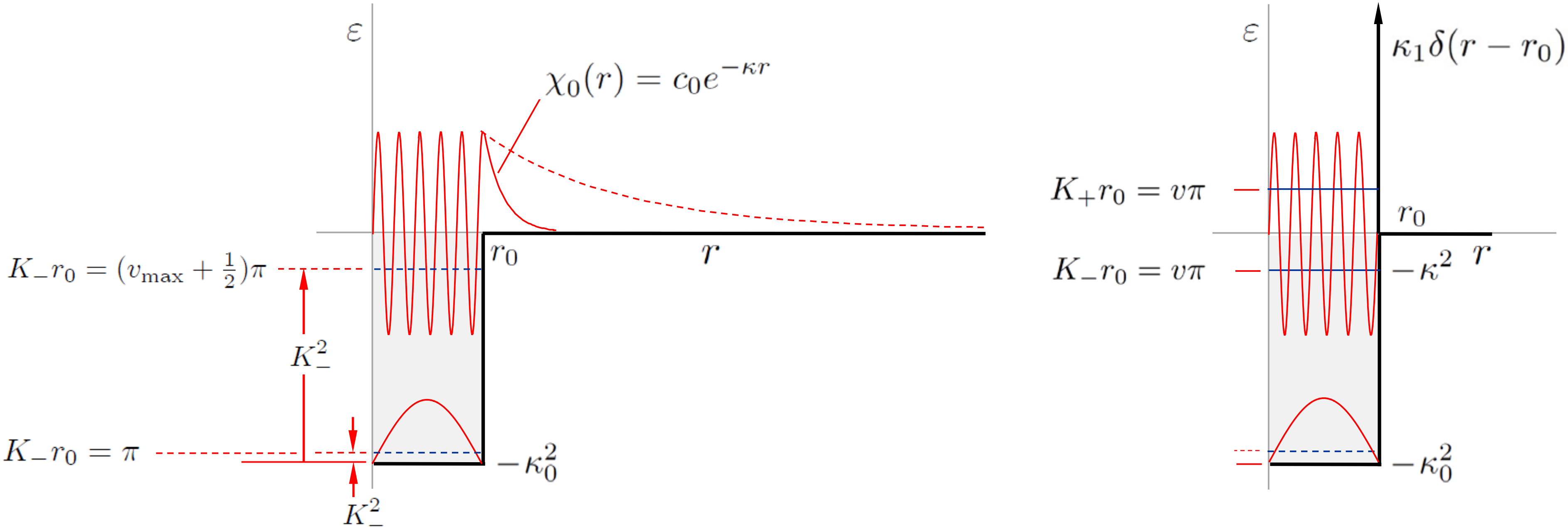
screened flat-bottom potential

narrow *s*-wave resonances

from halo state to narrow resonance

Bound states ($\varepsilon < 0$):

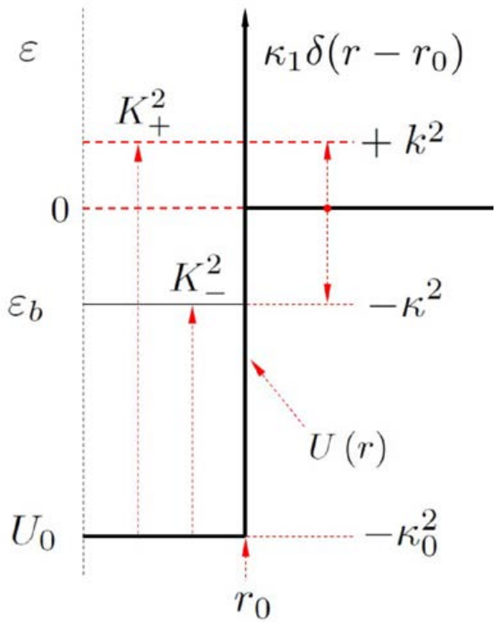
$$\chi_0 = Ae^{-\kappa r}$$



‘open’ channel

‘closed’ channel

screened flat-bottom potential



$$\chi_0'' + \left[k^2 - U(r)\right]\chi_0 = 0$$

$r > r_0 \quad U_0(r) = 0$

$$\chi_0'' + k^2\chi_0 = 0$$
$$\chi_0 = A\sin(kr + \eta_0)$$
$$\chi_0' = kA\cos(kr + \eta_0)$$

$r < r_0 \quad U_0(r) = -\kappa_0^2$

$$\chi_0'' + K_+^2\chi_0 = 0$$
$$\chi_0 = A'\sin(K_+ r + \cancel{\eta_0})$$
$$\chi_0' = K_+ A'\cos(K_+ r)$$

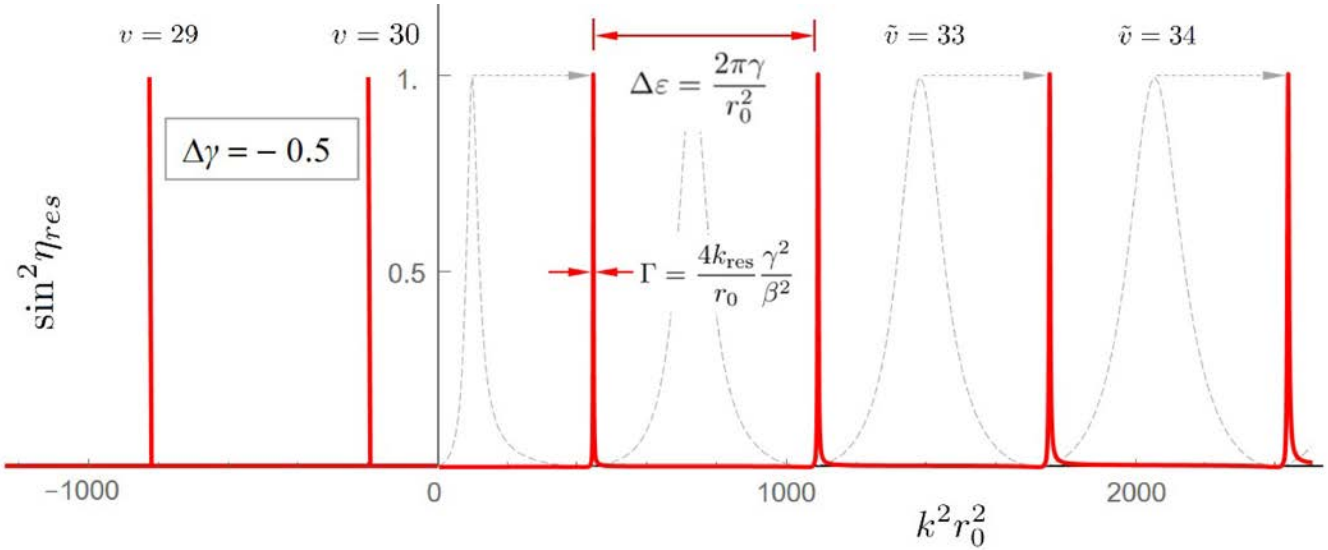
boundary condition: $\chi_0(r)$ continuous, $\chi_0'(r)$ jumps at $r = r_0$

$$\chi_0'/\chi_0|_{r=r_0} = \underbrace{K_+\cot K_+ r_0}_{r<r_0} = \underbrace{k\cot(kr_0 + \eta_0) + \kappa_1}_{r>r_0}$$

Barrier parameter: $\beta \equiv \kappa_1 r_0$ $\longrightarrow$

$$\boxed{a(k) = r_0 - \frac{1}{k}\arctan\left(\frac{kr_0}{K_+ r_0\cot K_+ r_0 - \beta}\right)}$$

screened flat-bottom potential



expand about resonance ($k \simeq k_{\rm res}$)

$$\tan\eta_{\rm res} = \frac{k}{K_+\cot K_+ r_0 - \kappa_1} \simeq -\frac{1}{\delta k\, r_0}\frac{\gamma^2}{\beta^2} = \frac{-(k+k_{\rm res})}{(k^2-k_{\rm res}^2)r_0}\frac{\gamma^2}{\beta^2} \simeq \frac{-\Gamma/2}{E-E_{\rm res}}$$

$$\boxed{\sin^2\eta_{\rm res} = \frac{(\Gamma/2)^2}{(E-E_{\rm res})^2+(\Gamma/2)^2}}$$

$$\boxed{\Gamma = (\hbar^2/m_r)(2k_{\rm res}/R^*)}$$

$$\boxed{R^* \equiv \tfrac{1}{2} r_0(\beta^2/\gamma^2)}$$

effective range r_e

We now analyze the *energy dependence* of the phase shift

$$\eta_0(k) = \underbrace{-kr_0}_{\text{“regular”}} + \underbrace{\arctan\left(\frac{kr_0}{K_+r_0\cot K_+r_0 - \beta}\right)}_{\text{“resonant”}}$$

evaluate phase shift:

$$k\cot\eta_0 = \frac{(K_+\cot K_+r_0 - \kappa_1) + k^2r_0 + \cdots}{1 - r_0\left(1 + \frac{1}{3}k^2r_0^2 + \cdots\right)(K_+\cot K_+r_0 - \kappa_1)}$$

$$K_+^2 = k^2 + \kappa_0^2 \longrightarrow \quad K_+r_0 = \kappa_0 r_0[1 + k^2/\kappa_0^2]^{1/2} = \gamma + \tfrac{1}{2}k^2r_0^2/\gamma + \cdots$$

$$\boxed{a = r_0\,(1 - \tan\gamma/\gamma)}$$

effective range r_e

Effective range expansion: r_e : measure for energy dependence of the phase shift

$$\boxed{k \cot \eta_0 = -\frac{1}{a} + \tfrac{1}{2} k^2 r_e + \cdots}$$

Effective range: $$r_e = r_0 \left(1 - \frac{3ar_0 + \gamma^2 r_0^2 - 3\beta(a^2 - r_0^2) + 3\beta^2 (a - r_0)^2}{3a^2\gamma^2} \right)$$

broad resonance – open channel ($r_e > 0$ for large $|a|$):

$$\beta \to 0 \quad \rightarrow \quad r_e = r_0 \left(1 - \frac{3ar_0 + \gamma^2 r_0^2}{3a^2\gamma^2} \right) = r_0 \left(1 - \frac{r_0}{a\gamma^2} - \frac{r_0^2}{3a^2} \right)$$

narrow resonance – closed ($r_e < 0$ for large $|a|$):

$$|\beta| \to \infty \quad \rightarrow \quad r_e \simeq -r_0 \frac{\beta^2}{\gamma^2} \frac{(a - r_0)^2}{a^2} = -r_0 \frac{\beta^2}{\gamma^2} \left(1 - 2\frac{r_0}{a} + \frac{r_0^2}{a^2} \right)$$

Resonance-width parameter R^*

$$k\cot\eta_0 = -\frac{1}{a} + \tfrac{1}{2}k^2 r_e + \cdots$$

narrow resonance – closed channel ($r_e < 0$ for large $|a|$):

$$|\beta| \to \infty \quad \rightarrow \quad r_e = -r_0\frac{\beta^2}{\gamma^2}\left(1 - 2\frac{r_0}{a} + \frac{r_0^2}{a^2}\right) \simeq -2R^*$$

$$\boxed{R^* = -\tfrac{1}{2}r_e \simeq \tfrac{1}{2}r_0\frac{\beta^2}{\gamma^2}}$$

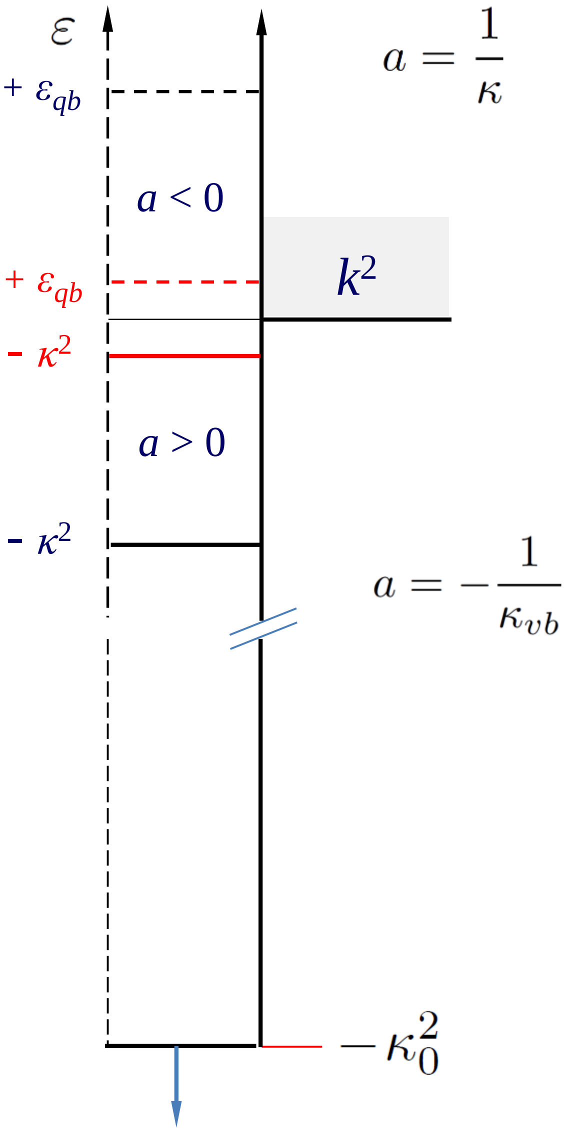
positive

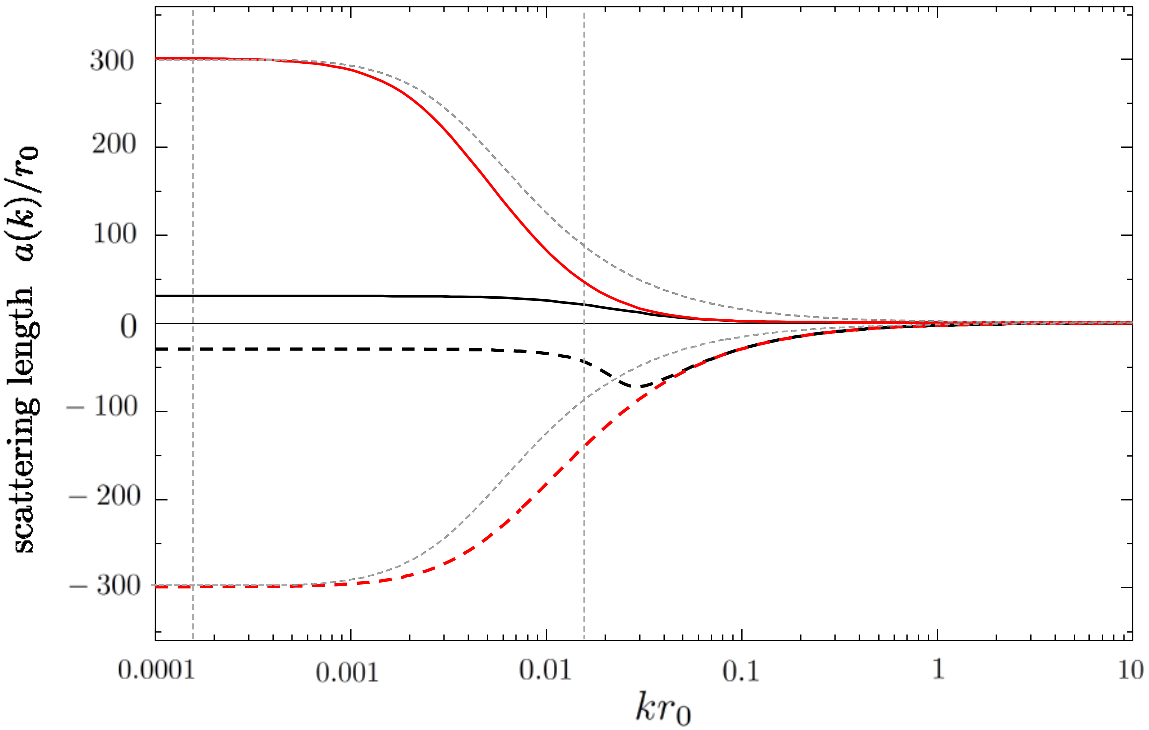
$$\boxed{k\cot\eta_0 = -\frac{1}{a} - k^2 R^*}$$

strong barrier – weak coupling – narrow resonance – closed channel $R^* >> r_0$

weak barrier – strong coupling – broad resonance – open channel $R^* << r_0$

narrow resonance near threshold





$$a(k) = r_0 - \frac{1}{k}\arctan\left(\frac{kr_0}{K_+ r_0 \cot K_+ r_0 - \beta}\right)$$

$$K_+^2 = k^2 + \kappa_0^2$$

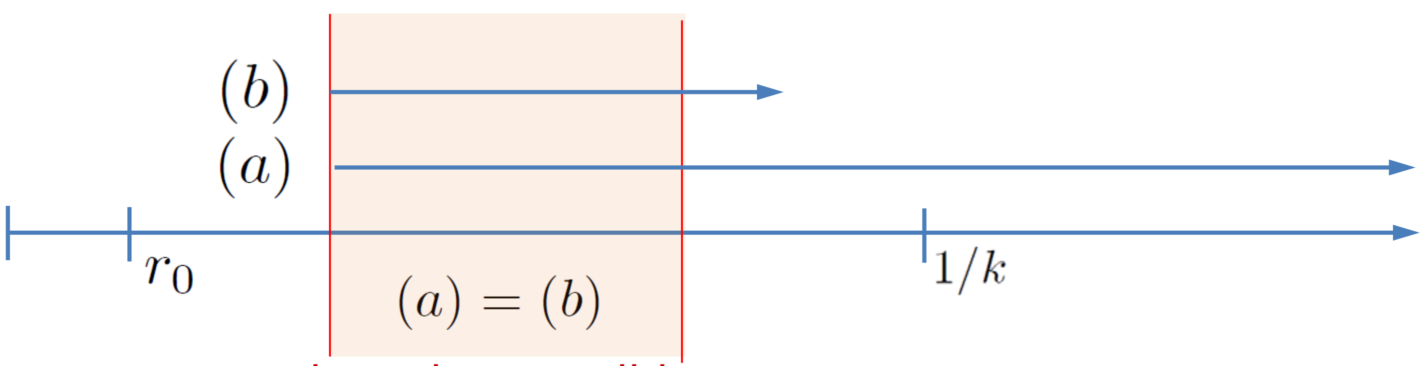
generalization: short-range potentials

$$R_l'' + \frac{2}{r}R_l' + \left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right]R_l = 0$$

$r \gg r_0 \longrightarrow$ $$R_l'' + \frac{2}{r}R_l' + \left[k^2 - \frac{l(l+1)}{r^2}\right]R_l = 0 \qquad \rightarrow (a)$$

$kr \ll 1 \longrightarrow$ $$R_l'' + \frac{2}{r}R_l' - \frac{l(l+1)}{r^2}R_l = 0 \qquad \rightarrow (b)$$

$$\longrightarrow kr_0 \ll kr \ll 1 \longrightarrow r_0 \ll r \ll 1/k$$



boundary condition

generalization: short-range potentials

$r \gg r_0$ $$R_l'' + \frac{2}{r}R_l' + \left[k^2 - \frac{l(l+1)}{r^2}\right]R_l = 0 \quad \rightarrow (a)$$

$kr \ll 1$ $$R_l'' + \frac{2}{r}R_l' - \frac{l(l+1)}{r^2}R_l = 0 \quad \rightarrow (b)$$

$$\rightarrow (a) \xrightarrow{r \gg r_0} R_l(r) = \alpha_l j_l(kr) + \beta_l n_l(kr)$$

$$kr \ll 1 \longrightarrow \quad = \alpha_l \frac{(kr)^l}{(2l+1)!!} + \beta_l \frac{(2l+1)!!}{(2l+1)}\left(\frac{1}{kr}\right)^{l+1}$$

$$\rightarrow (b) \xrightarrow{kr \ll 1} R_l(r) = c_{1l}\, r^l + c_{2l}\left(\frac{1}{r}\right)^{l+1}$$

$$(a) + (b) \longrightarrow \begin{cases} \alpha_l = A_l \cos\eta_l \simeq c_{1l}\dfrac{(2l+1)!!}{k^l} \\ \beta_l = A_l \sin\eta_l \simeq c_{2l}\dfrac{2l+1}{(2l+1)!!}k^{l+1} \end{cases}$$

generalization: short-range potentials

$$\alpha_l = A_l \cos\eta_l \simeq c_{1l}\frac{(2l+1)!!}{k^l}$$

$$\beta_l = A_l \sin\eta_l \simeq c_{2l}\frac{2l+1}{(2l+1)!!}k^{l+1}$$

generalization: short-range potentials

$$\left.\begin{aligned}\alpha_l &= A_l \cos\eta_l \simeq c_{1l}\,\frac{(2l+1)!!}{k^l}\\ \beta_l &= A_l \sin\eta_l \simeq c_{2l}\,\frac{2l+1}{(2l+1)!!}\,k^{l+1}\end{aligned}\right\} \rightarrow$$

$$\rightarrow \frac{\beta_l}{\alpha_l} = \boxed{\tan\eta_l \simeq -\frac{2l+1}{[(2l+1)!!]^2}(ka_l)^{2l+1}}$$

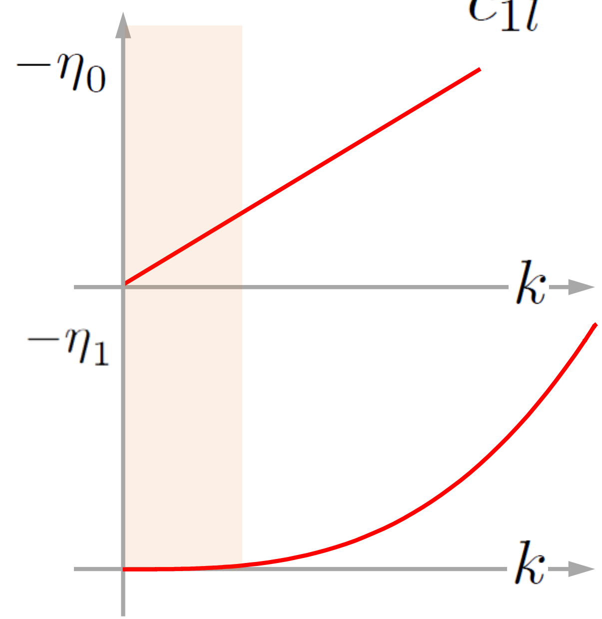
$$a_l^{2l+1} \equiv -\frac{c_{2l}}{c_{1l}}$$

$$l = 0 \;\rightarrow\; \tan\eta_0 \stackrel{ka \ll 1}{\simeq} -ka$$

$$l = 1 \;\rightarrow\; \tan\eta_1 \stackrel{ka_1 \ll 1}{\simeq} -\tfrac{1}{3}(ka_1)^3$$

But ...

$$\boxed{\mathcal{V}(r) = -\frac{C_s}{r^s} \;\rightarrow\; l < \tfrac{1}{2}(s-3)}$$



s-wave regime

Existance of short range

$$R_l'' + \frac{2}{r}R_l' + \left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right]R_l = 0$$

$kr \ll 1$

$$R_l'' + \frac{2}{r}R_l' - \frac{l(l+1)}{r^2}R_l = U(r)R_l \qquad \to (b)$$

$\to (b)$

$$R_l(r) = c_{1l}\, r^l + c_{2l}\left(\frac{1}{r}\right)^{l+1}$$

$$\mathcal{V}(r) = -\frac{C_s}{r^s}$$

$r \to \infty$

$$U(r)\left\{c_{1l}r^l + c_{2l}(\frac{1}{r})^{l+1}\right\} \sim c_{1l}r^{l-s} \to 0$$

compare with $c_{2l}\dfrac{l(l+1)}{r^2}(\dfrac{1}{r})^{l+1}$

$$c_{1l}r^{l-s} \ll c_{2l}(\frac{1}{r})^{l+3} \qquad \boxed{l < \tfrac{1}{2}(s-3)}$$

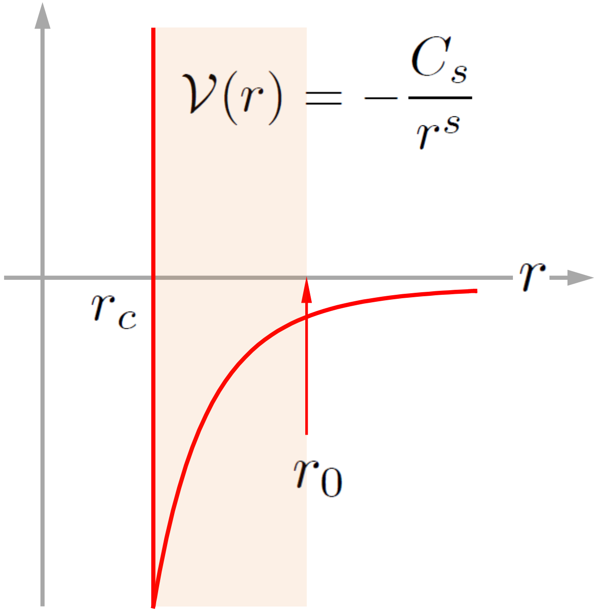
power-law potentials

Short range r_0 can only be defined for $l < \frac{1}{2}(s-3)$

$$l = 0 \longrightarrow s > 2l + 3 = 3$$

Van der Waals potential: $s = 6 \longrightarrow l < 1.5 \longrightarrow l = 0, 1$

Let us analyze the case of power-law potentials:



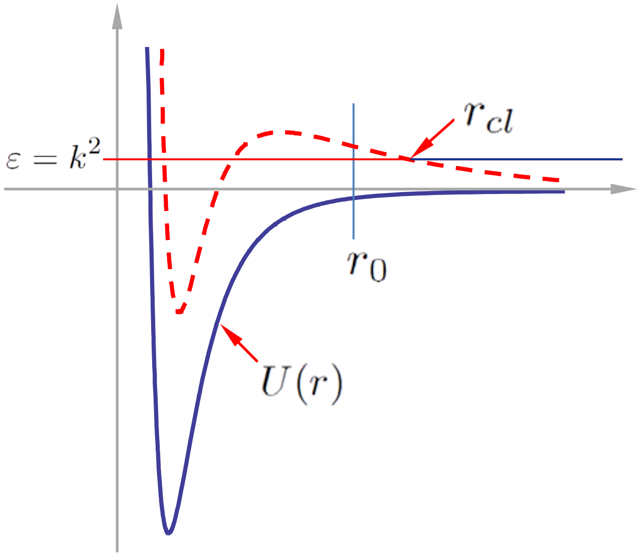
range: heuristic argument

range exist if $E_{\text{conf}} > E_{\text{pot}}$

$$\frac{\hbar^2}{2\mu r_0^2} > \frac{C_s}{r_0^s}$$

$$r_0^{s-2} \simeq 2\mu C_s/\hbar^2 \xrightarrow{s=6} \boxed{r_0 \simeq [2\mu C_6/\hbar^2]^{1/4}}$$

s-wave regime



Conclusion:

phase shift for short-range potentials:

$$l < \frac{1}{2}(s-3) \qquad \boxed{\tan \eta_l \underset{k\to 0}{\simeq} -\frac{2l+1}{[(2l+1)!!]^2}(ka_l)^{2l+1}}$$

$$l > \frac{1}{2}(s-3) \qquad \boxed{\sin \eta_l \underset{k\to 0}{\simeq} \kappa_c^2 r_c^2 \frac{3\pi(2l+3-s)!!}{(2l+5)!!}(kr_c)^{s-2}}$$

Relative motion of interacting particles II

1. We analyzed the energy dependence of the scattering length
2. We introduced the effective range r_e
3. We analyzed when the effective range is important
4. We discussed anomalously large and small scattering lengths
5. We analyzed s-wave resonances
6. We noticed that negative effective ranges did not appear
7. We introduced a tunnel barrier
8. We found that for a weak tunnel coupling $r_e < 0$
9. We introduced the width parameter R^* to discriminate between broad and narrow s-wave resonances
10. We generalized to arbitrary short range potentials
11. We derived a criterion for the existence of a short range

integral equation for the phase shift

$$\chi_l'' + \left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right]\chi_l = 0 \longrightarrow \chi_l(k,r) = kr R_l(k,r)$$

$$\chi_l'' + \left[k^2 - \frac{l(l+1)}{r^2}\right]\chi_l = 0 \longrightarrow \hat{j}_l(kr) = kr j_l(kr)$$

$$\left.\begin{aligned} \hat{j}_l\,\chi_l'' + \left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right]\hat{j}_l\,\chi_l = 0 \\ \chi_l\,\hat{j}_l'' + \left[k^2 - \frac{l(l+1)}{r^2}\right]\hat{j}_l\,\chi_l = 0 \end{aligned}\right\} \quad \hat{j}_l\,\chi_l'' - \chi_l\,\hat{j}_l'' = U(r)\,\hat{j}_l\,\chi_l$$

$$\int_0^\infty \left[\hat{j}_l(r)\chi_l''(r) - \chi_l(r)\hat{j}_l''(r)\right]dr = \int_0^\infty U(r)\hat{j}_l(r)\chi_l(r)dr$$

$$\left[\hat{j}_l(r)\chi_l'(r) - \chi_l(r)\hat{j}_l'(r)\right]_0^\infty = \int_0^\infty U(r)\hat{j}_l(r)\chi_l(r)dr$$

integral equation for the phase shift

$$\left[\hat{j}_l(r)\chi_l'(r) - \chi_l(r)\hat{j}_l'(r)\right]_0^\infty = \int_0^\infty U(r)\hat{j}_l(r)\chi_l(r)dr$$

integral equation for the phase shift

$$\left[\hat{j}_l(r)\chi_l'(r) - \chi_l(r)\hat{j}_l'(r)\right]_0^\infty = \int_0^\infty U(r)\hat{j}_l(r)\chi_l(r)dr$$

$$\chi_l(k,r) = krR_l(k,r) \underset{r\to 0}{\longrightarrow} 0$$

$$\hat{j}_l(kr) = krj_l(kr) \underset{r\to 0}{\longrightarrow} 0$$

$$\chi_l(k,r) \underset{r\to\infty}{\longrightarrow} \sin(kr - \tfrac{1}{2}l\pi + \eta_l)$$

$$\hat{j}_l(kr) \underset{r\to\infty}{\longrightarrow} \sin(kr - \tfrac{1}{2}l\pi)$$

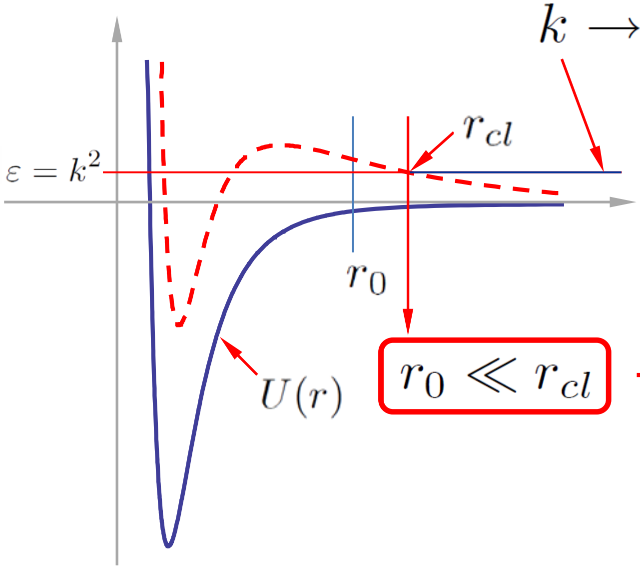
$$k[\sin(kr - \tfrac{1}{2}l\pi + \eta_l)\cos(kr - \tfrac{1}{2}l\pi) - \cos(kr - \tfrac{1}{2}l\pi + \eta_l)\sin(kr - \tfrac{1}{2}l\pi)]$$

$$\downarrow$$

$$-k\sin\eta_0$$

$$\boxed{\sin\eta_l = -\frac{1}{k}\int_0^\infty U(r)\chi_l(k,r)\hat{j}_l(kr)dr}$$

s-wave regime



$$\boxed{r_0 \ll r_{cl}}$$

$$\sin\eta_l = -\frac{1}{k}\int_0^\infty U(r)\chi_l(k,r)\hat{\jmath}_l(kr)dr$$

outside shape resonances

no phase shift for $l > 0$: $\chi_l(k,r) \to \hat{\jmath}_l(kr)$

Born approximation

$$\hat{\jmath}_l(kr) = \sqrt{\tfrac{\pi}{2}kr}\,J_{l+1/2}(kr)$$

$$U(r) = \frac{\kappa_c^2 r_c^s}{r^s}$$

$$\sin\eta_l \simeq -\frac{\pi}{2}\int_0^\infty \frac{\kappa_c^2 r_c^s}{r^s}\left[J_{l+1/2}(kr)\right]^2 r dr$$

$$= 6\kappa_c^2 r_c^2 (kr_c)^{s-2}\frac{(2l+3-s)!!}{(2l+5)!!}$$

$$\boxed{l > \tfrac{1}{2}(s-3)}$$