Basics of Fermi gas theory (M. Baranov)

Fermi :– fermionic statistics $\leftrightarrow$ Pauli principle : not more than one fermion in a quantum state.

Math. formulation : a_ν, a_ν^+ – annihilation/creation operator of a particle for the q. state ν:

ν – complete set of q. numbers for a single-particle state :

$$\{a_\nu, a_{\nu'}^+\} = \delta_{\nu\nu'} ; \quad \{a_\nu^+, a_{\nu'}^+\} = 0 \quad ; \{a_\nu, a_{\nu'}\} = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(a_\nu^+)^2 = 0 \qquad (a_\nu)^2 = 0$$ – Pauli principle.

Gas :– dilute system : range of the interparticle interaction $r_0 \sim a_s$ is much smaller than the interparticle separation $n^{-1/3}$ (n – concentration) :

$$a_s \ll n^{-1/3} \text{ or } a_s n^{1/3} \ll 1 \text{ or } n a_s^3 \ll 1 .$$

Popular model for a short range interaction :

$$V(\vec r) = g\,\delta(\vec r) \quad , \text{with } g = \frac{4\pi\hbar^2}{m} a_s$$

Short-range interaction $\Longrightarrow$ – only s-wave scattering (with a_s) is relevant for low energies or temperatures.

One needs at least two components to have interaction effects in a Fermi gas.

Ideal Fermi gas (single component)

$\hat{H}_0 = \sum_{\vec p} \varepsilon_p a^+_{\vec p} a_{\vec p}$; $\varepsilon_p = \frac{p^2}{2m}$ ‖ N particles in a large box V: $N \to \infty$; $V \to \infty$; $n = \frac{N}{V}$ - fixed. ‖

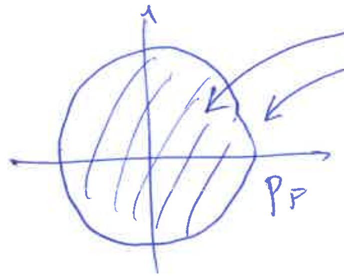
$\hat{N} = \sum_{\vec p} \underbrace{a^+_{\vec p} a_{\vec p}}_{\hat n_p}$ - number operator (fixed to N)

<u>Ground state</u>: $|G\rangle$ - state with the lowest energy

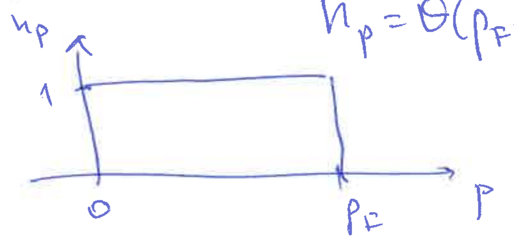
$|G\rangle = \prod_{\vec p: p < p_F} a^+_{\vec p} |0\rangle$ - filled Fermi sphere

$n_p = \langle G | a^+_{\vec p} a_{\vec p} | G\rangle = 1$

$n_p = 0$



$n_p = \Theta(p_F - p)$



p_F - Fermi momentum; ε_F ($= \frac{p_F^2}{2m}$) - Fermi energy; $|\vec p| = p_F$ - Fermi surface.

p_F - ? $\quad N = \sum_{p<p_F} 1 = V \int \frac{d\vec p}{(2\pi\hbar)^3} n_p = V \frac{4\pi}{(2\pi\hbar)^3} \frac{p_F^3}{3}$

$\Rightarrow \boxed{n = \frac{p_F^3}{6\pi^2\hbar^3}}$ or $\boxed{p_F = \hbar (6\pi^2 n)^{1/3}}$ $\sim n^{1/3}$ - depends only on n

Basis of the local-density approximation.

<u>Ground state energy</u>: $E_0 = N \frac{3}{5} \varepsilon_F$ - large kinetic energy (Pauli principle)

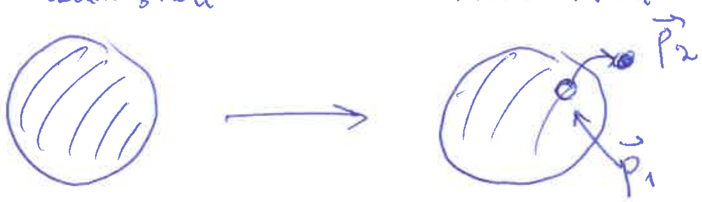
<u>Pressure</u>: $p = -\frac{\partial E_0}{\partial V} = \frac{2}{5} n \varepsilon_F \neq 0$

<u>Chemical potential</u>: $\mu = \frac{\partial E_0}{\partial N} = \varepsilon_F$ (at $T=0$!)

Density of states at the Fermi surface per unit volume: $\nu_F = \frac{1}{V} \sum_{\vec p} \delta(\varepsilon_p - \varepsilon_F) = \frac{m p_F}{2\pi^2\hbar^3}$ - constant ~~[illegible]~~

Excitations:

Ground state $\longrightarrow$ Excited state



$p_1 < p_F$ and $p_2 > p_F$! (Pauli principle!)

$$E - E_0 = \frac{p_2^2}{2m} - \frac{p_1^2}{2m} = \underbrace{\frac{p_2^2 - p_F^2}{2m}}_{\text{particle exc.}} + \underbrace{\frac{p_F^2 - p_1^2}{2m}}_{\text{hole excitation.}} = E_{p_1} + E_{p_2}$$

Particle excitation: $a^+_{\vec{p}}|G\rangle$ ($\neq 0$ only for $p > p_F$!) – has $N+1$ particles

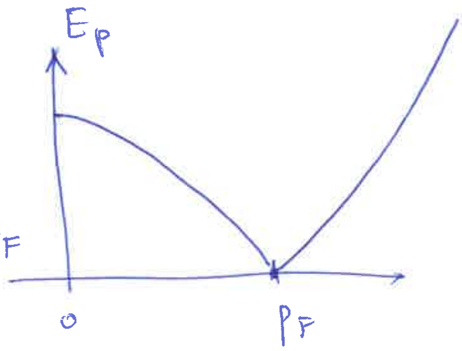
$$E_p = \varepsilon_p + \underbrace{E_0(N) - E_0(N+1)}_{-\mu} = \varepsilon_p - \mu > 0 \; ; \; \mu = \varepsilon_F$$

Hole excitation: $a_{\vec{p}}|G\rangle$ ($\neq 0$ only for $p < p_F$!) – – has $N-1$ particles.

$$E_p = \overbrace{E_0(N) - \varepsilon_p - E_0(N-1)}^{\mu} = \mu - \varepsilon_p > 0$$

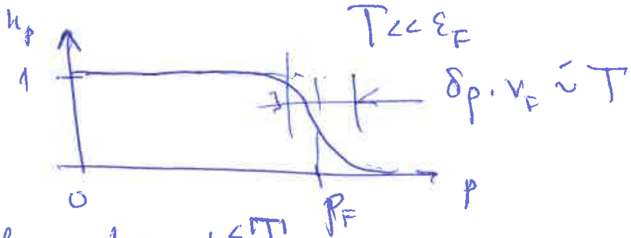
In general $E_p = \left| \frac{p^2 - p_F^2}{2m} \right|$ – – gapless!

$\approx \frac{p_F}{m}|p - p_F|$ for $p \approx p_F$



Finite temperature $T \ll \varepsilon_F$

$$n_p = \frac{1}{e^{\frac{\varepsilon_p - \mu}{T}} + 1} \; -$$



– Smooth and essentially differs from $\Theta(p_F - p)$ only for $|\varepsilon_p - \mu| \lesssim T$

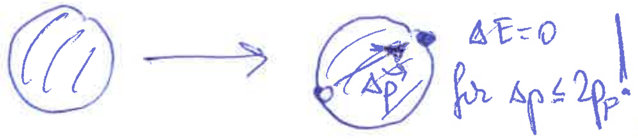
$$\nu_F = \frac{m p_F}{2\pi^2 \hbar^3}$$

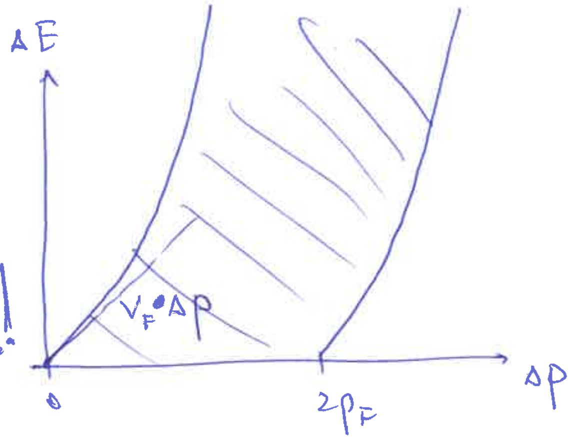
$$\mu(T) = \varepsilon_F \left(1 - \frac{\pi^2}{12} \frac{T^2}{\varepsilon_F^2}\right)$$

$$E(T) = E_0 + \frac{\pi^2}{6} \nu_F T^2 \Longrightarrow c_V = \frac{\pi^2}{3} \nu_F T \sim T \; !$$

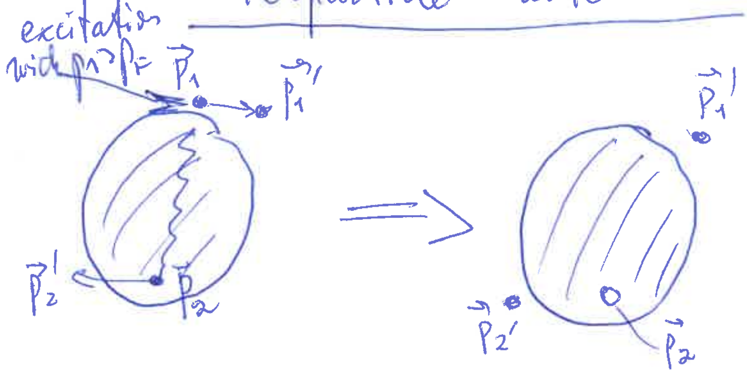
Consequences of the Fermi sphere/surface.

1. Specific heat $C_v \sim T$
2. Transferring momentum without transferring energy





3. Slow decay (small collisional rate) of single-particles excitations near the Fermi surface, due to interparticle interactions (=> Landau Fermi liquid) :
(need two component).



Pauli principle : $p_1' > p_F$
$p_2 < p_F$; $p_2' > p_F$

Energy conservation: $\varepsilon_{p_1} + \varepsilon_{p_2} = \varepsilon_{p_1'} + \varepsilon_{p_2'}$

or $\quad \varepsilon_{p_1} - \varepsilon_F = \underbrace{(\varepsilon_{p_1}' - \varepsilon_F)}_{>0} + \underbrace{(\varepsilon_{p_2'} - \varepsilon_F)}_{>0} + \underbrace{(\varepsilon_F - \varepsilon_{p_2})}_{>0}$

or $\quad E_{p_1} = E_{p_1'} + E_{p_2'} + E_{p_2}$ where all $E_p > 0$

These conditions strongly reduce the available collisional final states (Pauli blocking) due to Pauli principle.

As a result : $\frac{1}{\tau} = \frac{1}{\tau_{cl}} \max\left\{\left(\frac{E_p}{\varepsilon_F}\right)^2, \left(\frac{T}{\varepsilon_F}\right)^2\right\} \ll \frac{1}{\tau_{cl}}$; $\frac{1}{\tau_{cl}} = n a_s^2 v_F$

lifetime $\to \tau$

Consequences : 1. Excitations near F. S. are long-lived.
2. Long times required to reach local equilibrium and, therefor, hydrodynamic regime $\omega\tau \ll 1$ ~~[illegible]~~ (no hydrodynamics!)

4. BCS

Weakly interacting Fermi gas (two component)

For simplicity: two components $m_+ = m_- = m$; $N_+ = N_- = N$
in a large "box" of volume V: ($V \to \infty$; $N \to \infty$; $n = \frac{N}{V}$ - fixed)

$n_+ = n_- = n = \frac{N}{V}$ - fixed.

For contact interaction

$$\hat{H} = \hat{H}_0 + \underbrace{g \int d\bar{r}\, \hat{n}_+(\bar{r})\, \hat{n}_-(\bar{r})}_{\hat{H}_{int}} \qquad \nu \equiv \{\vec{p}, \sigma\};\ \sigma = \pm$$

$$= \sum_{\bar{p},\sigma} \varepsilon_p a^+_{\bar{p}\sigma} a_{\bar{p}\sigma} + \frac{g}{V} \sum_{\bar{p}_1 \bar{p}_2 \bar{q}} a^+_{\bar{p}_1+\bar{q},+} a^+_{\bar{p}_2-\bar{q},-} a_{\vec{p}_2 -} a_{\vec{p}_1 +}$$

Ground state energy to the 1^{st} order:

$|G\rangle = \prod_{\bar{p},\sigma;\ p<p_F} a^+_{\bar{p}\sigma}|0\rangle$ - two filled F. spheres:

$$E = \langle G|\hat{H}|G\rangle = N_+ \frac{3}{5}\varepsilon_{F_+} + N_- \frac{3}{5}\varepsilon_{F_-} + gV n_+ n_- \qquad \text{mean-field}$$

$$= 2N \frac{3}{5}\varepsilon_F + gVn^2 = 2N\left\{\frac{3}{5}\varepsilon_F + \frac{1}{2}gn\right\}$$

For chemical potentials $\mu_\pm = \frac{\partial E}{\partial n_\pm} = \varepsilon_{F_\pm} + g n_\mp = \varepsilon_F + gn$

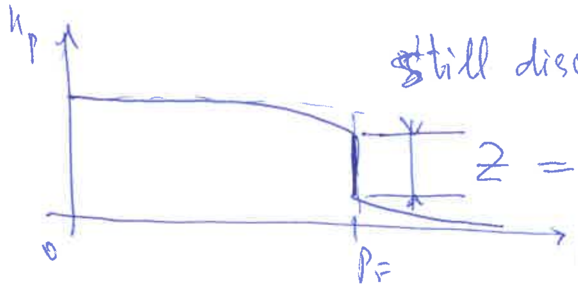
Small parameter: $\frac{gn}{\varepsilon_F} \approx \frac{4\pi\hbar^2}{m} a_s \frac{p_F^3}{6\pi^2\hbar^3} \frac{2m}{p_F^2} \sim \frac{a p_F}{\hbar} \sim a n^{1/3} \ll 1$!

Interaction energy is small compared to the large kinetic energy in the Fermi sea!

But: effects are different for $g > 0$ and $g < 0$.

Repulsive interaction : $g>0$ (or $a_s>0$) Normal F. gas

Fermi-liquid renormalizations of single-particle properties and collective mode – Landau zero sound:

1\. 

still discontinuous at p_F

$$Z = 1 - \frac{8\ln 2}{\pi^2}\left(\frac{ap_F}{\hbar}\right)^2$$

2\. Single-particle excitation for $p \simeq p_F$:

$$E_p \simeq \frac{p_F}{m_*}|p-p_F| \; ; \; m_* = m\left[1+\frac{8}{15\pi^2}(7\ln 2-1)\left(\frac{ap_F}{\hbar}\right)^2\right]$$

Single-particle properties are equivalent to those of the ideal gas of quasiparticles with m_* and p_F (same density)

3\. Collective mode – Landau zero sound:

$$\omega_k = ck \quad \text{with} \quad c = v_F\left(1+2e^{-2/\lambda}\right); \quad \lambda \equiv \nu_F g = \frac{2a_s p_F}{\pi\hbar} \ll 1.$$

For this collective mode: $\omega\tau \gg 1$ (high-frequency sound)

This is therefore not a thermodynamic sound with $\omega\tau \ll 1$ and $c_{th} = v_F/\sqrt{3}$

Landau zero sound = coherent motion of particle-hole excitations near the Fermi surface.

It is not the density wave.

Attractive interaction $g<0$ $(a_s<0)$

Cooper pairing and superfluidity.

For the F.S. ground state, interaction leads to exp. growth of the correlators: $\langle \hat\psi_+(\bar r,t)\hat\psi_-(\bar r,t)\hat\psi_-^+(\bar r,0)\hat\psi_+^+(\bar r,0)\rangle \sim \exp\{\frac{\Delta}{\hbar}t\}$

$\Rightarrow$ ground state in the form of a Fermi surface is unstable.

Phase transition: $\Delta = g\langle\hat\psi_-(\bar r)\hat\psi_+(\bar r)\rangle = \frac{g}{V}\sum_p\langle a_{-\bar p-}a_{\bar p+}\rangle$ – the order parameter.

Hamiltonian in the Bogoliubov approximation:

$$g\hat\psi_-^+(\bar r)\hat\psi_-(\bar r)\hat\psi_+^+(\bar r)\hat\psi_+(\bar r) \rightarrow \Delta\hat\psi_-^+(\bar r)\hat\psi_+^+(\bar r)+\Delta^*\hat\psi_-(\bar r)\hat\psi_+(\bar r)$$

real Δ

$$\hat H_{Bog} = \hat H - \mu\hat N \Longrightarrow \sum_{\bar p,\sigma}(\varepsilon_p-\mu)a^+_{\bar p\sigma}a_{\bar p\sigma} + \Delta\sum_{\bar p}\left(a_{-\bar p-}a_{\bar p+}+a^+_{\bar p+}a^+_{-\bar p-}\right)$$

pairs of particles with opposite momentum appear and disappear in the collective degree of freedom Δ.

Diagonalization via Bogoliubov transformation

$$\hat H_{Bog} = E_0 + \sum_{\bar p,\sigma}E_p\,\alpha^+_{\bar p\sigma}\alpha_{\bar p\sigma}$$

New ground state $\alpha_{\bar p\sigma}|G\rangle = 0$

where $\alpha_{\bar p+} = u_p a_{\bar p+} + v_p a^+_{-\bar p-}$; $\alpha_{\bar p-} = u_p a_{\bar p-} - v_p a^+_{-\bar p+}$

$\alpha^+_{\bar p+} = u_p a^+_{\bar p+} + v_p a_{-\bar p-}$; $\alpha^+_{\bar p-} = u_p a^+_{\bar p-} - v_p a_{-\bar p+}$

new fermionic annihilation/creation operators for excitations with energies

$$E_p = \sqrt{(\varepsilon_p-\mu)^2+\Delta^2}\ ; \quad u_p^2 = \frac12\left[1+\frac{\varepsilon_p-\mu}{E_p}\right],\quad v_p^2 = \frac12\left[1-\frac{\varepsilon_p-\mu}{E_p}\right]$$

Gap equation (equation on Δ)

$$\Delta = \underbrace{-g}_{|g|} \int \frac{dp}{(2\pi\hbar)^3} \left[\frac{\tanh E_p/2T}{2E_p} - \frac{m}{p^2} \right] \Delta \quad \text{(BCS model)}$$

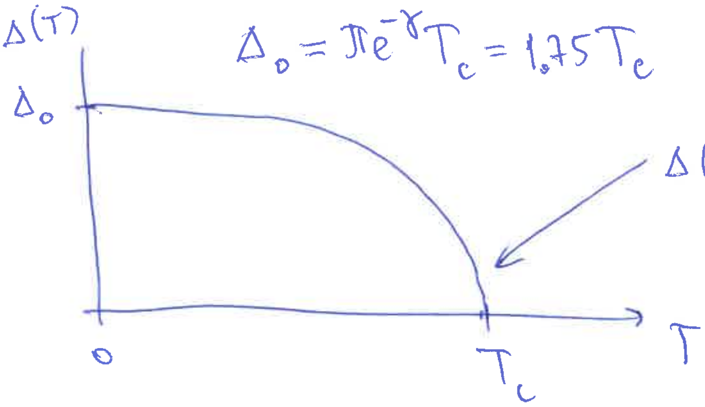
Non-trivial solution $\Delta = 0$ exists for $T < T_c$

$$T_c^{BCS} = \frac{e^{\gamma}}{\pi} 8 e^{-2} \varepsilon_F e^{-1/\lambda} = 0.61\, \varepsilon_F e^{-1/\lambda} \quad ; \quad \lambda = |g| \nu_F = \frac{2|a_s| p_F}{\pi \hbar}$$

$$T_c^{Gor} = \frac{e^{\gamma}}{\pi} \left(\frac{2}{e}\right)^{7/3} \varepsilon_F e^{-1/\lambda} = 0.28\, \varepsilon_F e^{-1/\lambda}$$

$\gamma = 0.5772$ - Euler constant

$$\Delta_0 = \pi e^{-\gamma} T_c = 1.75\, T_c \quad (T_c = 0.57\, \Delta_0)$$



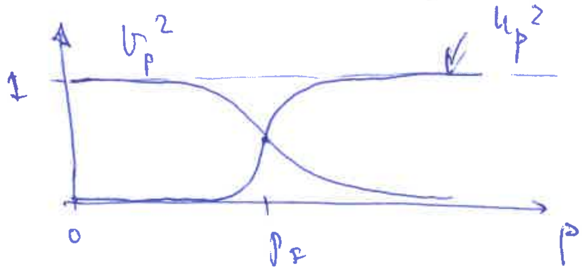
$$\Delta(T) = 3.06\, T_c \sqrt{1 - T/T_c} \quad \left\{ \sqrt{\frac{8\pi^2}{7\zeta(3)}} = 3.06 \right.$$

Second order phase transition!

New ground state: $\alpha_{\bar p\sigma}|G\rangle = 0$!

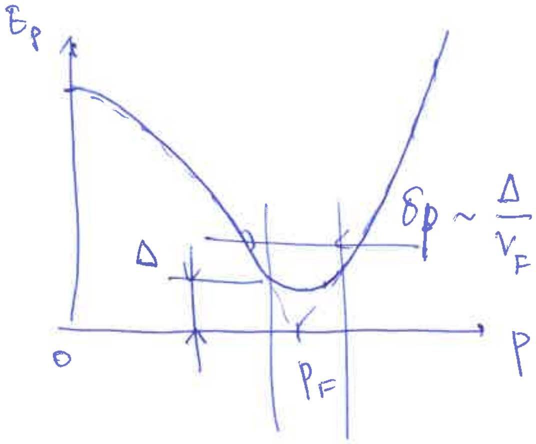
$|G\rangle = \prod_{\bar p}(u_p + v_p a^+_{-\bar p -} a^+_{\bar p +})|0\rangle$ - pair population.

E_0 - energy of the ground state.



$n_{p\sigma} = \langle a^+_{\bar p\sigma} a_{\bar p\sigma}\rangle \xrightarrow{T=0} v_p^2$ - continuous.

Excitation energy E_p: - gapped with Δ



Correlation length

$$\xi \sim \frac{\hbar}{\delta p} = \frac{\hbar v_F}{\Delta} \sim \frac{\hbar}{p_F}\frac{v_F p_F}{\Delta} \sim n^{-1/3}\frac{\varepsilon_F}{\Delta}$$

$\gg n^{-1/3}$ - strongly overlapping Cooper pairs.

Collective excitations (Goldstone modes)

are phase fluctuations of $\Delta(\bar r, t) \simeq \Delta\, e^{i\varphi(\bar r, t)}$

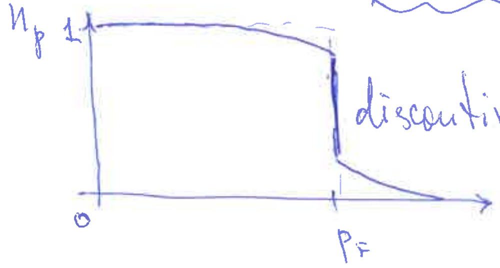
$\omega_k = ck$ with $c \simeq v_F/3$ - Bogolinbov-Anderson sound.

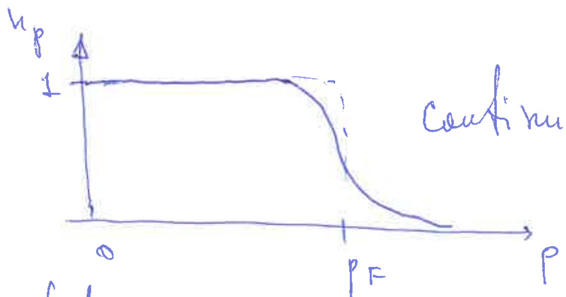
Superfluidity (Landau criterion)

$\min_p \frac{E_p}{p} \approx \frac{\Delta}{p_F} \neq 0 \Rightarrow$ superfluid.

Comparison of normal and superfluid Fermi gases

normal | n_p for $T=0$ | **superfluid**

1. 



2. single-particle excitations

gapless | gapped

3. Collective modes

Landau zero sound | Bogoliubov-Anderson sound (Goldstone mode)

$c = v_F\left(1+2e^{-2/\lambda}\right)$ | $c = v_F/\sqrt{3}$

4. Excitation density of states

continuous | energy gap 2Δ

5. Specific heat

$C_V \sim T$ | fermionic contribution $\sim e^{-\Delta/T}$ (collective modes $\sim T^3$)