Heavy Quark Masses from QCD Sum Rules
(with calibrated uncertainty)

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Work ongoing in collaboration with
Jens Erler and Hubert Spiesberger
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fundamental QCD parameters
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Outline

- Motivation and Introduction
- Using Sum Rules to extract m_Q
  - overview
  - our proposal for *charm* and *bottom*
- Conclusions and outlook

Motivation: why precise m_Q?

Higgs decay $\sim \overline{m_b}(M_H)^2$

$\Gamma(B \to X_u l\nu) \sim G_F^2 m_b^5 |V_{ub}|^2$

$\Gamma(B \to X_c l\nu) \sim G_F^2 m_b^5 f(m_c^2/m_b^2)|V_{cb}|^2$

$B \to K(^*)\ell\ell$
$B \to D(^*)\ell\nu$ (pQCD contributions on FFs depend on m_q)

Yukawa unification [Baer et al '00]

$$\frac{\delta m_b}{m_b} \sim \frac{\delta m_t}{m_t} \qquad \text{if } \delta m_t \sim 1\text{GeV} \Rightarrow \delta m_b \sim 25\text{MeV}$$

Motivation: why precise m_Q?

Υ-spectroscopy

$$m(\Upsilon(1S)) = 2M_b - \mathcal{C}\alpha^2 M_b + \cdots$$

[Ayala et al '14]

lattice: HPQCD '14

$$\overline{m_c}(3\mathrm{GeV}) = 986(6)\mathrm{MeV}$$

$$\overline{m_b}(10\mathrm{GeV}) = 3617(25)\mathrm{MeV}$$

QCD Sum Rules

$$\int \frac{\mathrm{d}s}{s^{n+1}} R_q(s) \sim \left(\frac{1}{m_q}\right)^{2n}$$

Motivation: why precise m_Q?

$\overline{m_c}(\overline{m_c})$MeV	method	reference
1223 ± 33	N^3LO quarkonium	Peset et al, 1806.05197
1273 ± 10	lattice ($N_f = 4$) + HQET	Fermilab-MILC-TUMQCD 1802.04248
$1335 \pm 43^{+40}_{-11}$	HERA DIS	xFitter, 1605.01946
1246 ± 23	quarkonium 1S	Kiyo et al, 1510.07072
1288 ± 20	1st moment SR $J/\Psi, \Psi, R$	Dehnadi et al, 1504.07638
1271.5 ± 9.5	lattice ($N_f = 4$), PS current	HPQCD, 1408.4169
1348 ± 46	lattice (2+1+1), M_D	ETM, 1403.4504
1274 ± 36	lattice ($N_f = 2$), f_D	ALPHA, 1312.7693
1240 ± 50	$c\bar{c}$ X-section DIS	Alekhin et al, 1310.3059
1260 ± 65	$c\bar{c}$ X-section NLO fit	HI and ZEUS, 1211.1182
1262 ± 17	SR $J/\Psi, \Psi(2S - 6S)$	Narison, 1105.5070
1260 ± 36	lattice (2+1), f_D	PACS-CS, 1104.4600
1278 ± 9	SR $J/\Psi, \Psi, R$	Bodenstain et al, 1102.3835
1282 ± 24	1st moment SR $J/\Psi, \Psi, R$	Dehnadi et al, 1102.2264
1280 ± 70	lattice + pQCD in static potential	Laschka et al, 1102.0945
1279 ± 13	1st moment SR $J/\Psi, \Psi, R$	Chetyrkin et al, 1010.6157
$1.275^{0.025}_{-0.035}$GeV	PDG average	PDG 2018

Motivation: why precise m_Q?

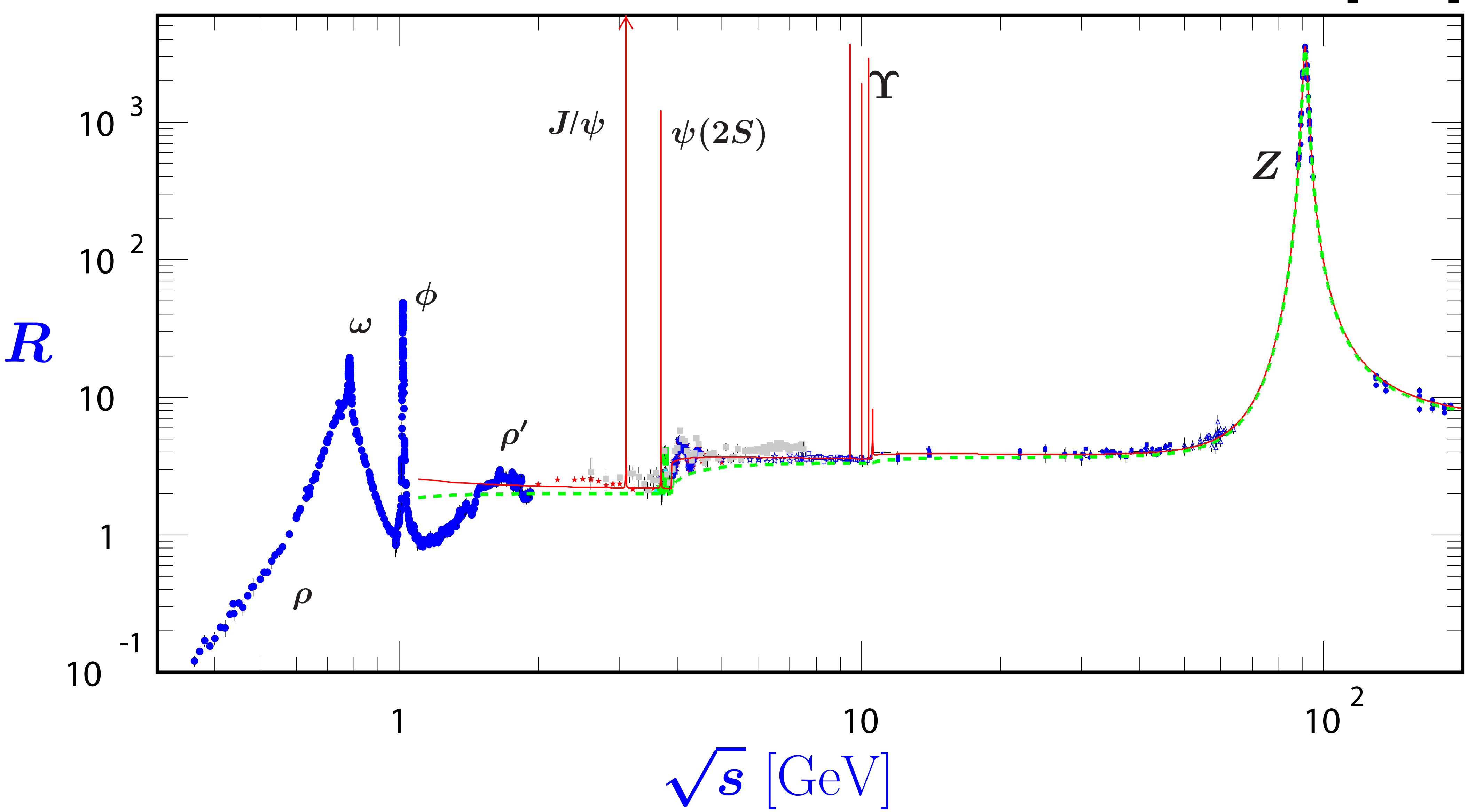
$\overline{m_b}(\overline{m_b})$	method	reference
4186 ± 37	N^3LO quarkonium	Peset et al, 1806.05197
4195 ± 14	lattice ($N_f = 4$) + HQET	Fermilab-MILC-TUMQCD 1802.04248
4197 ± 22	N^2LO pQCD, M_Υ	Kiyo et al, 1510.07072
4176 ± 23	SR $\Upsilon(1S - 4S)$, R	Dehnadi et al, 1504.07638
4183 ± 37	B decays	Alberti et al, 1411.6560
4203^{+16}_{-34}	N^3LO pQCD, M_Υ	Beneke et al, 1411.3132
4174 ± 24	lattice ($N_f = 4$), PS current	HPQCD, 1408.4169
4201 ± 43	N^3LO pQCD, M_Υ	Ayala et al, 1407.2128
4070 ± 170	ZEUS Coll.	Abramowicz et al, 1405.6915
4169 ± 9	SR $\Upsilon(1S - 6S)$	Penin, Zerf, 1401.7035
4247 ± 34	SR, f_B	Lucha et al, 1305.7099
4166 ± 43	lattice + pQCD, M_Υ, M_{B_s}	HPQCD, 1302.3739
4235 ± 55	SR $\Upsilon(1S - 6S)$, R	Hoang et al, 1209.0450
4171 ± 9	SR $\Upsilon(1S - 6S)$, R	Bodenstain et al, 1111.5742
4177 ± 11	SR $\Upsilon(1S - 6S)$	Narison, 1105.5070
4180 ± 50	lattice + pQCD in static potential	Laschka et al, 1102.0945
4163 ± 16	2nd moment SR $\Upsilon(1S - 6S)$, R	Chetyrkin et al, 1010.6157
$4.18^{+0.04}_{-0.03}$	PDG average	PDG 2018

QCD Sum Rules

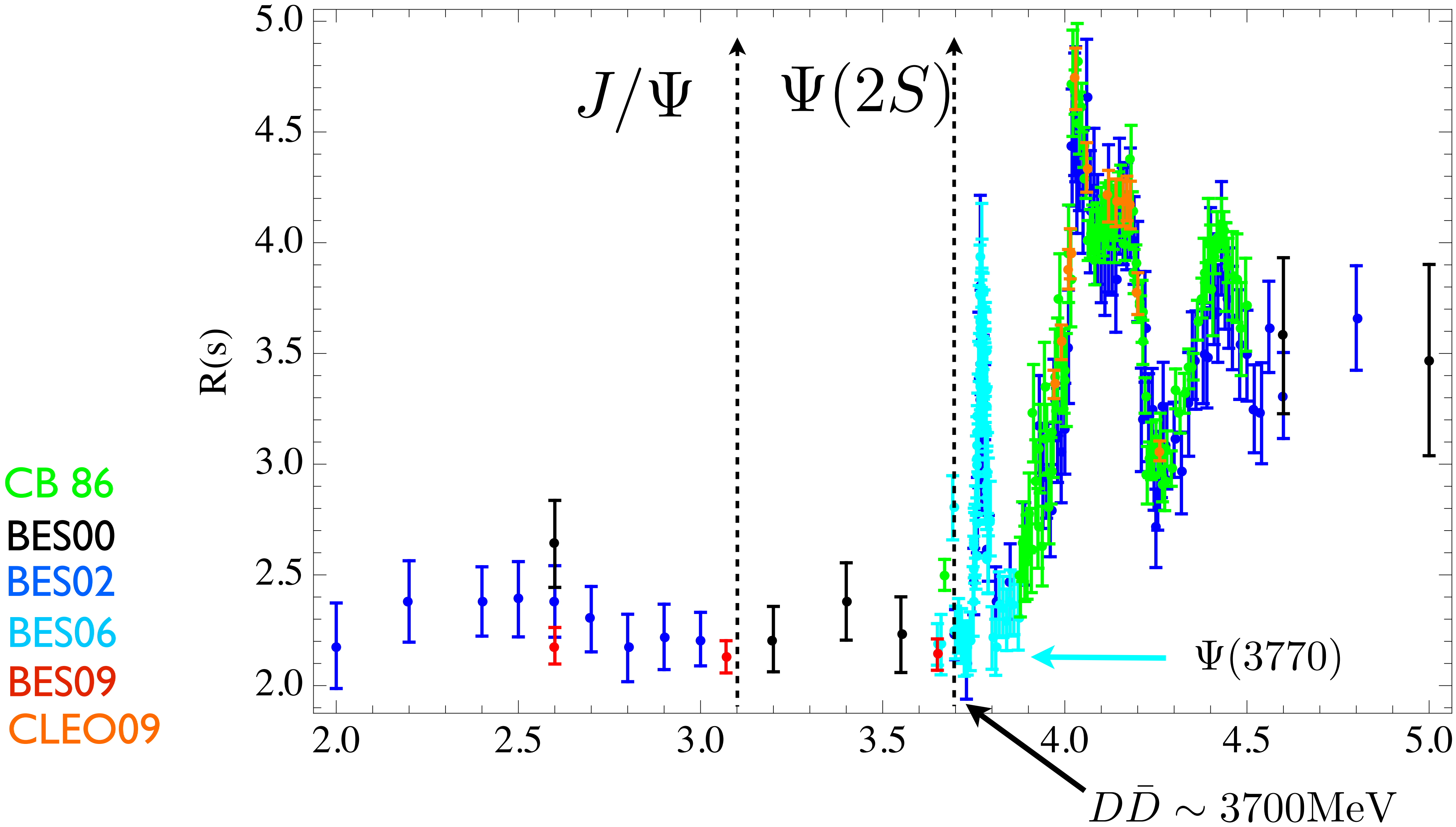
QCD Sum Rules

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \qquad \sigma(e^+e^- \to \mu^+\mu^-) = 4\pi\alpha_{\rm em}(s)^2/3s$$

[PDG]

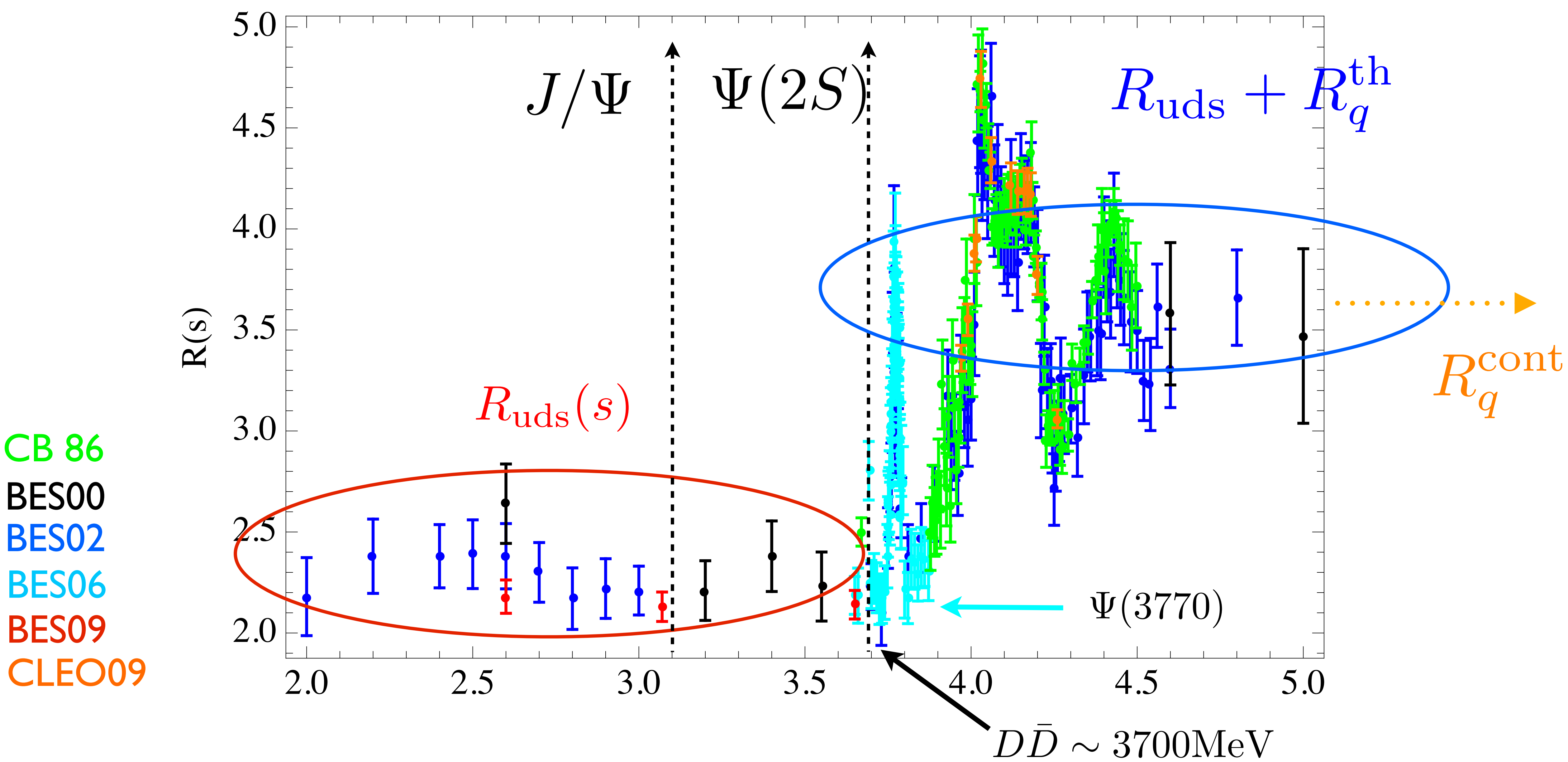


QCD Sum Rules



QCD Sum Rules

$$R(s) = R_{\rm uds}(s) + R_q(s) \qquad R_q(s) = R_q^{\rm Res}(s) + R_q^{\rm th}(s) + R_q^{\rm cont}(s)$$



QCD Sum Rules

Using the optical theorem: [SVZ,'79]

$$R(s) = 12\pi \mathrm{Im}[\Pi(s + i\epsilon)]$$

$\Pi_q(s)$ is the correlator of two heavy-quark vector currents which can be calculated in pQCD order by order and satisfies a Dispersion Relation:

$$12\pi^2 \frac{\hat{\Pi}_q(0) - \hat{\Pi}_q(-t)}{t} = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q(s)}{s+t}$$

$\hat{\Pi}_q(s)$ in $\overline{MS}$

For t→0

$$\boxed{\mathcal{M}_n := \frac{12\pi^2}{n!} \frac{d^n}{dt^n} \hat{\Pi}_q(t)\Big|_{t=0} = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1}} R_q(s)}$$

QCD Sum Rules

$\hat{\Pi}_q(s)$ can be Taylor expanded:

$$\Pi_q(t) = Q_q^2 \frac{3}{16\pi^2} \sum_{n\geq 0} \bar{C}_n \left(\frac{t}{4\hat{m}_q^2}\right)^n$$

QCD Sum Rules

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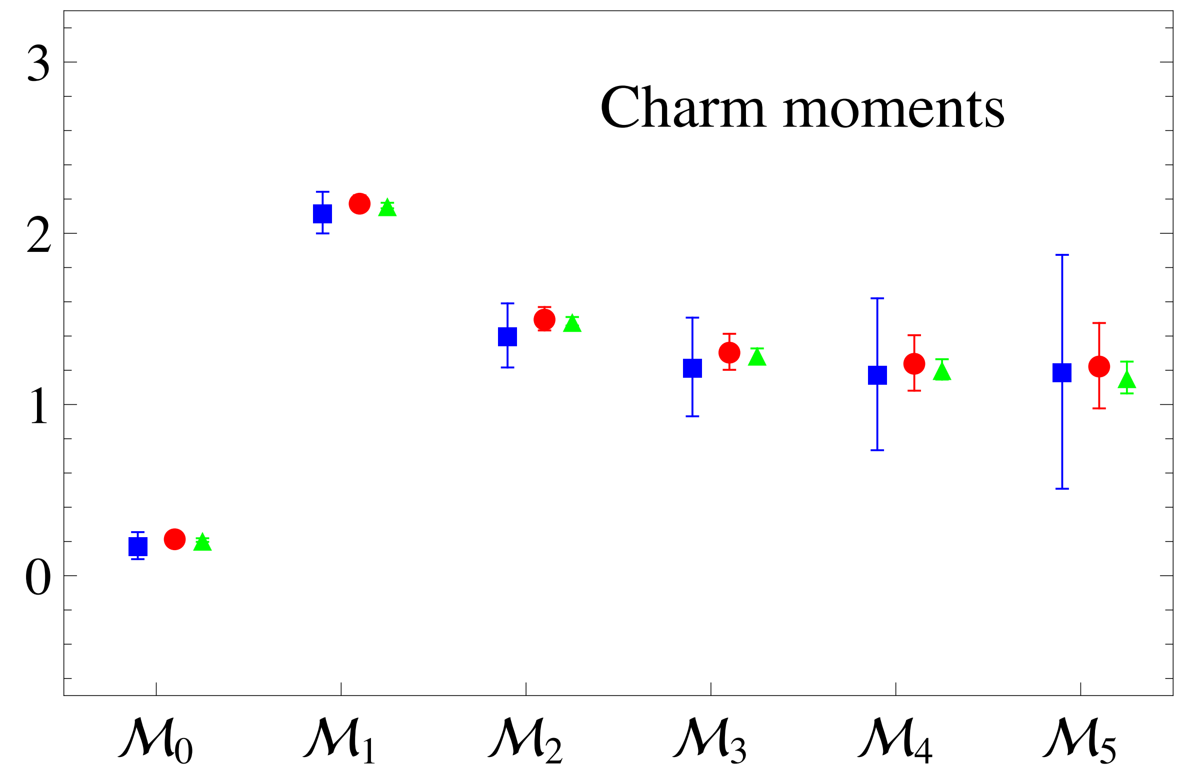


$$\mathcal{M}_n^{\mathrm{pQCD}} = \frac{9}{4} Q_q^2 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n} \bar{C}_n$$

$$\hat{\alpha} = \hat{\alpha}(\overline{m_q})$$

$$\bar{C}_n = \bar{C}_n^{(0)} + \left(\frac{\hat{\alpha}}{\pi}\right) \bar{C}_n^{(1)} + \left(\frac{\hat{\alpha}}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\hat{\alpha}}{\pi}\right)^3 \bar{C}_n^{(3)} + \mathcal{O}\left(\frac{\hat{\alpha}}{\pi}\right)^4$$

[Maier et al, '08]
[Chetyrkin, Steinhauser'06]
[Melnikov, Ritberger'03]

[Kiyo et al '09]
[Hoang et al '09]
[Greynat et al '09]

QCD Sum Rules

Sum Rules:

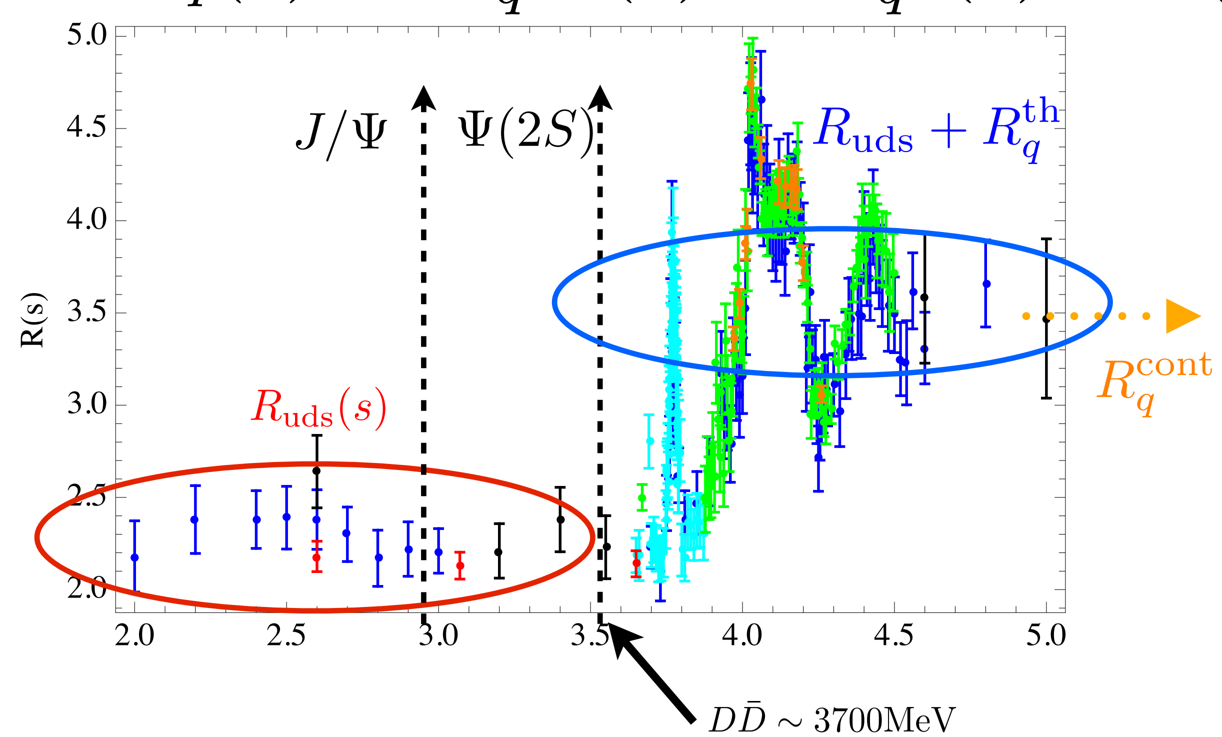
$$\boxed{\mathcal{M}_n = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1}} R_q(s)}$$

L.h.s. from theory

$$\mathcal{M}_n^{\mathrm{pQCD}} = \frac{9}{4} Q_q^2 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)} \right)^{2n} \bar{C}_n$$

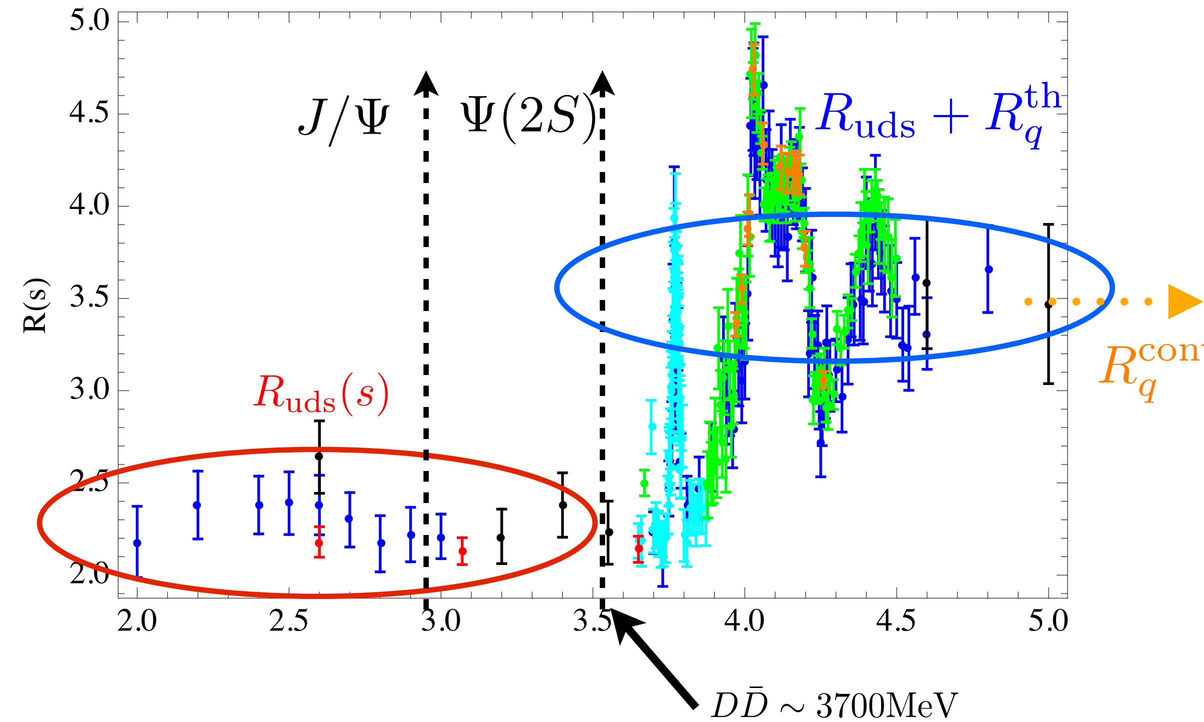
R.h.s. from experiment

$$R_q(s) = R_q^{\mathrm{Res}}(s) + R_q^{\mathrm{th}}(s) + R_q^{\mathrm{cont}}(s)$$



QCD Sum Rules

$$R_q(s) = R_q^{\rm Res}(s) + R_q^{\rm th}(s) + R_q^{\rm cont}(s)$$



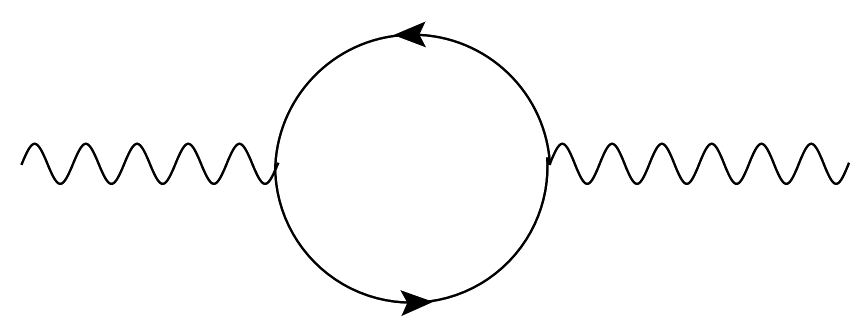
$$R_q^{\rm Res}(s) = \frac{9\pi M_R \Gamma_R^e}{\alpha_{\rm em}^2(M_R)} \delta(s - M_R^2)$$

$$R_q^{\rm th}(s) = R_q(s) - R_{\rm background}$$

$$(2M_D \leq \sqrt{s} \leq 4.8{\rm GeV})$$

$R_q^{\rm cont}(s)$ **calculated using pQCD**

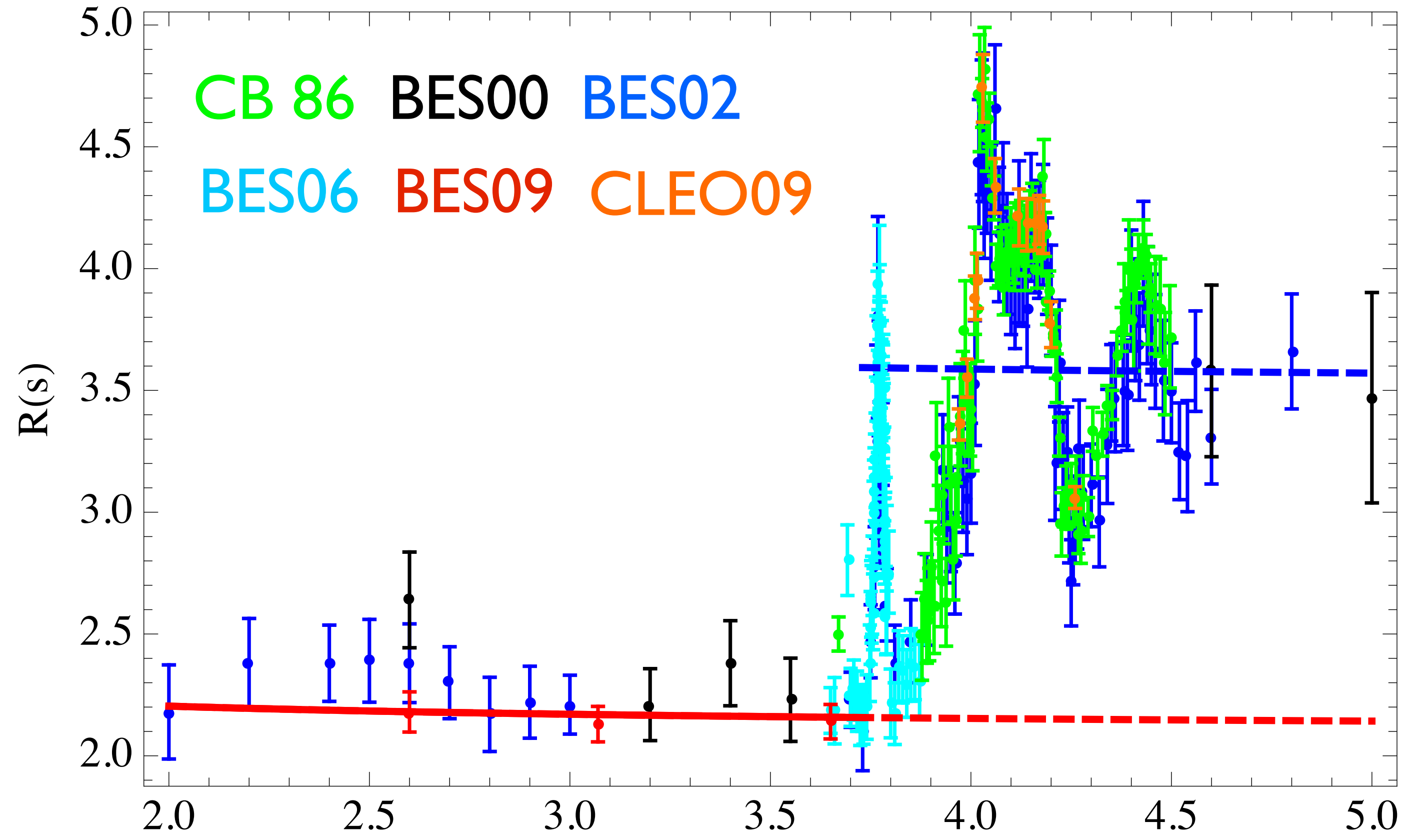
$$(\sqrt{s} \geq 4.8{\rm GeV})$$



Background

$$R_{\text{background}} = R_{\text{uds}} + R_{\text{uds(cb)}} + R_{\text{sing}} + R_{\text{QED}}$$

Light flavor contribution in charm region

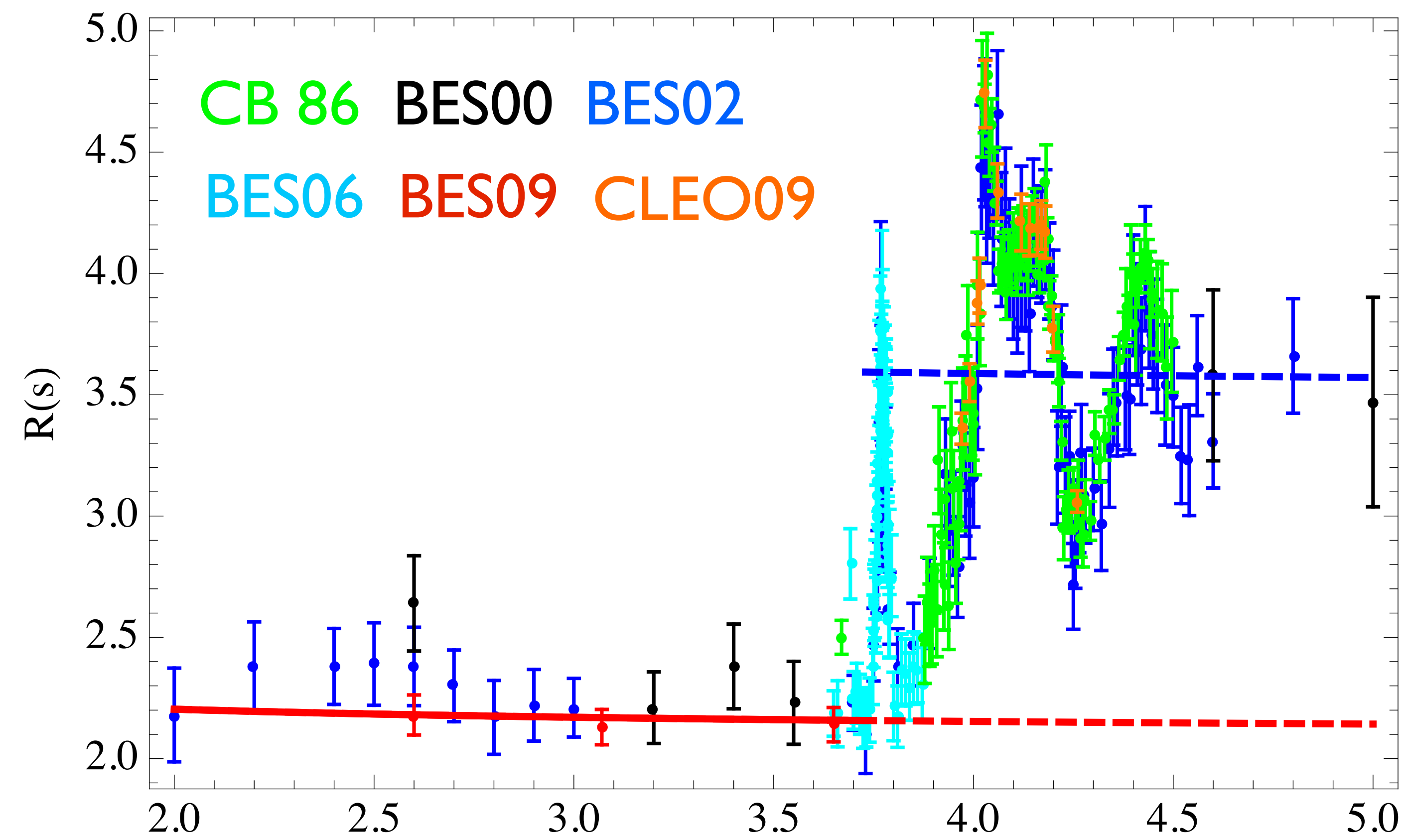


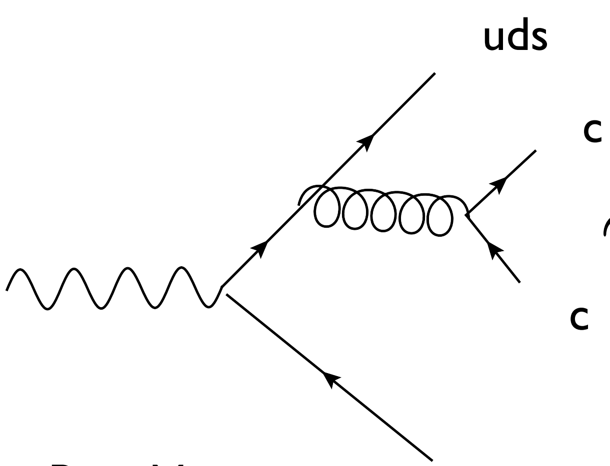
Using pQCD below threshold, calculate R, and extrapolate

Background

$$R_{\rm background} = R_{\rm uds} + R_{\rm uds(cb)} + R_{\rm sing} + R_{\rm QED}$$

Light flavor contribution in charm region
+
secondary production



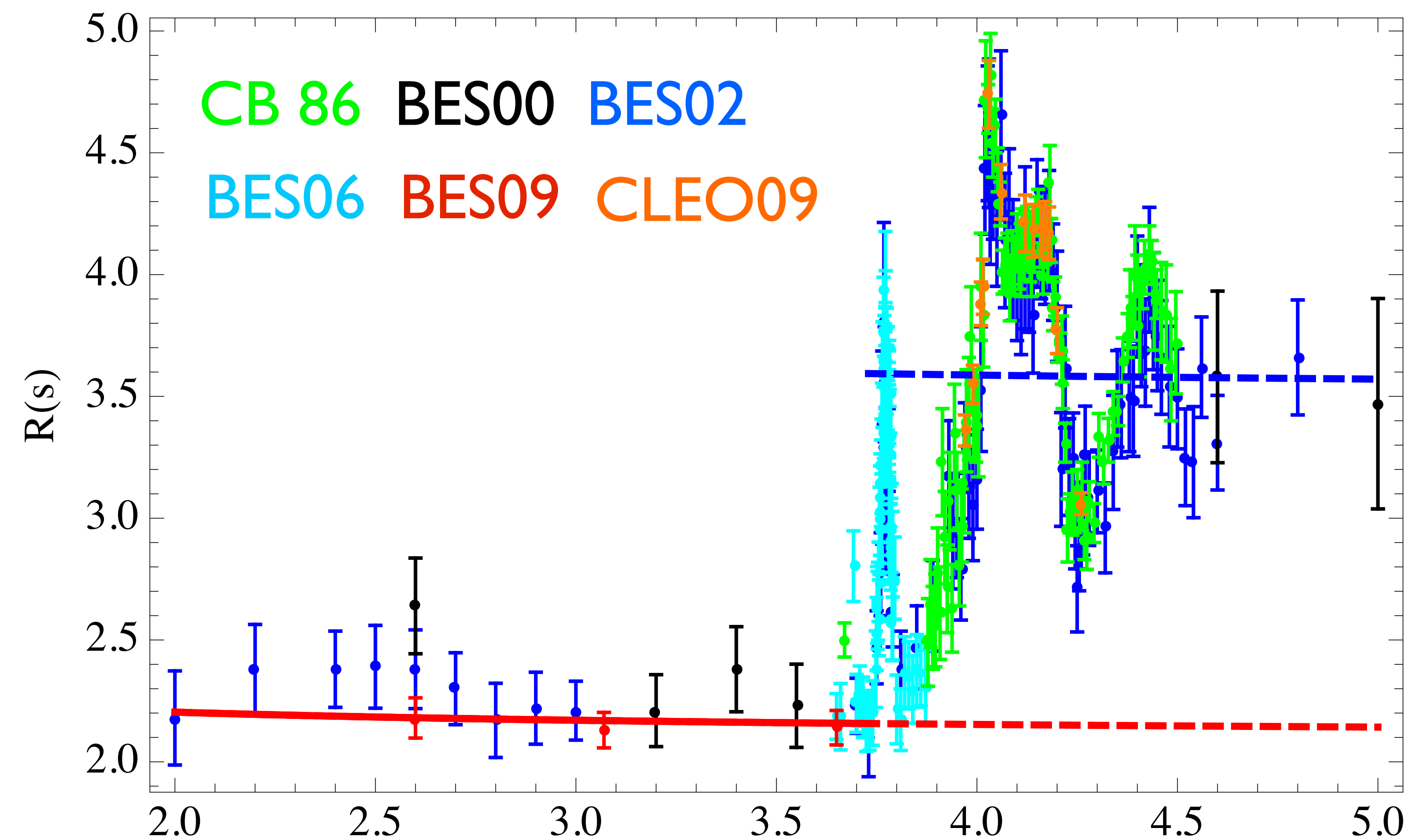


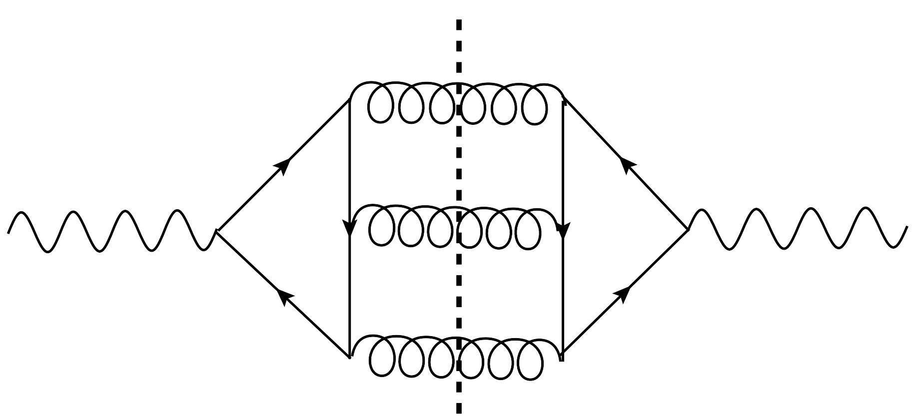
$\sim \alpha^2$

Background

$$R_{\text{background}} = R_{\text{uds}} + R_{\text{uds(cb)}} + R_{\text{sing}} + R_{\text{QED}}$$

Light flavor contribution in charm region
+
secondary production
+
singlet contribution



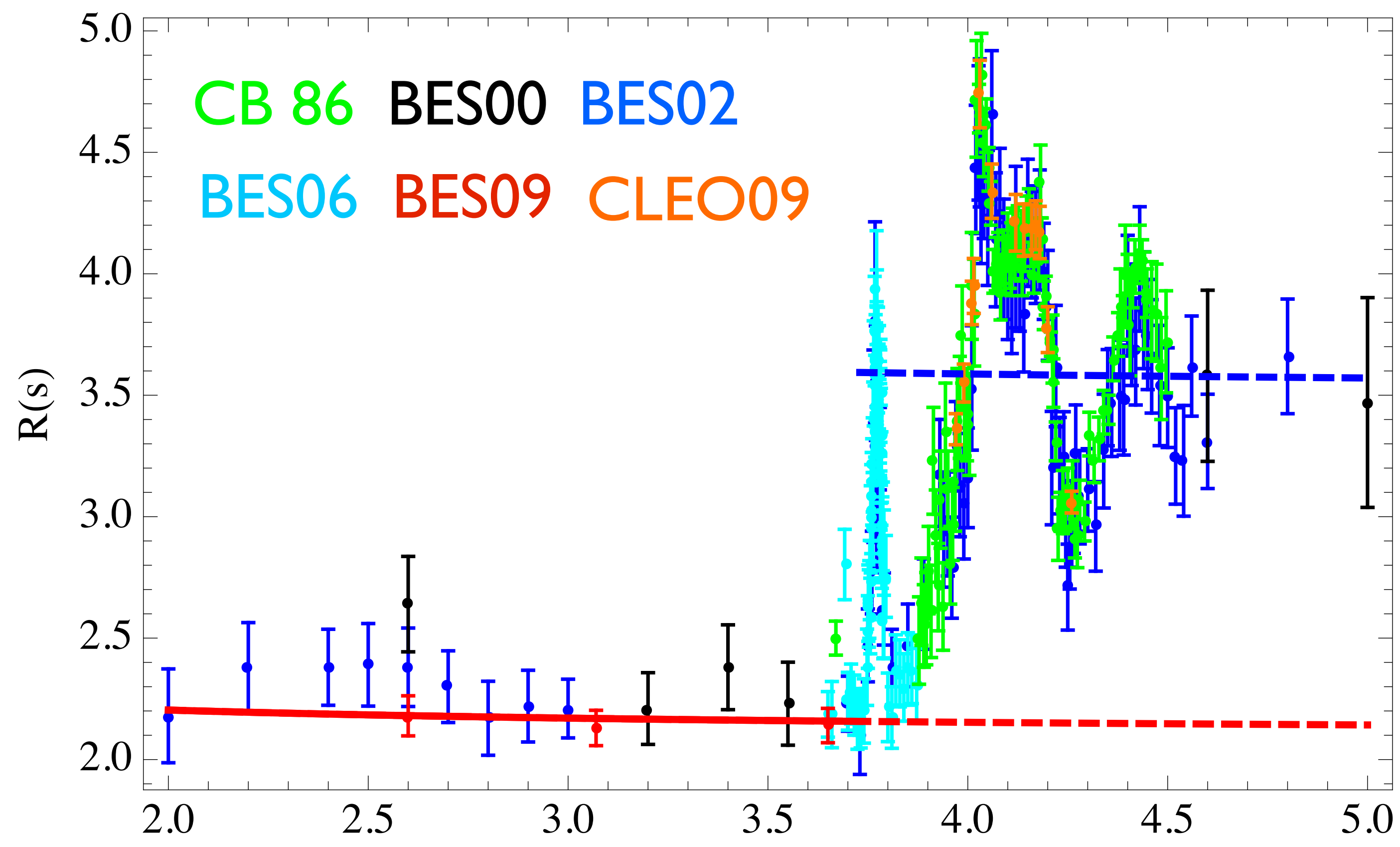


$\sim \alpha^3$

Background

$$R_{\text{background}} = R_{\text{uds}} + R_{\text{uds(cb)}} + R_{\text{sing}} + R_{\text{QED}}$$

Light flavor
contribution in
charm region
+
secondary
production
+
singlet contribution
+
2loop QED



Non-perturbative effects

Non-perturbative effects due to gluon condensates to the moments are:

[Chetyrkin et al '12]

$$\mathcal{M}_n^{\mathrm{nonp}}(\mu^2) = \frac{12\pi^2 Q_q^2}{(4\hat{m}_q^2)^{n+2}} \mathrm{Cond}\, a_n \left(1 + \frac{\alpha_s(\hat{m}_q^2)}{\pi} b_n\right)$$

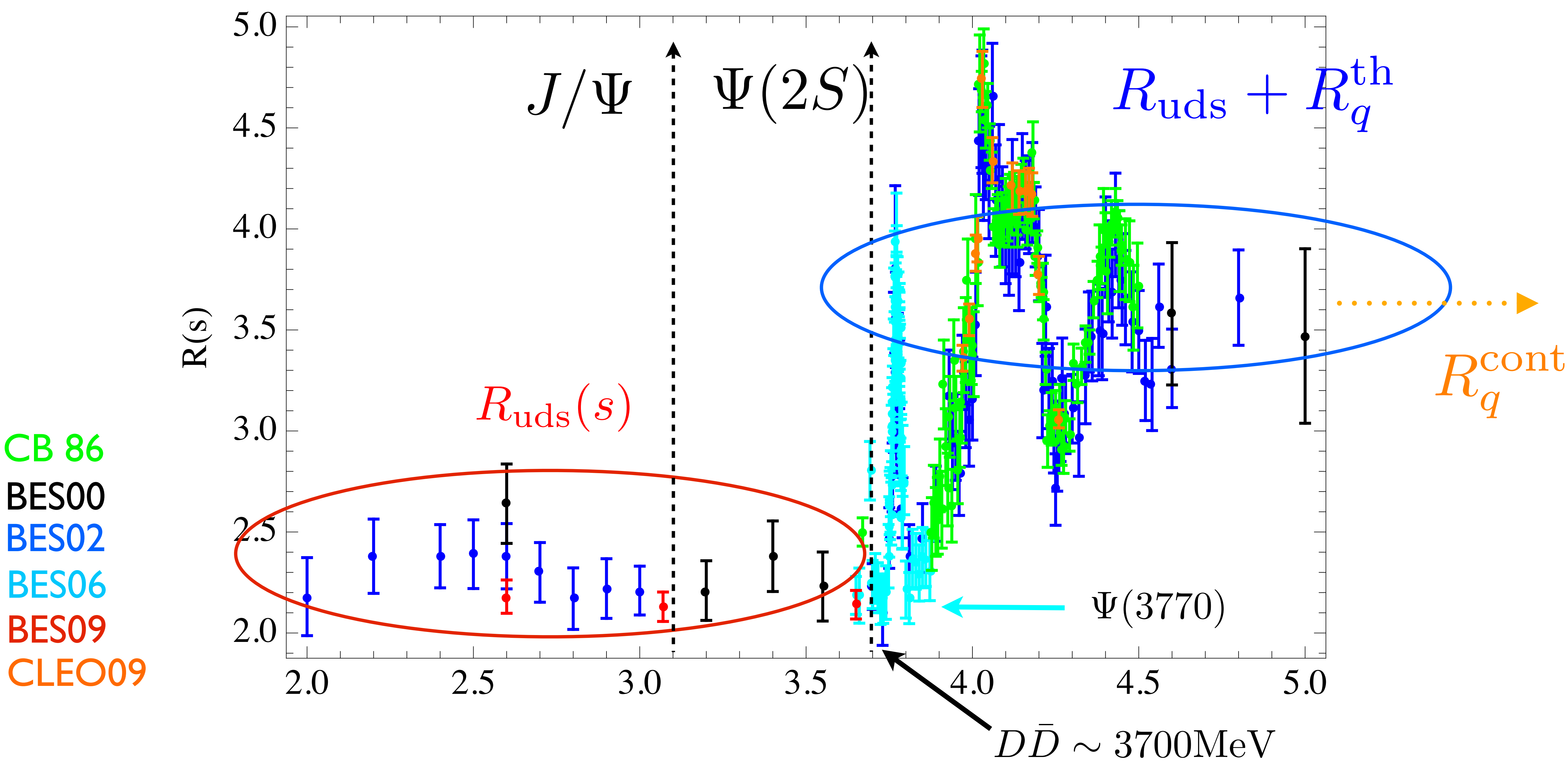
$a_n\ , b_n$ are numbers, and $\mathrm{Cond} = \langle \frac{\alpha_s}{\pi} G^2 \rangle = (5 \pm 5) \cdot 10^{-3}\mathrm{GeV}^4$ [Dominguez et al '14]

from fits to tau data

$$\frac{\mathcal{M}_n^{\mathrm{nonp}}(\hat{m}_c)}{\mathcal{M}_n^{\mathrm{th}}} \sim 0.5\% - 2\% \longrightarrow \Delta\hat{m}_c(\hat{m}_c) \sim 2\mathrm{MeV} - 8\mathrm{MeV}$$

QCD Sum Rules

$$R(s) = R_{\rm uds}(s) + R_q(s) \qquad R_q(s) = R_q^{\rm Res}(s) + R_q^{\rm th}(s) + R_q^{\rm cont}(s)$$



QCD Sum Rules

Our approach is different

- We try to avoid *local* duality: consider *global* duality
- Then, we do *not use experimental data* on threshold region, only resonances below threshold
  - Exp data in threshold only for error estimation
- How you do it then? Use *two different moment* equations to determine the continuum requiring self-consistency:
  - extract the quark mass

Charm

QCD Sum Rules

Our approach

For a global duality:

$\hat{\Pi}_q(s)$ in $\overline{MS}$

$$12\pi^2 \frac{\hat{\Pi}_q(0) - \hat{\Pi}_q(-t)}{t} = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q(s)}{s+t}$$

$t \to \infty \quad \text{define the } \mathcal{M}_0$

[Erler, Luo '03]

QCD Sum Rules

Our approach

For a global duality:

$\hat{\Pi}_q(s)$ in $\overline{MS}$

$$12\pi^2 \frac{\hat{\Pi}_q(0) - \hat{\Pi}_q(-t)}{t} = \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q(s)}{s+t}$$

$t \to \infty$ $\quad$ define the $\mathcal{M}_0$ (but has a divergent part) [Erler, Luo '03]

$$\lim_{t\to\infty} \hat{\Pi}_q(-t) \sim \log(t) \quad \longleftrightarrow \quad \int_{4m_q^2}^{\infty} \frac{\mathrm{d}s}{s} R_q(s) \sim \log(\infty)$$

Fortunately, divergence given by the zero-mass limit of R(s), can be easily subtracted

[Chetyrkin, Harlander, Kühn,'00]

QCD Sum Rules

Our approach

zero-mass limit of R(s)

[Chetyrkin, Harlander, Kühn,'00]

$$\lambda_1^q(s) = 1 + \frac{\alpha_s(s)}{\pi}$$
$$+ \left[\frac{\alpha_s(s)}{\pi}\right]^2 \left[\frac{365}{24} - 11\zeta(3) + n_q\left(\frac{2}{3}\zeta(3) - \frac{11}{12}\right)\right]$$
$$+ \left[\frac{\alpha_s(s)}{\pi}\right]^3 \left[\frac{87029}{288} - \frac{121}{8}\zeta(2) - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)\right.$$
$$+ n_q\left(-\frac{7847}{216} + \frac{11}{6}\zeta(2) + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right)$$
$$\left. + n_q^2\left(\frac{151}{162} - \frac{1}{18}\zeta(2) - \frac{19}{27}\zeta(3)\right)\right]$$

n_q active flavors

QCD Sum Rules

Our approach

Zeroth Sum Rule:

$$\sum_{\text{resonances}} \frac{9\pi\Gamma_R^e}{3Q_q^2 M_R \hat{\alpha}_{em}^2(M_R)} + \int_{4M^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q^{\text{cont}}}{3Q_q^2} - \int_{\hat{m}_q^2}^{\infty} \frac{\mathrm{d}s}{s} \lambda_1^q(s)$$

$$= -\frac{5}{3} + \frac{\hat{\alpha}_s}{\pi}\left[4\zeta(3) - \frac{7}{2}\right] \qquad \hat{\alpha}_s = \alpha_s(\hat{m}_q^2)$$

$$+ \left(\frac{\hat{\alpha}_s}{\pi}\right)^2 \left[\frac{2429}{48}\zeta(3) - \frac{25}{3}\zeta(5) - \frac{2543}{48} + n_q\left(\frac{677}{216} - \frac{19}{9}\zeta(3)\right)\right]$$

$$+ \left(\frac{\hat{\alpha}_s}{\pi}\right)^3 \left[-9.86 + 0.40\, n_q - 0.01\, n_q^2\right]$$

$$= -1.667 + 1.308\,\frac{\hat{\alpha}_s}{\pi} + 1.595\left(\frac{\hat{\alpha}_s}{\pi}\right)^2 - 8.427\left(\frac{\hat{\alpha}_s}{\pi}\right)^3$$

QCD Sum Rules

Our approach

Zeroth Sum Rule:

$$\boxed{\sum_{\text{resonances}} \frac{9\pi\Gamma_R^e}{3Q_q^2 M_R \hat{\alpha}_{em}^2(M_R)}} + \int_{4M^2}^{\infty} \frac{\mathrm{d}s}{s} \frac{R_q^{\text{cont}}}{3Q_q^2} - \int_{\hat{m}_q^2}^{\infty} \frac{\mathrm{d}s}{s} \lambda_1^q(s)$$

$$\hat{\alpha}_s = \alpha_s(\hat{m}_q^2)$$

[PDG]

R	M_R [GeV]	Γ_R^e [keV]
J/Ψ	3.096916	5.55(14)
$\Psi(2S)$	3.686109	2.36(4)

QCD Sum Rules

Our approach

Zeroth Sum Rule:

$$\sum_{\text{resonances}} \frac{9\pi\Gamma_R^e}{3Q_q^2 M_R \hat{\alpha}_{em}^2(M_R)} + \int_{4M^2}^{\infty} \frac{\mathrm{d}s}{s}\frac{R_q^{\text{cont}}}{3Q_q^2} - \int_{\hat{m}_q^2}^{\infty} \frac{\mathrm{d}s}{s}\lambda_1^q(s)$$

$$\hat{\alpha}_s = \alpha_s(\hat{m}_q^2)$$

$$\hat{\alpha}_{em}(0) \sim 0.98\hat{\alpha}_{em}(M_{J/\Psi})$$

$$\Delta\hat{\alpha}_{em} \to \Delta m_c \sim 12\text{MeV}$$

QCD Sum Rules

Our approach: ansatz

Zeroth Sum Rule: invoke global quark-hadron duality

[Erler, Luo '03]

$$R_q^{\mathrm{cont}}(s) = 3Q_q^2\lambda_1^q(s)\sqrt{1-\frac{4\,\hat{m}_q^2(2M)}{s'}}\left[1+\lambda_3^q\frac{2\,\hat{m}_q^2(2M)}{s'}\right]$$

Simpler version of analytic reconstruction [Greynat, PM, Peris'12]

$$s' = s + 4(\hat{m}_q^2(2M) - M^2)$$

Two parameters to determine: m_q , λ_3^q

QCD Sum Rules

Our approach: ansatz

Zeroth Sum Rule: invoke global quark-hadron duality

[Erler, Luo '03]

$$R_q^{\text{cont}}(s) = 3Q_q^2\lambda_1^q(s)\sqrt{1-\frac{4\,\hat{m}_q^2(2M)}{s'}}\left[1+\lambda_3^q\frac{2\,\hat{m}_q^2(2M)}{s'}\right]$$

Simpler version of analytic reconstruction [Greynat, PM, Peris'12]

$$s' = s + 4(\hat{m}_q^2(2M) - M^2)$$

Two parameters to determine: m_q , λ_3^q

We need two equations: zeroth moment + nth moment

$$\frac{9}{4}Q_q^2\left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n}\bar{C}_n = \sum_{\text{resonances}}\frac{9\pi\Gamma_R^e}{M_R^{2n+1}\hat{\alpha}_{em}^2(M_R)} + \int_{4M^2}^{\infty}\frac{\mathrm{d}s}{s^{n+1}}R_q(s)$$

$$n \geq 1$$

QCD Sum Rules

Our approach: ansatz

Zeroth Sum Rule: invoke global quark-hadron duality

[Erler, Luo '03]

$$R_q^{\rm cont}(s) = 3Q_q^2\lambda_1^q(s)\sqrt{1-\frac{4\,\hat{m}_q^2(2M)}{s'}}\left[1+\lambda_3^q\frac{2\,\hat{m}_q^2(2M)}{s'}\right]$$

Simpler version of analytic reconstruction [Greynat, PM, Peris'12]

$$s' = s + 4(\hat{m}_q^2(2M) - M^2)$$

Two parameters to determine: m_q , λ_3^q

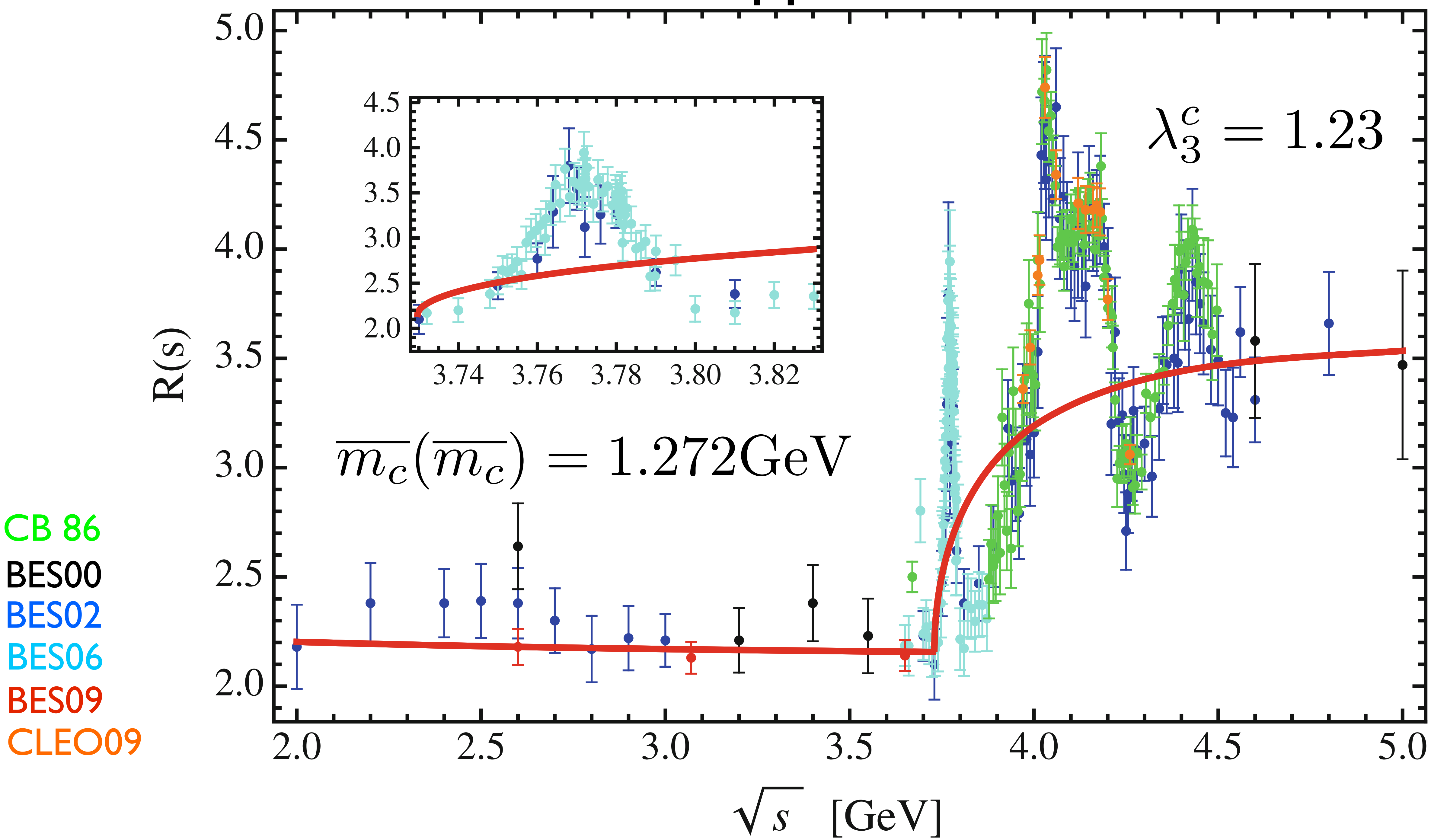
We use Zeroth + 2nd moments
(no experimental data on R(s) so far)

we require self-consistency among the 2 moments

n	Resonances	Continuum	Total	Theory
0	1.231 (24)	$-3.229(+28)(43)(1)$	$-1.999(56)$	**Input** (11)
1	1.184 (24)	0.966(+11)(17)(4)	2.150(33)	2.169(16)
2	1.161 (25)	0.336(+5)(8)(9)	1.497(28)	**Input** (25)
3	1.157 (26)	0.165(+3)(4)(16)	1.322(31)	1.301(39)
4	1.167 (27)	0.103(+2)(2)(26)	1.270(38)	1.220(60)
5	1.188 (28)	0.080(+1)(1)(38)	1.268(47)	1.175(95)

QCD Sum Rules

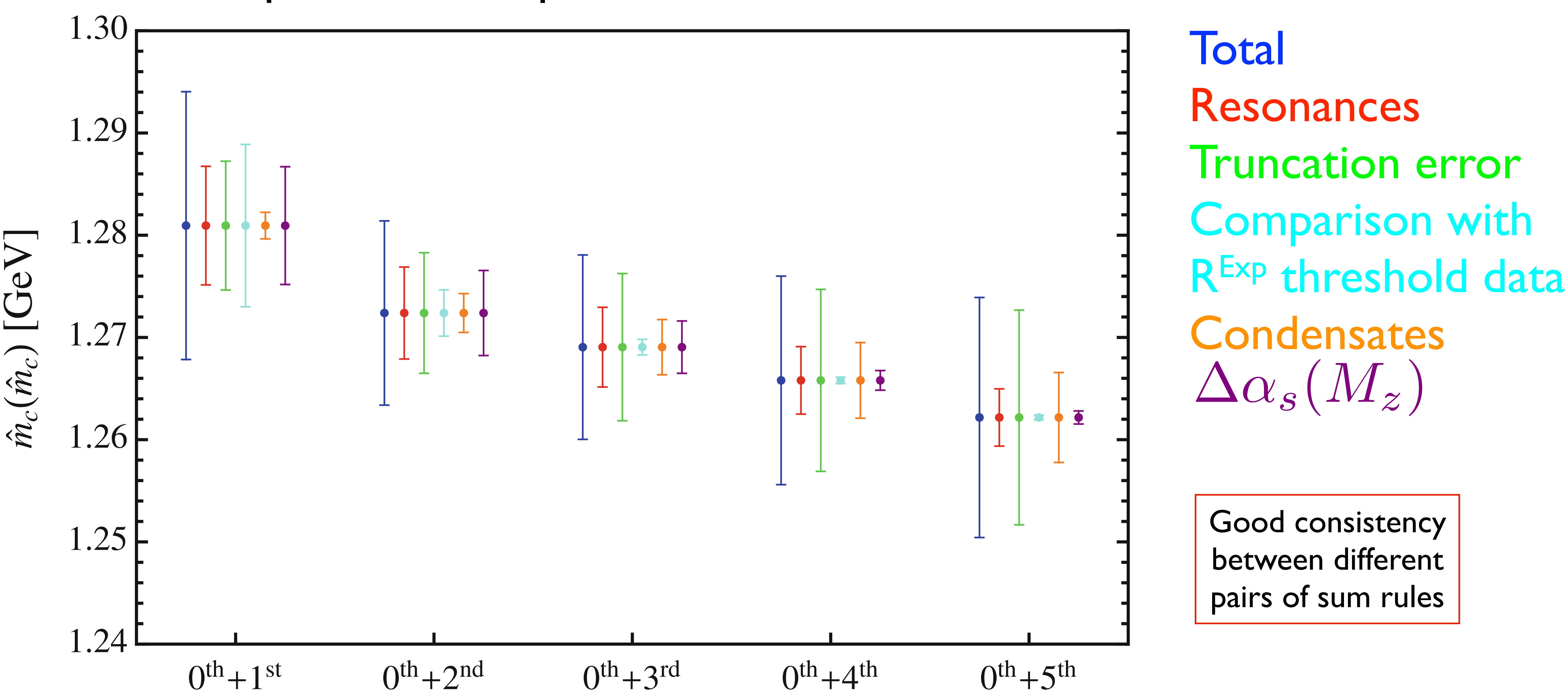
Our approach



QCD Sum Rules

Our approach

Repeat for each pair Zeroth+nth moment



Good consistency between different pairs of sum rules

QCD Sum Rules

Our approach: error budget

Resonances:

$$\frac{9}{4}Q_q^2\left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n}\bar{C}_n = \sum_{\text{resonances}} \frac{9\pi\Gamma_R^e}{M_R^{2n+1}\hat{\alpha}_{em}^2(M_R)} + \int_{4M^2}^{\infty}\frac{\mathrm{d}s}{s^{n+1}}R_q(s)$$

R	M_R [GeV]	Γ_R^e [keV]
J/Ψ	3.096916	5.55(14)
$\Psi(2S)$	3.686109	2.36(4)

from 6 MeV to 3 MeV

(0th+1st) (0th+5th)

(completely dominated by J/Ψ)

QCD Sum Rules

Our approach: error budget

Truncation Error (theory error):

$$\mathcal{M}_n^{\rm pQCD} = \frac{9}{4} Q_q^2 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n} \bar{C}_n$$

$$\bar{C}_n = \bar{C}_n^{(0)} + \left(\frac{\hat{\alpha}}{\pi}\right) \bar{C}_n^{(1)} + \left(\frac{\hat{\alpha}}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\hat{\alpha}}{\pi}\right)^3 \bar{C}_n^{(3)} + \mathcal{O}\left(\frac{\hat{\alpha}}{\pi}\right)^4$$

(use the largest group th. factor in the next uncalculated pert. order)

[Erler, Luo '03]

$$\boxed{\Delta\mathcal{M}_n^{(4)} = \pm N_C C_F C_A^3\, Q_q^2 \left[\frac{\hat{\alpha}_s(\hat{m}_q)}{\pi}\right]^4 \left(\frac{1}{2\hat{m}_q(\hat{m}_q)}\right)^{2n}}$$

from 5 MeV to 10 MeV
(0th+1st) (0th+5th)

Example known orders

n	$\frac{\Delta\mathcal{M}_n^{(2)}}{\lvert\mathcal{M}_n^{(2)}\rvert}$	$\frac{\Delta\mathcal{M}_n^{(3)}}{\lvert\mathcal{M}_n^{(3)}\rvert}$
0	1.88	3.03
1	2.14	2.84
2	1.92	4.58
3	3.25	5.63
4	6.70	4.30
5	19.18	3.62

More conservative than varying the renorm. scale within a factor of 4

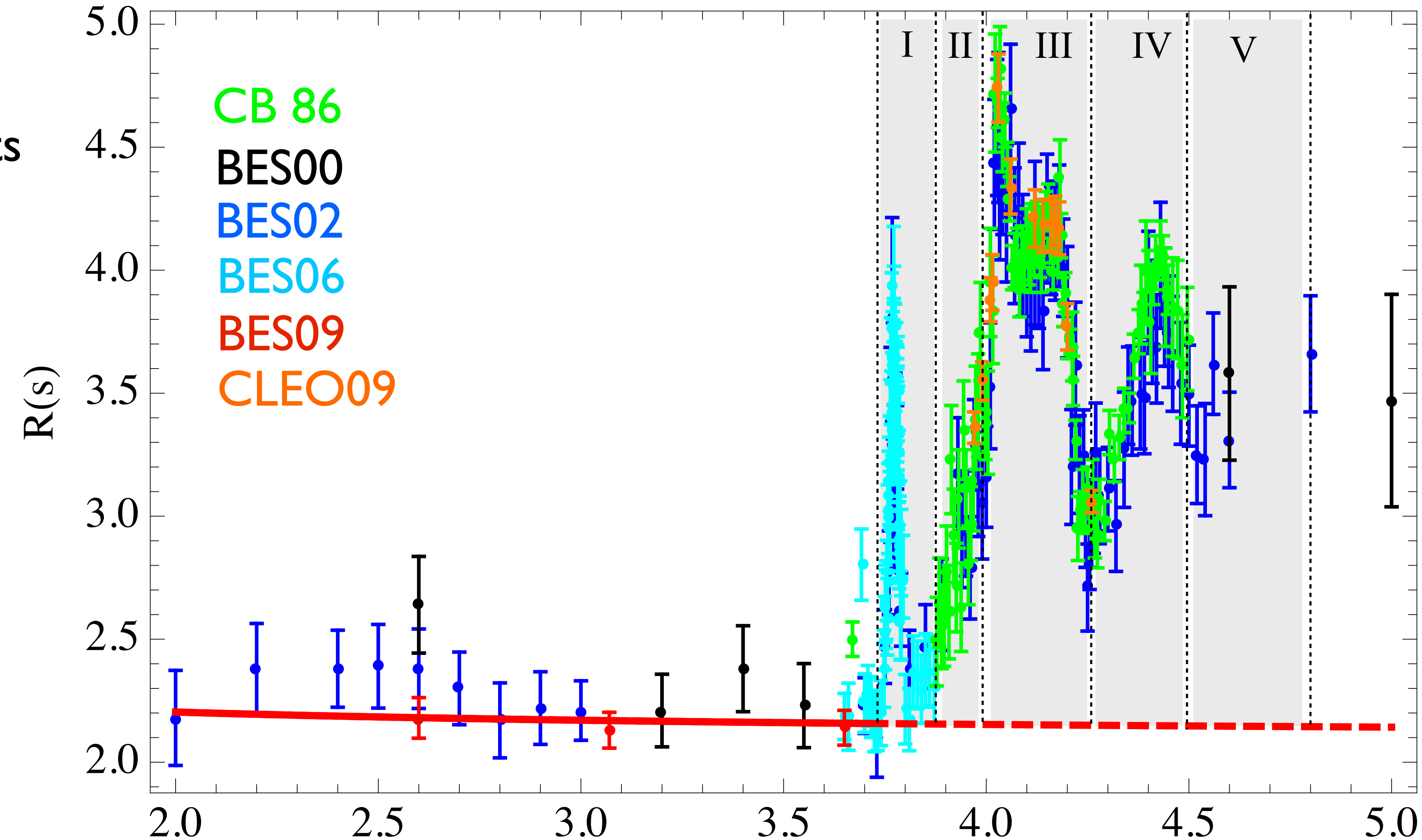
QCD Sum Rules

Our approach: error budget

Comparison with R^{Exp} threshold data:

$$(2M_D \leq \sqrt{s} \leq 4.8\text{GeV})$$

Calculate Exp moments



QCD Sum Rules

Our approach: error budget

Comparison with R^{Exp} threshold data:

Collab.	n	$[2M_{D^0}, 3.872]$	[3.872, 3.97]	[3.97, 4.26]	[4.26, 4.496]	[4.496, 4.8]
CB86	0	–	0.0339(22)(24)	0.2456(25)(172)	0.1543(27)(108)	–
	1	–	0.0220(14)(15)	0.1459(16)(102)	0.0801(14)(56)	–
	2	–	0.0143(9)(10)	0.0868 (9)(61)	0.0416(7)(29)	–
BES02	0	0.0334(24)(17)	0.0362(29)(18)	0.2362(41)(118)	0.1399(38)(70)	0.1705(63)(85)
	1	0.0232(17)(12)	0.0235(19)(12)	0.1401(24)(70)	0.0726(20)(36)	0.0788(30)(39)
	2	0.0161(12)(8)	0.0152(13)(8)	0.0832(15)(42)	0.0378(10)(19)	0.0365(14)(18)
BES06	0	0.0311(16)(15)	–	–	–	–
	1	0.0217(11)(11)	–	–	–	–
	2	0.0151(8)(7)	–	–	–	–
CLEO09	0	–	–	0.2591(22)(52)	–	–
	1	–	–	0.1539(13)(31)	–	–
	2	–	–	0.0915(8)(18)	–	–
Total	0	0.0319(14)(11)	0.0350(18)(15)	0.2545(18)(46)	0.1448(27)(59)	0.1705(63)(85)
	1	0.0222(9)(8)	0.0227(12)(10)	0.1511(11)(27)	0.0752(14)(31)	0.0788(30)(39)
	2	0.0155(6)(6)	0.0147(8)(6)	0.0899(6)(16)	0.0391(7)(16)	0.0365(14)(18)

QCD Sum Rules

Our approach: error budget

Comparison with R^{Exp} threshold data:

$$\int_{(2M_{D^0})^2}^{(4.8\,\mathrm{GeV})^2} \frac{ds}{s} R_c^{\mathrm{cont}}(s)\Big|_{\hat{m}_c=1.272\,\mathrm{GeV}} = \mathcal{M}_0^{\mathrm{Data}} = 0.6367(195) \longrightarrow \lambda_3^{\mathrm{c,exp}} = 1.34(17)$$

$$(2M_D \leq \sqrt{s} \leq 4.8\mathrm{GeV})$$

Error induced to Quark mass:

I) $\lambda_3^c = 1.23 \to \lambda_3^{\mathrm{c,exp}} = 1.34$

from $+6.4$ MeV to $+0.2$ MeV

II) $\Delta\lambda_3^{\mathrm{c,exp}} = 0.17$

from 4.7 MeV to 0.1 MeV

n	Data	$\lambda_3^c = 1.34(17)$	$\lambda_3^c = 1.23$
0	0.6367(195)	0.6367(195)	0.6239
1	0.3500(102)	0.3509(111)	0.3436
2	0.1957(54)	0.1970(65)	0.1928
3	0.1111(29)	0.1127(38)	0.1102
4	0.0641(16)	0.0657(23)	0.0642
5	0.0375(9)	0.0389(14)	0.0380

QCD Sum Rules

Our approach: error budget

Comparison with R^{Exp} threshold data:

$$\int_{(2M_{D^0})^2}^{(4.8\,\mathrm{GeV})^2} \frac{\mathrm{d}s}{s} R_c^{\mathrm{cont}}(s)\Big|_{\hat{m}_c=1.272\,\mathrm{GeV}} = \mathcal{M}_0^{\mathrm{Data}} = 0.6367(195) \longrightarrow \lambda_3^{c,\mathrm{exp}} = 1.34(17)$$

Error induced to Quark mass:

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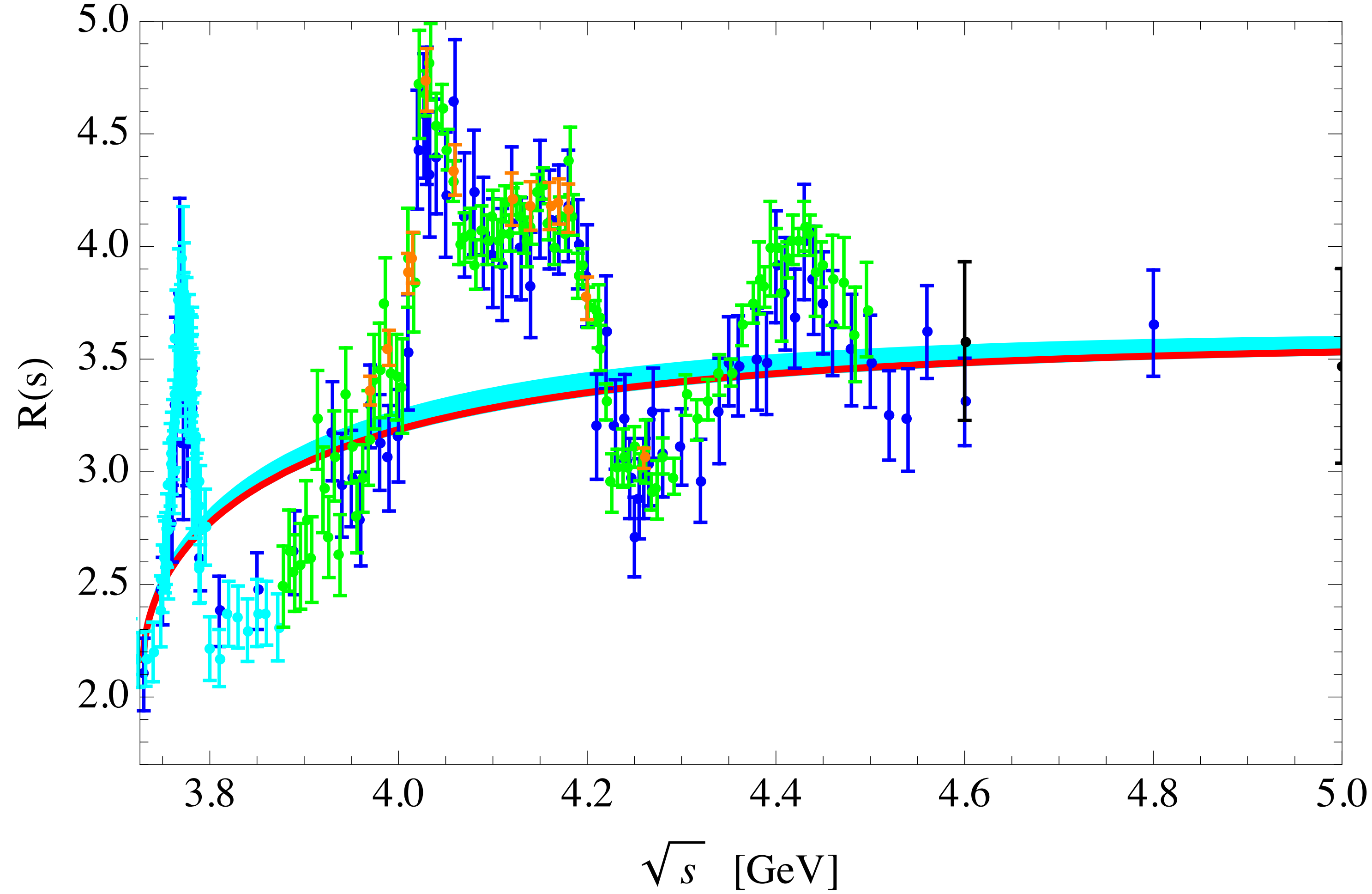
II) $\Delta\lambda_3^{c,\mathrm{exp}} = 0.17$

from 4.7 MeV to 0.1 MeV



QCD Sum Rules

Our approach: error budget

Condensates:

Non-perturbative effects due to gluon condensates to the moments are: [Chetyrkin et al '12]

$$\mathcal{M}_n^{\text{nonp}}(\mu^2) = \frac{12\pi^2 Q_q^2}{(4\hat{m}_q^2)^{n+2}} \text{Cond}\, a_n \left(1 + \frac{\alpha_s(\hat{m}_q^2)}{\pi} b_n\right)$$

$a_n\,, b_n$ are numbers, and $\text{Cond} = \langle \frac{\alpha_s}{\pi} G^2 \rangle = (5 \pm 5) \cdot 10^{-3} \text{GeV}^4$ [Dominguez et al '14]

$$\Delta \langle \frac{\alpha_s}{\pi} G^2 \rangle = 5 \cdot 10^{-3} \text{GeV}^4 \longrightarrow$$

from 1 MeV to 4 MeV

(0th+1st) (0th+5th)

Parametric error:

$$\Delta \overline{m_c}(\overline{m_c})[\text{MeV}] = -0.5 \cdot 10^3 \frac{\text{MeV}}{\text{GeV}^4} \Delta \langle \frac{\alpha_s}{\pi} G^2 \rangle$$

(but this is only the first condensate)

QCD Sum Rules

Our approach: error budget

$\Delta\alpha_s(M_z)$ $\qquad$ $\alpha_s(M_z) = 0.1182(16)$ $\qquad$ from PDG16

$$\Delta\alpha_s(M_z) = 0.0016 \longrightarrow \text{from 6 MeV to 1 MeV}$$

Parametric error:

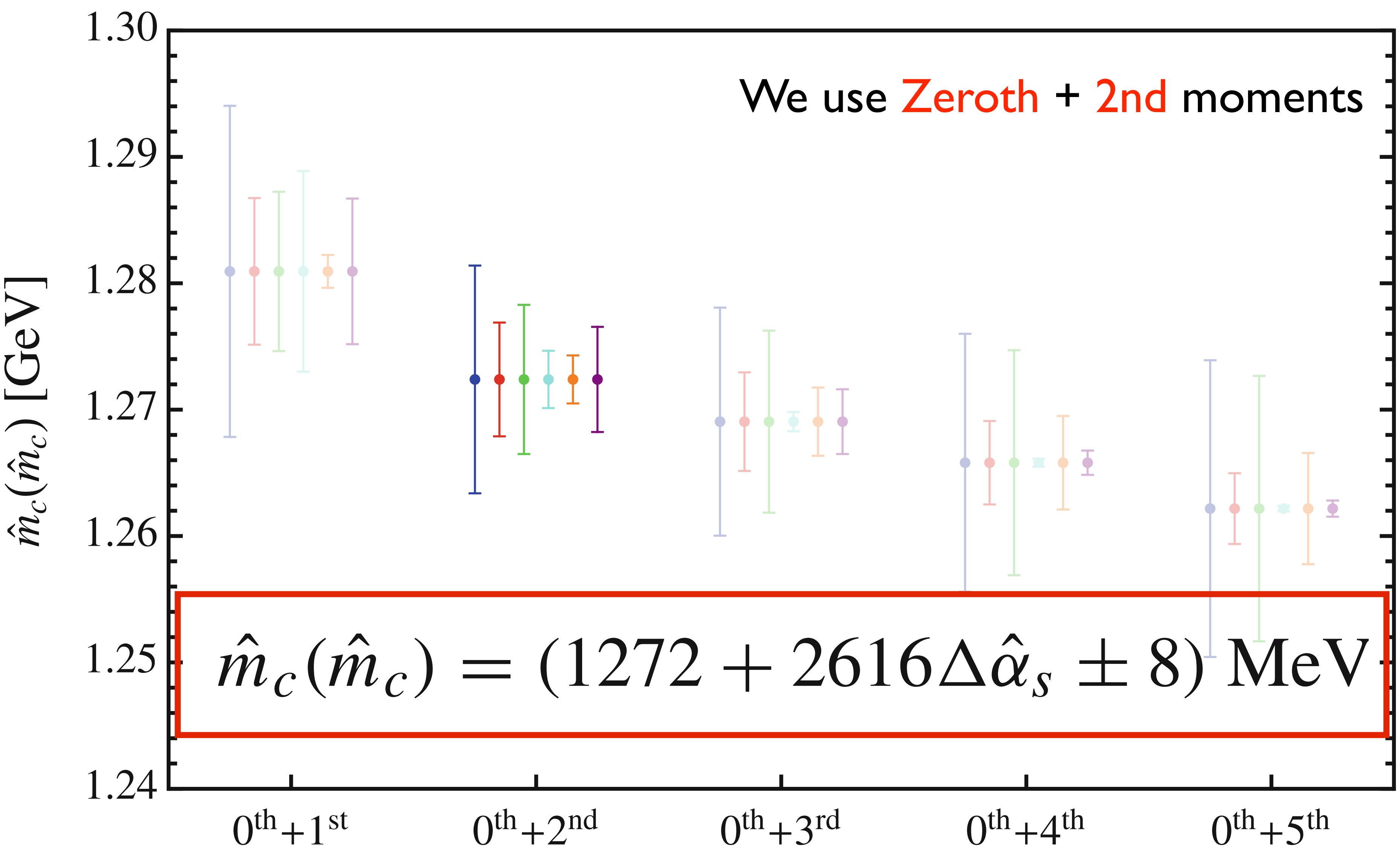
(0th+1st) $\quad \Delta\overline{m_c}(\overline{m_c})[\text{MeV}] = 3.6 \cdot 10^3 \Delta\alpha_s(M_z)$

(0th+5th) $\quad \Delta\overline{m_c}(\overline{m_c})[\text{MeV}] = -0.4 \cdot 10^3 \Delta\alpha_s(M_z)$

QCD Sum Rules

Our approach: final result

[J.Erler, P.M., H. Spiesberger'17]

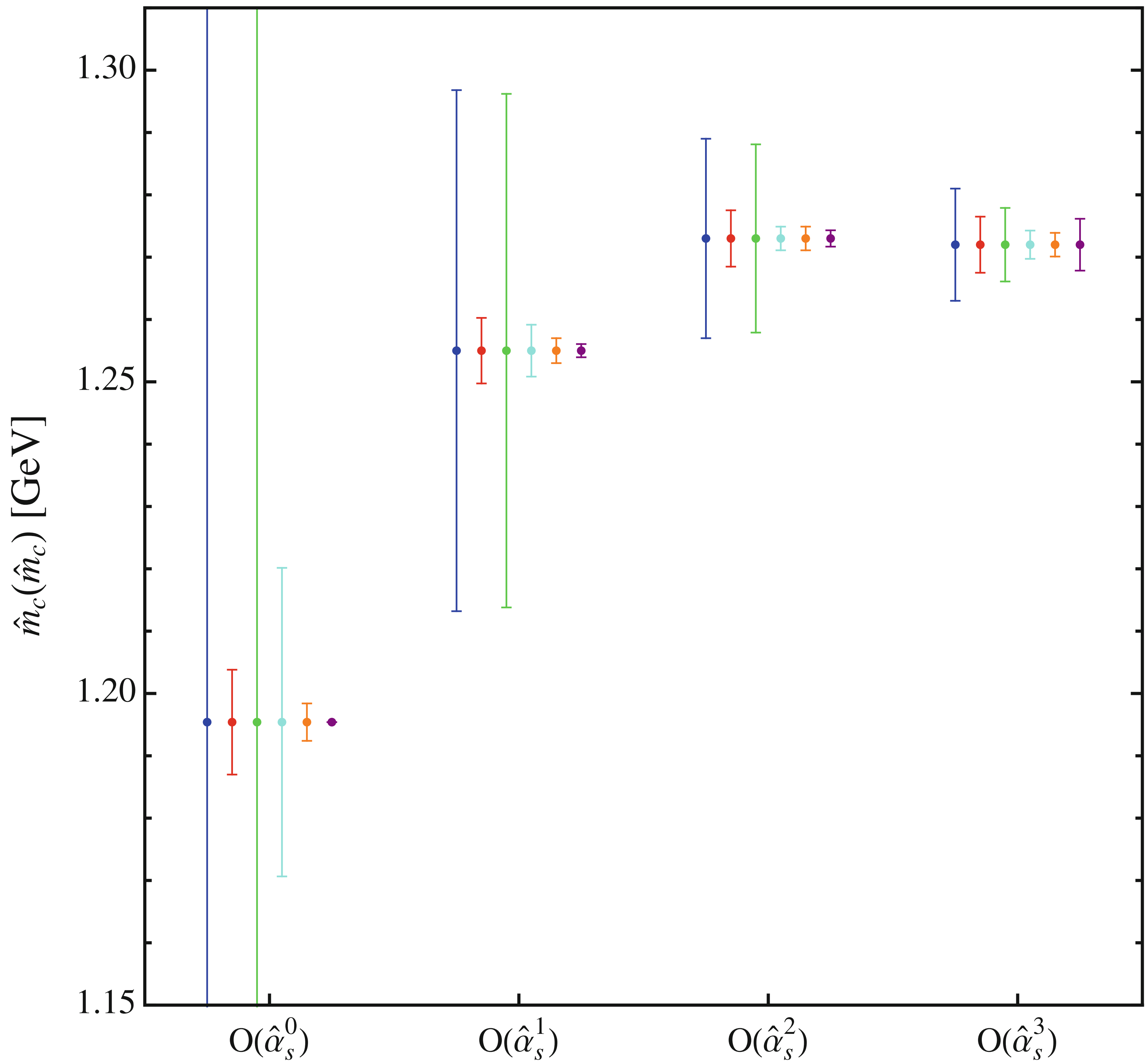


$$\boxed{\hat{m}_c(\hat{m}_c) = (1272 + 2616\Delta\hat{\alpha}_s \pm 8)\ \text{MeV}}$$

Resonances
Truncation error
Comparison with
R^{Exp} threshold data
Condensates
$\Delta\alpha_s(M_z)$

$$\Delta\hat{\alpha}_s = \hat{\alpha}_s(M_z) - 0.1182$$

QCD Sum Rules



α_s expansion

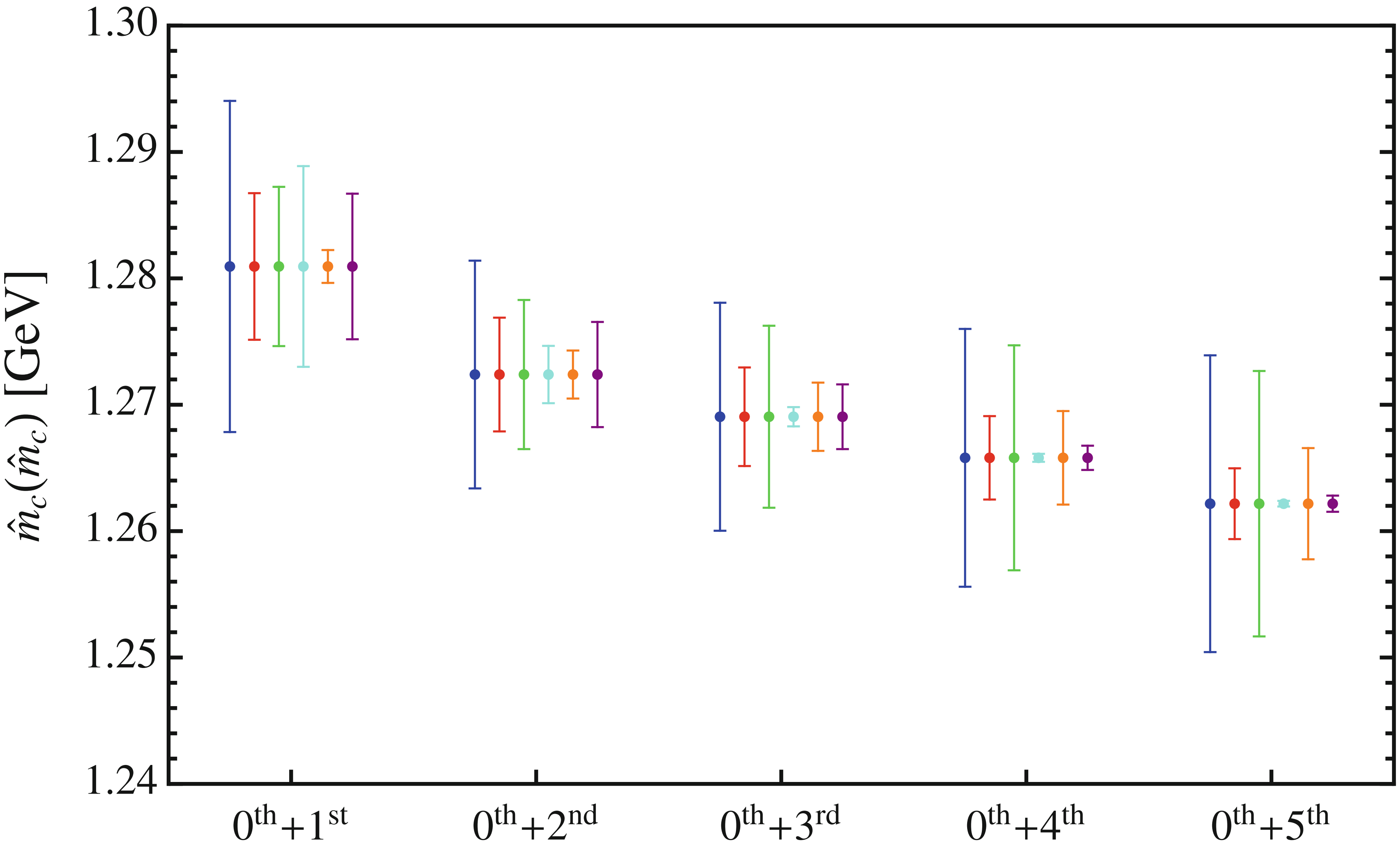
for the Zeroth + 2nd moments

Total
Resonances
Truncation error
Comparison with
R^{Exp} threshold data
Condensates
$\Delta\alpha_s(M_z)$

QCD Sum Rules

Our approach

What pair/result to choose?

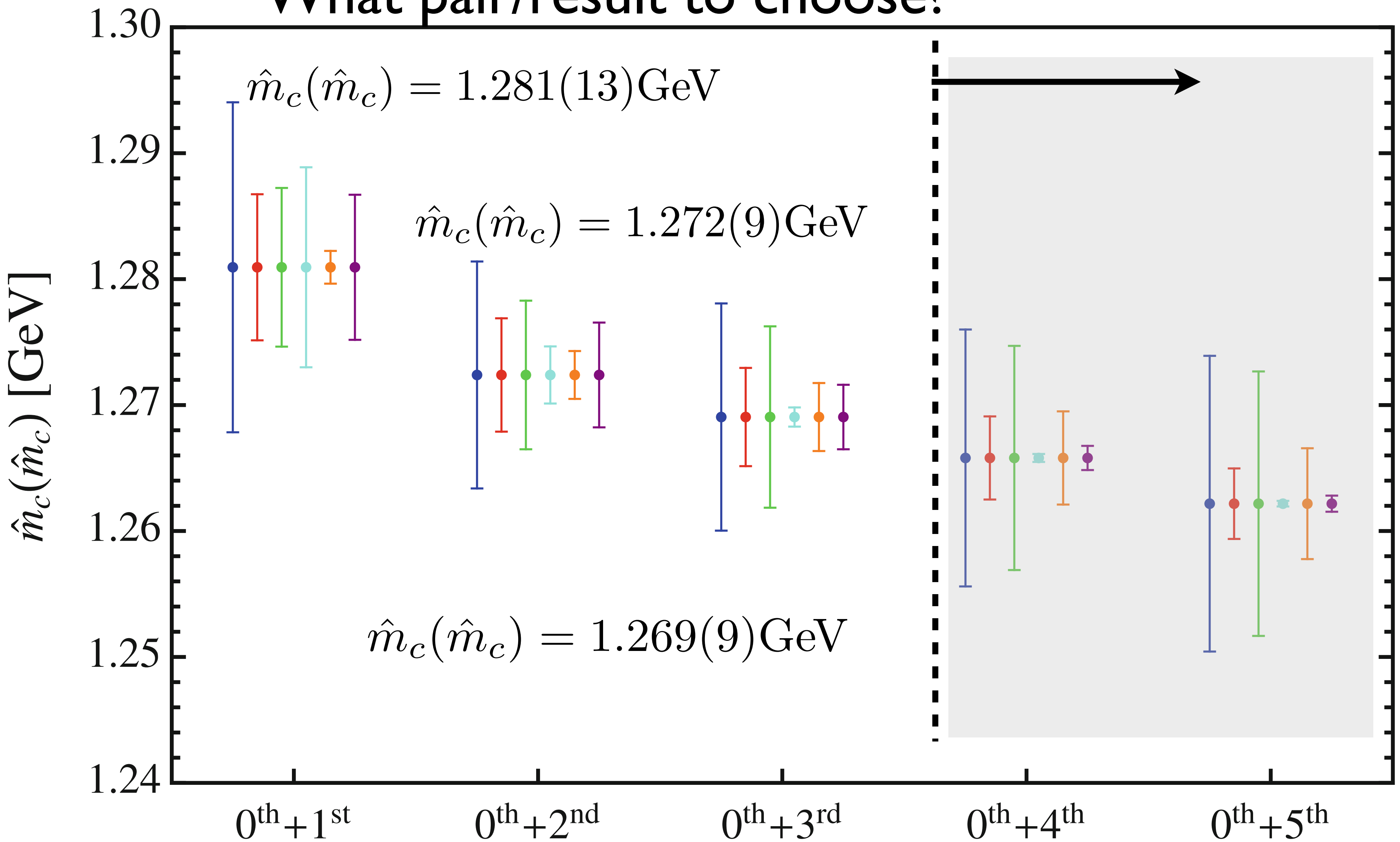


Resonances
Truncation error
Comparison with R^{Exp} threshold data
Condensates
$\Delta\alpha_s(M_z)$

QCD Sum Rules

Our approach

What pair/result to choose?



Resonances

Truncation error

Comparison with R^{Exp} threshold data

Condensates

$\Delta\alpha_s(M_z)$

Large condensate effects
+
new condensates will matter

QCD Sum Rules

Our approach: more than two moments?

Define a χ^2 function:

$$\chi^2 = \frac{1}{2}\sum_{n,m}\left(\mathcal{M}_n - \mathcal{M}_n^{\rm pQCD}\right)\left(\mathcal{C}^{-1}\right)^{nm}\left(\mathcal{M}_m - \mathcal{M}_m^{\rm pQCD}\right) + \chi_c^2$$

$$\mathcal{C} = \frac{1}{2}\sum_{n,m}\rho^{{\rm Abs}(n-m)}\Delta\mathcal{M}_n^{(4)}\Delta\mathcal{M}_m^{(4)}$$

ρ a correlation parameter

$$\chi_c^2 = \left(\frac{\Gamma^e_{J/\Psi(1S)} - \Gamma^{e,{\rm exp}}_{J/\Psi(1S)}}{\Delta\Gamma^e_{J/\Psi(1S)}}\right)^2 + \left(\frac{\Gamma^e_{\Psi(2S)} - \Gamma^{e,{\rm exp}}_{\Psi(2S)}}{\Delta\Gamma^e_{\Psi(2S)}}\right)^2 + \left(\frac{\hat{\alpha}_s(M_z) - \hat{\alpha}_s(M_z)^{\rm exp}}{\Delta\hat{\alpha}_s(M_z)}\right)^2 + \left(\frac{\langle\frac{\alpha_s}{\pi}G^2\rangle - \langle\frac{\alpha_s}{\pi}G^2\rangle^{\rm exp}}{\Delta\langle\frac{\alpha_s}{\pi}G^2\rangle}\right)^2$$

QCD Sum Rules

Our approach: more than two moments?

Define a χ^2 function:

ρ	Constraints	$(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)_\rho$ -0.06	$\mathcal{M}_0,\ (\mathcal{M}_1, \mathcal{M}_2)_\rho$ -0.05	$\mathcal{M}_0,\ (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)_\rho$ 0.32
$\hat{m}_c(\hat{m}_c)$ [GeV]		1.275(8)	1.275(8)	1.271(7)
λ_3^c		1.19(8)	1.19(8)	1.19(7)
$\Gamma^e_{J/\Psi}$ [keV]	5.55(14)	5.57(14)	5.57(14)	5.59(14)
$\Gamma^e_{\Psi(2S)}$ [keV]	2.36(4)	2.36(4)	2.36(4)	2.36(4)
C_G [GeV^4]	0.005(5)	0.005(5)	0.005(5)	0.004(5)
$\hat{\alpha}_s(M_z)$	0.1182(16)	0.1178(15)	0.1178(15)	0.1173(15)

QCD Sum Rules

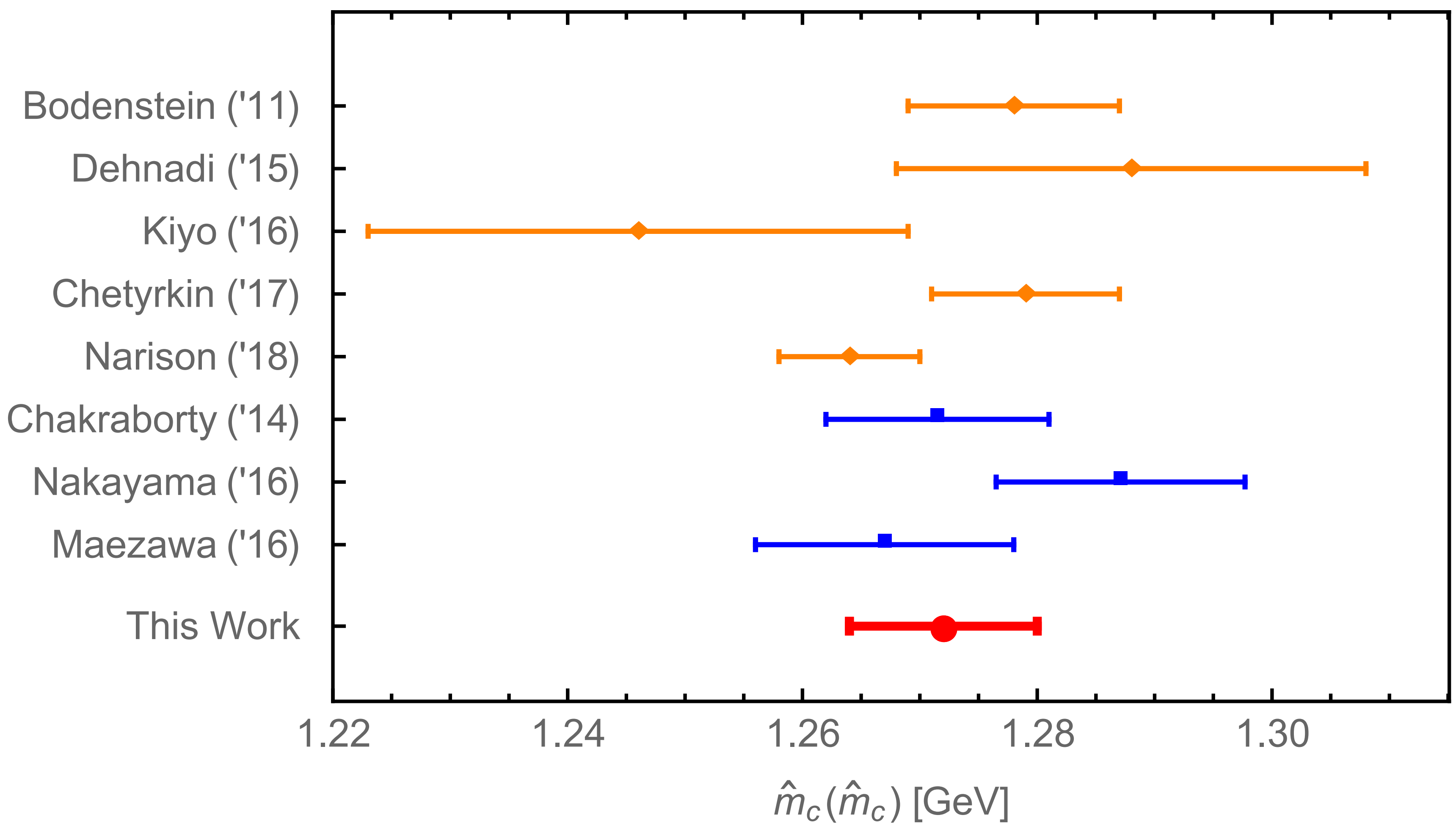
Our approach: more than two moments?

Preferred scenario:

	$0\text{th} + (1\text{st} + 2\text{nd})_\rho$ $\Delta\hat{m}_c(\hat{m}_c)$ [MeV]	$(0\text{th} + 2\text{nd})$ $\Delta\hat{m}_c(\hat{m}_c)$ [MeV]
Central value	1274.5	1272.4
$\Delta\Gamma^e_{J/\Psi}$	5.9	4.5
$\Delta\Gamma^e_{\Psi(2S)}$	1.4	0.4
Truncation	–	5.9
$\Delta\lambda^c_3$	3.0	2.3
Condensates	1.1	1.9
$\Delta\hat{\alpha}_s(M_Z)$	5.4	4.2
Total	8.7	9.0

QCD Sum Rules

results for the charm quark mass



Bottom

QCD Sum Rules

(preliminary)

zero-mass limit of R(s)

$$\lambda_1^q(s) = 1 + \frac{\alpha_s(s)}{\pi} + \left[\frac{\alpha_s(s)}{\pi}\right]^2 (\dots) + \left[\frac{\alpha_s(s)}{\pi}\right]^3 (\dots) + \dots$$
$$+ \frac{m_q^2}{s}\left(12\left[\frac{\alpha_s(s)}{\pi}\right] + \left[\frac{\alpha_s(s)}{\pi}\right]^2 (\dots) + \left[\frac{\alpha_s(s)}{\pi}\right]^3 (\dots)\right)$$
$$+ \frac{m_q^4}{s^2}\left(-6 + \left[\frac{\alpha_s(s)}{\pi}\right](\dots) + \left[\frac{\alpha_s(s)}{\pi}\right]^2 (\dots)\right) + \frac{m_q^6}{s^3}\left(-8 + \left[\frac{\alpha_s(s)}{\pi}\right](\dots)\right)$$

QCD Sum Rules

zero-mass limit of R(s)

(preliminary)

$$\lambda_1^q(s) = 1 + \frac{\alpha_s(s)}{\pi} + \left[\frac{\alpha_s(s)}{\pi}\right]^2 (\ldots) + \left[\frac{\alpha_s(s)}{\pi}\right]^3 (\ldots) + \ldots$$
$$+ \frac{m_q^2}{s}\left(12\left[\frac{\alpha_s(s)}{\pi}\right] + \left[\frac{\alpha_s(s)}{\pi}\right]^2 (\ldots) + \left[\frac{\alpha_s(s)}{\pi}\right]^3 (\ldots)\right)$$
$$+ \frac{m_q^4}{s^2}\left(-6 + \left[\frac{\alpha_s(s)}{\pi}\right] (\ldots) + \left[\frac{\alpha_s(s)}{\pi}\right]^2 (\ldots)\right) + \frac{m_q^6}{s^3}\left(-8 + \left[\frac{\alpha_s(s)}{\pi}\right] (\ldots)\right)$$

For charm:

$$12\frac{m_c^2}{s}\left(\frac{\alpha_s(s)}{\pi}\right) - 6\left(\frac{m_c^2}{s}\right)^2 \sim 0$$

For bottom:

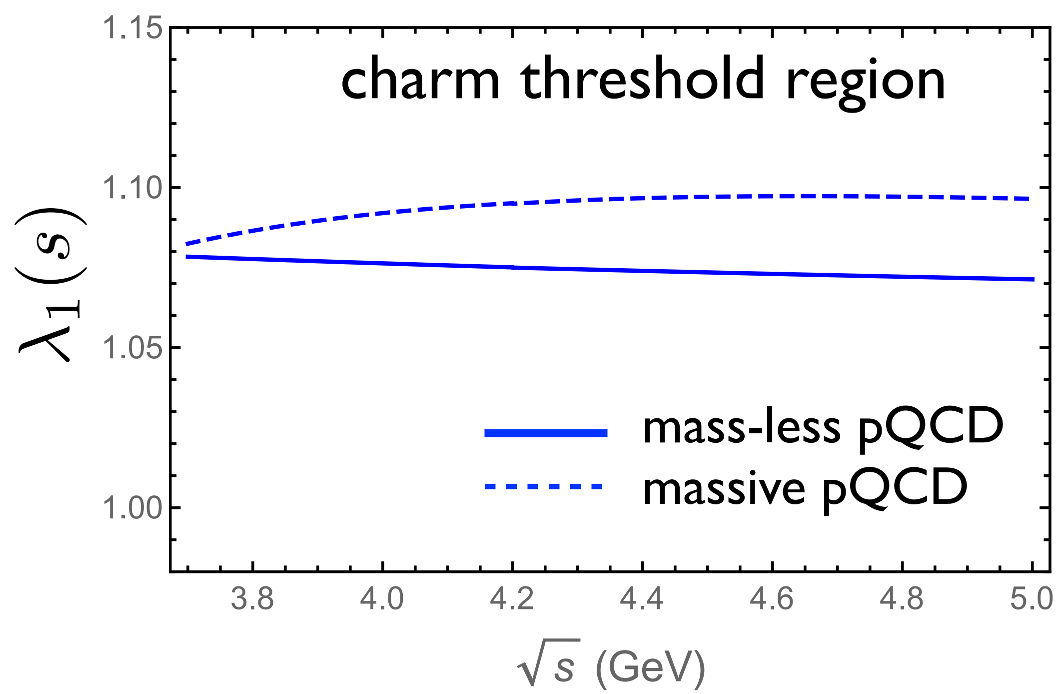
$$12\frac{m_b^2}{s}\left(\frac{\alpha_s(s)}{\pi}\right) < 6\left(\frac{m_b^2}{s}\right)^2$$

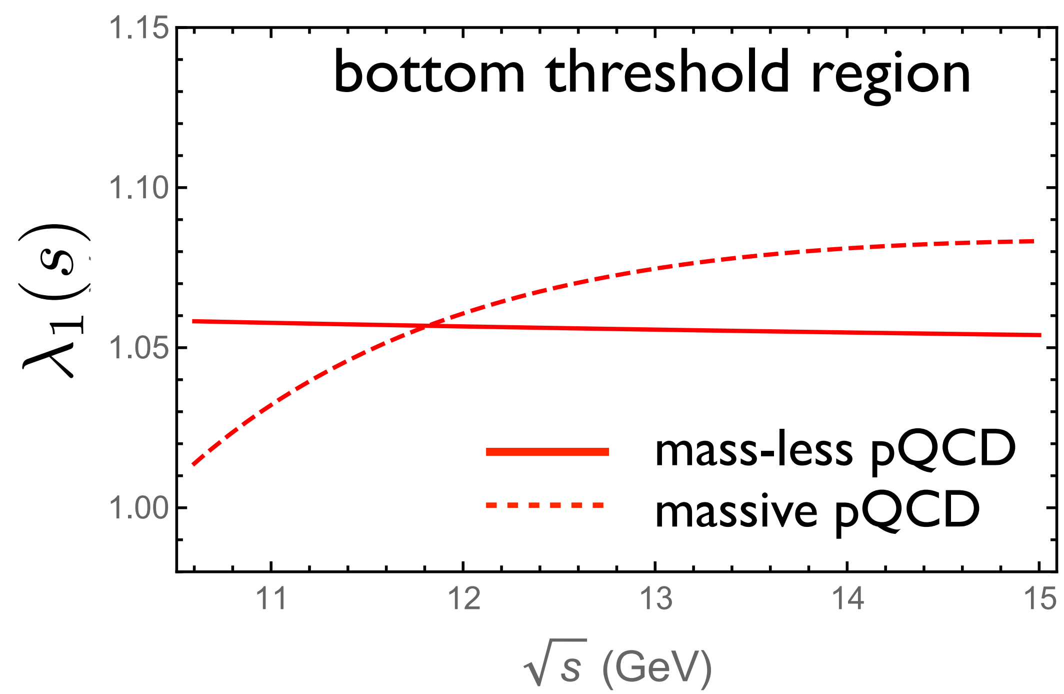
QCD Sum Rules

(preliminary)

massive limit of R(s)

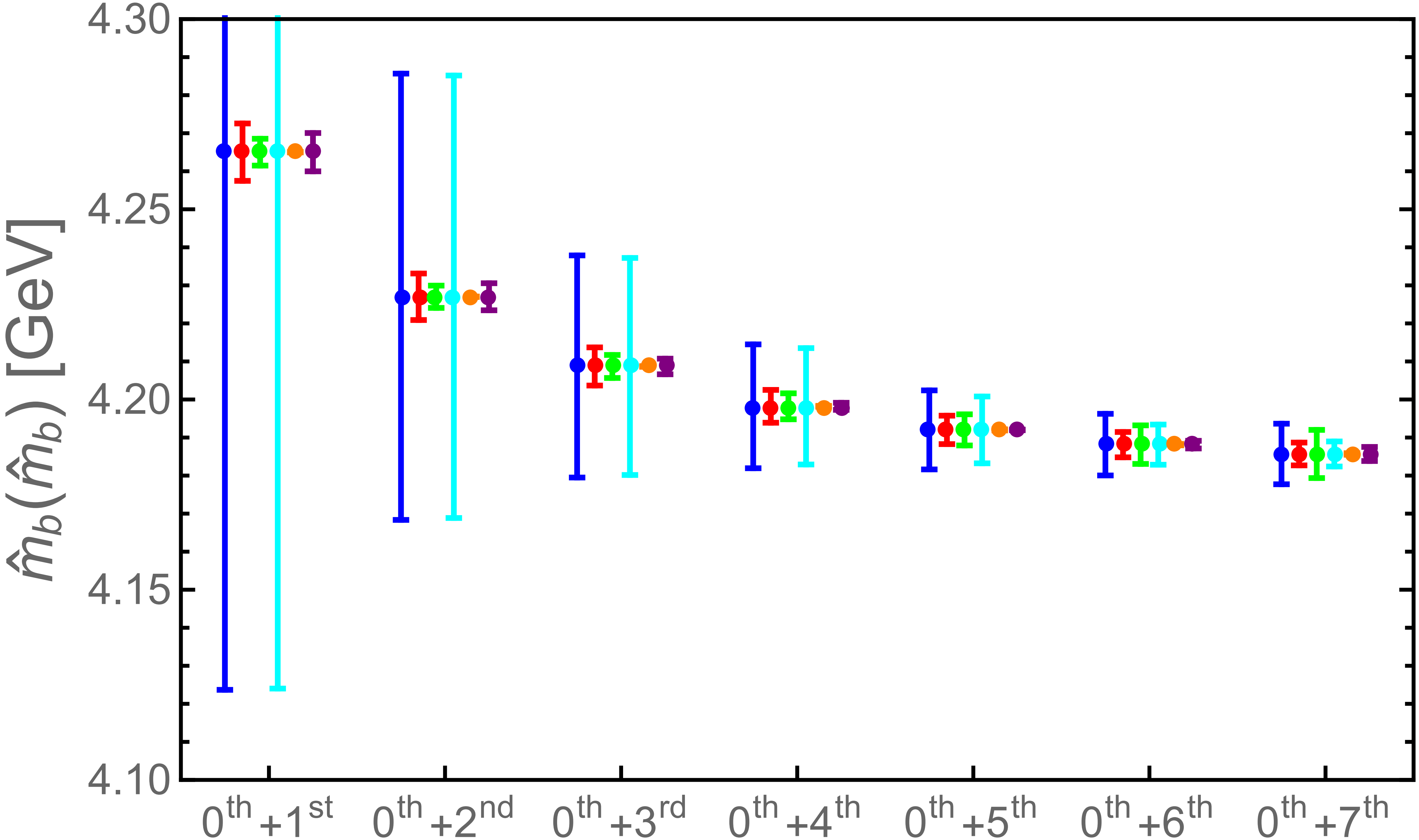
$$R_b^{\rm cont}(s) = 3Q_b^2 \lambda_1^{b,m}(s)\sqrt{1-\frac{4\,\hat{m}_b^2(2M)}{s'}}\left[1+\lambda_3^b\left(\frac{2\,\hat{m}_b^2(2M)}{s'}+\left(\frac{2\,\hat{m}_b^2(2M)}{s'}\right)^2\right)\right]$$





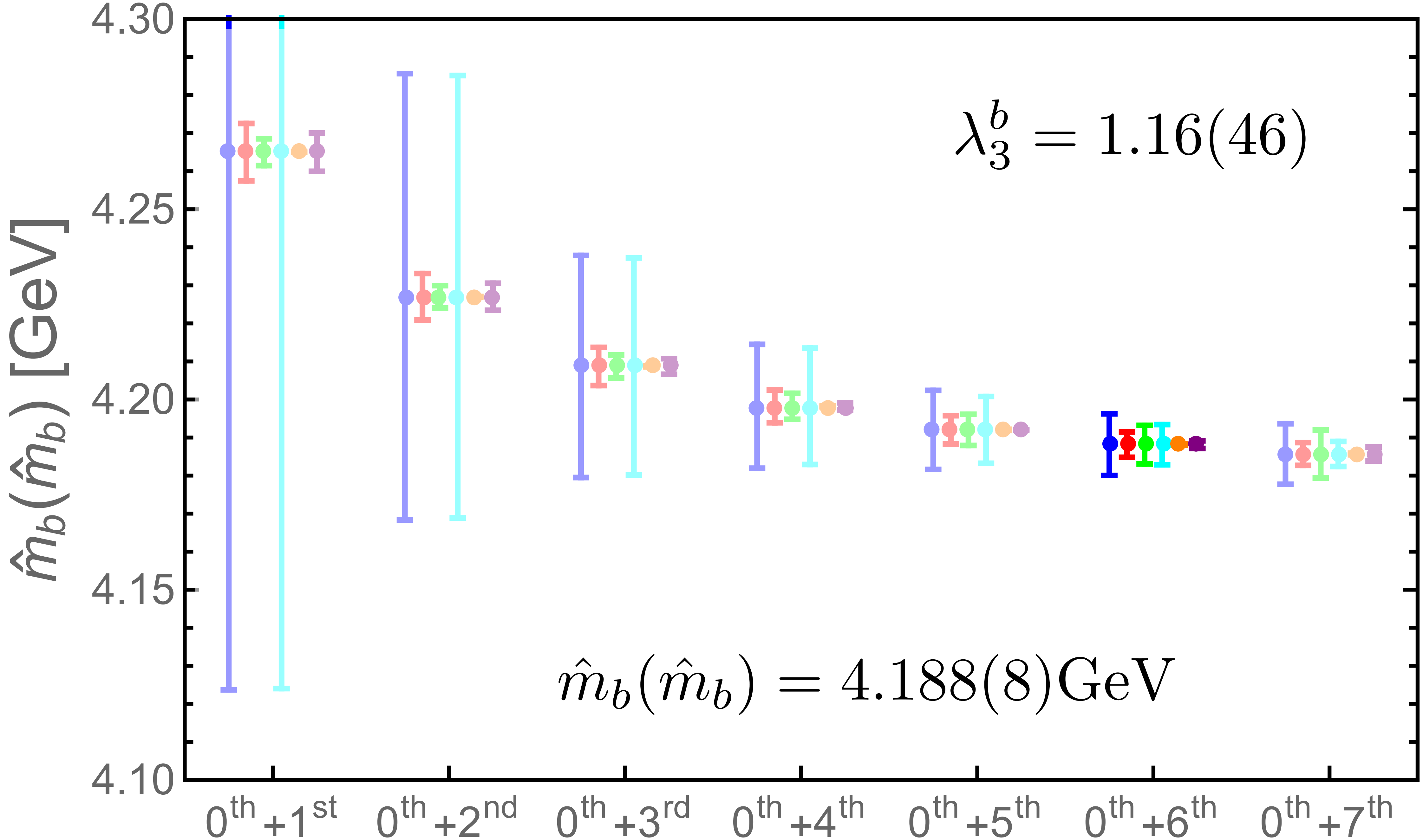
QCD Sum Rules

(preliminary)



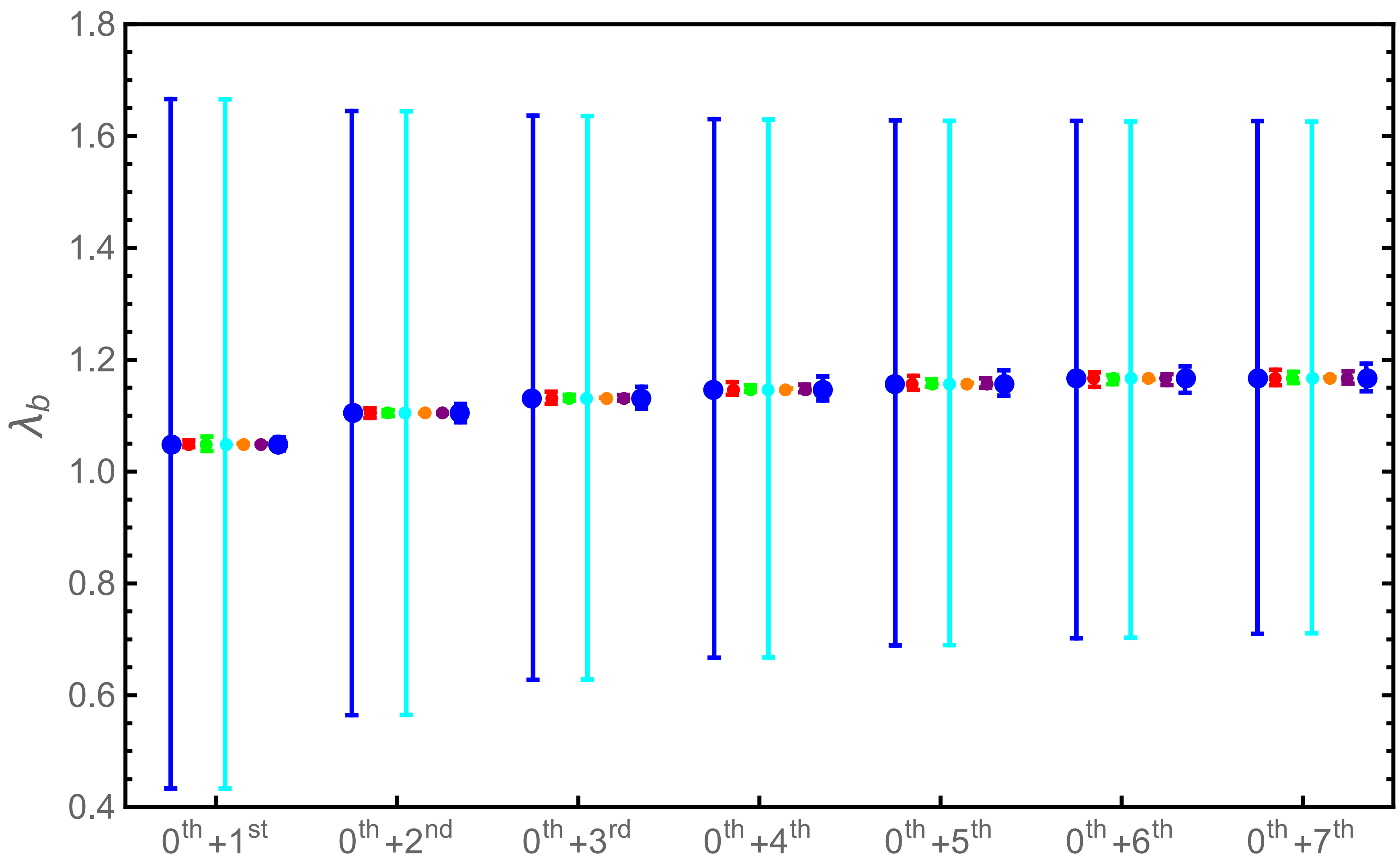
QCD Sum Rules

(preliminary)



QCD Sum Rules

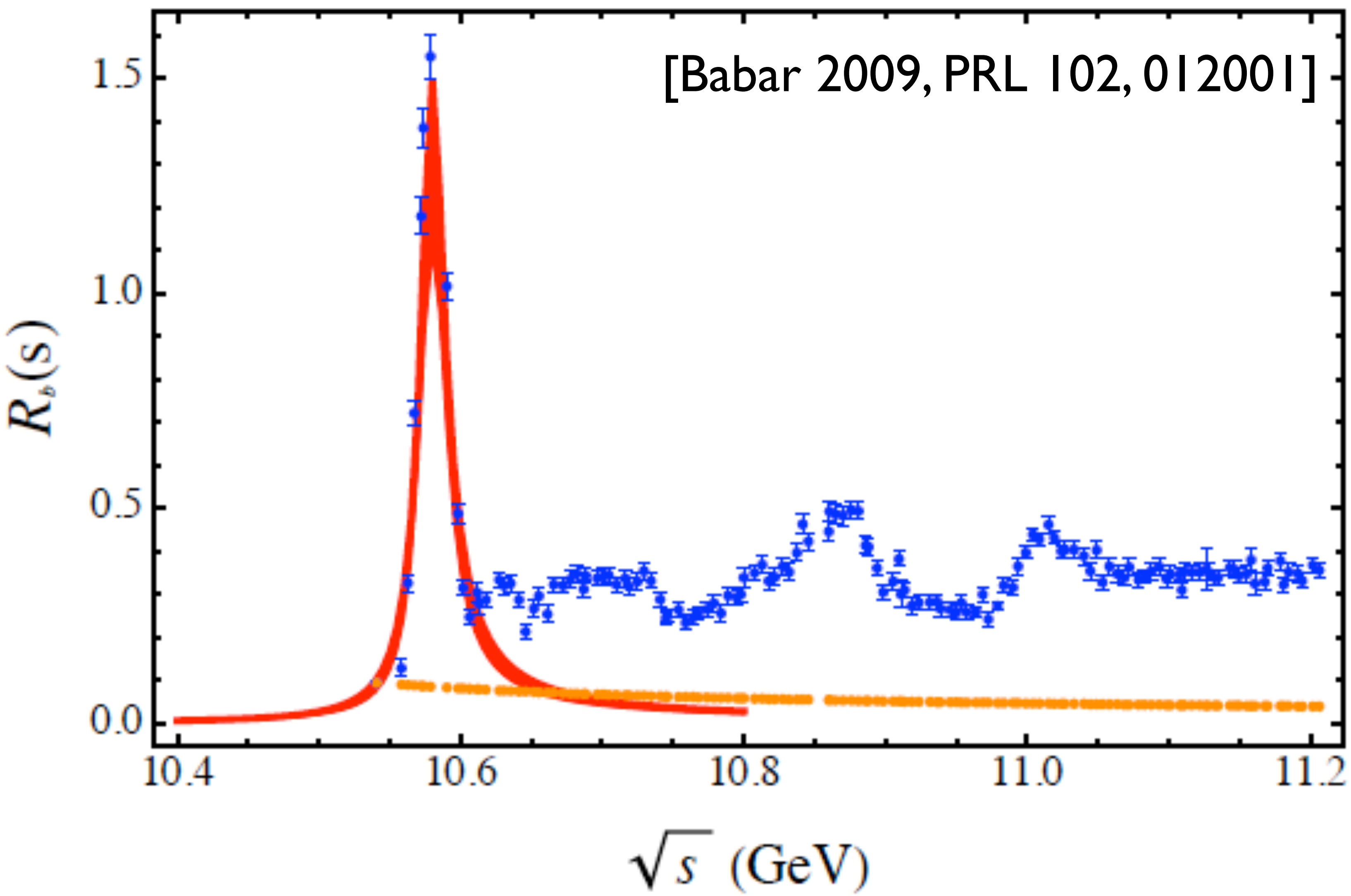
(preliminary)



Comparison with R^{Exp} threshold data: symmetrised asymmetric error

QCD Sum Rules

(preliminary)



Vacuum polarization

$$(\alpha(0)/\alpha(M_R))^2 \equiv 0.93$$

Radiative tails

ISR corrections

$$\hat{R}(s) = \int_{z_0}^{1} \frac{\mathrm{d}z}{z} G(z,s) R(zs)$$

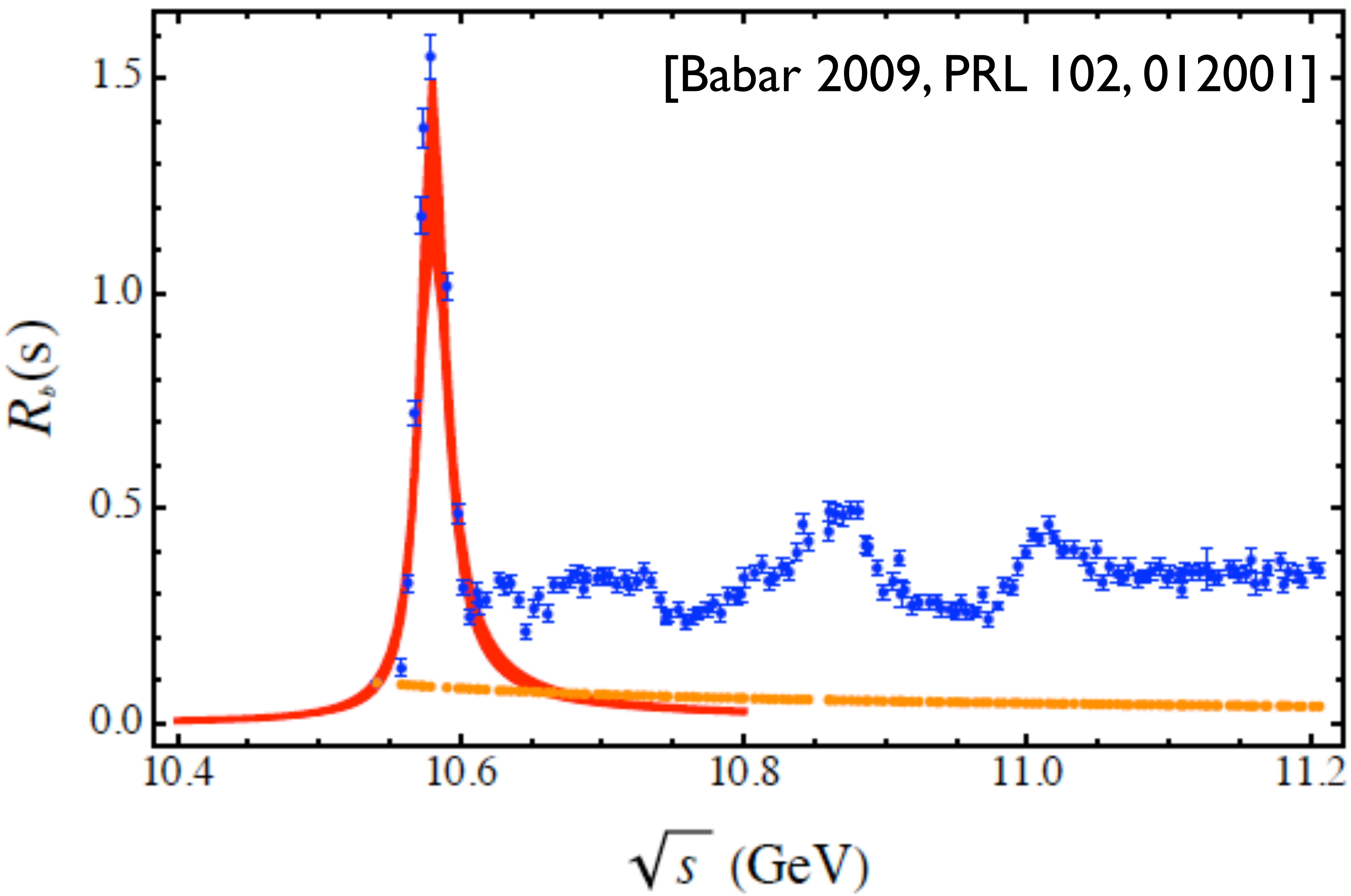
$$z_0 = 10.6^2/s$$

BW param for $\Upsilon(4S)$

$$BW(s) = \frac{9}{\alpha(M_R^2)^2} \frac{M_R^2 \Gamma \Gamma_R^e}{(s - M_R^2)^2 + \Gamma^2 M_R^2}$$

QCD Sum Rules

(preliminary)

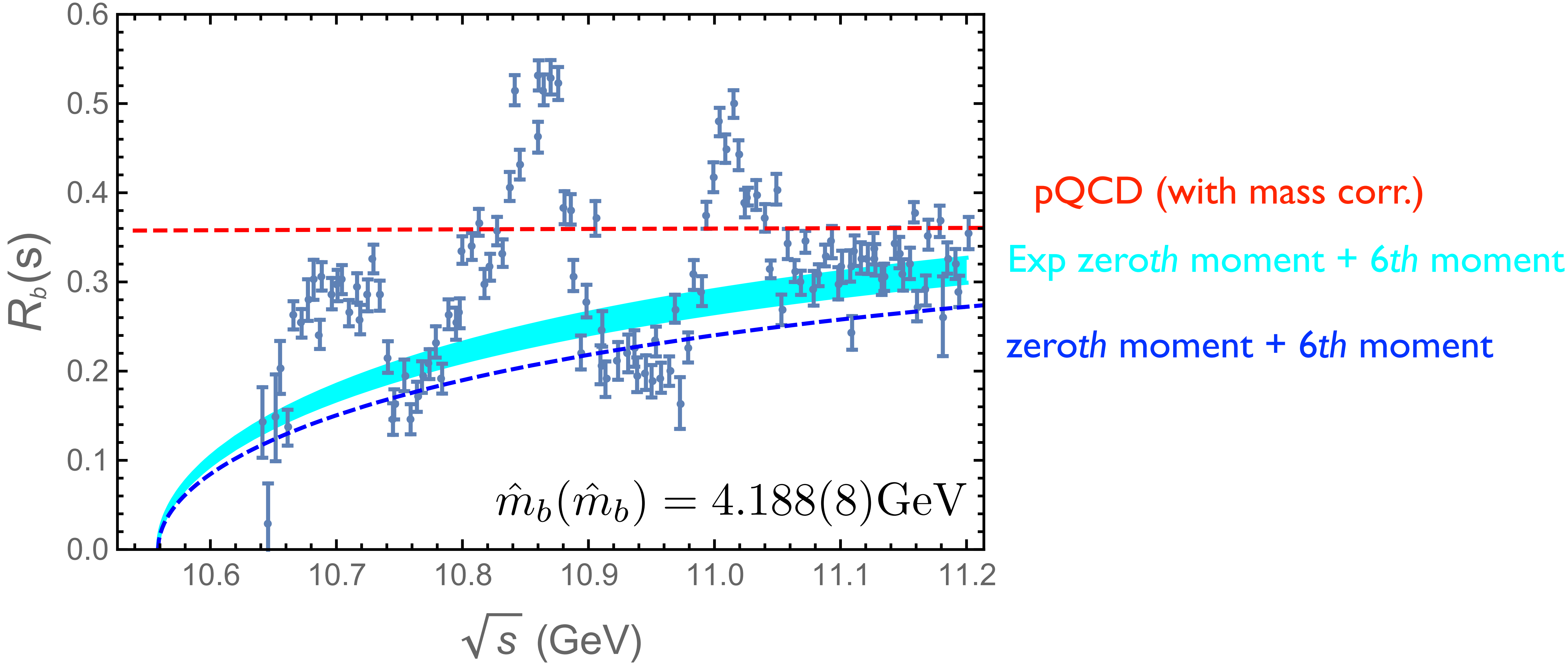


n	Hoang et al	This work
0	-	0.321(2)(11)(5)
1	0.270(2)(9)	0.269(2)(9)(4)
2	0.226(1)(8)	0.226(1)(8)(3)
3	0.190(1)(7)	0.189(1)(7)(3)
4	0.159(1)(6)	0.159(1)(6)(3)

Errors:
- Stat. error
- Sys. error
- BW inputs

QCD Sum Rules

(preliminary)



Conclusions and Outlook

- Using SR technique + *zeroth moment* (very sensitive to the continuum) + data on charm resonances below threshold + continuum exploiting self-consistency among different moments:

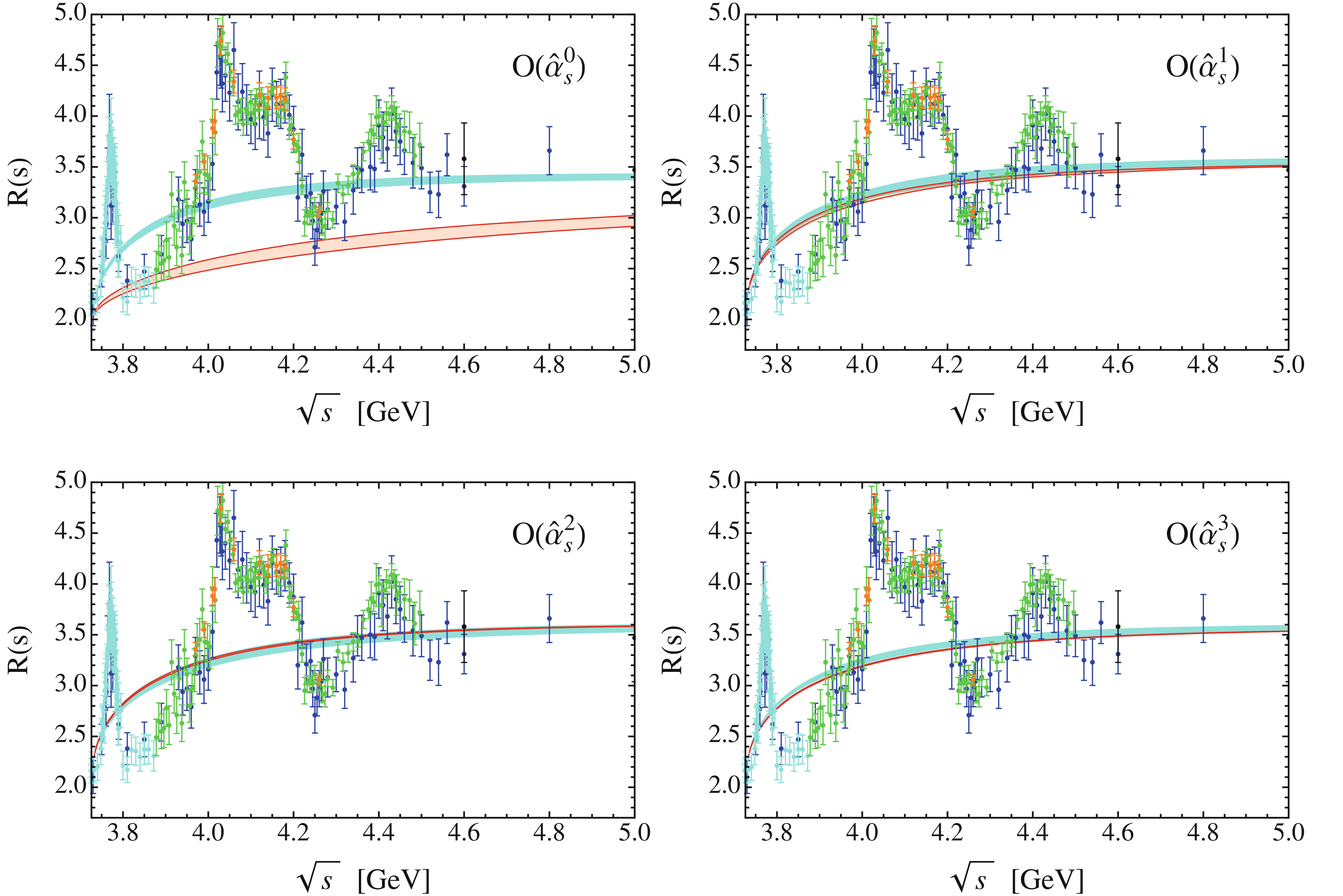
$$\hat{m}_c(\hat{m}_c) = 1.272(9)\mathrm{GeV} \qquad \hat{m}_b(\hat{m}_b) = 4.188(8)\mathrm{GeV}$$

(preliminary)

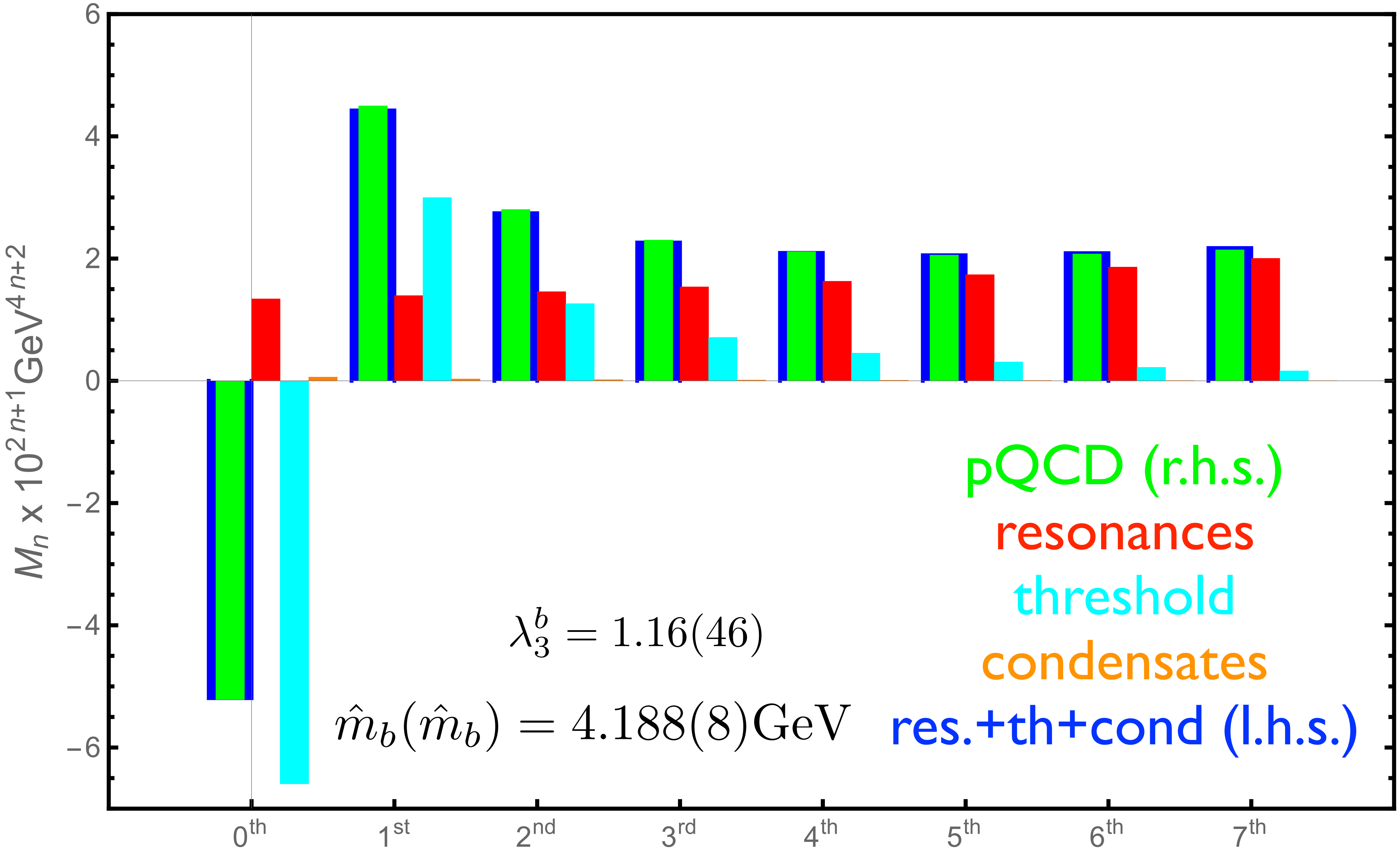
- We confirm the result using SR + global fit using *different* moments (χ^2)
 Good agreement with other determinations based on SRs and lattice!
- Error sources are understood: seems a clear roadmap for improvements
- Next step: improve the bottom case (more subtle than expected)

Thanks!

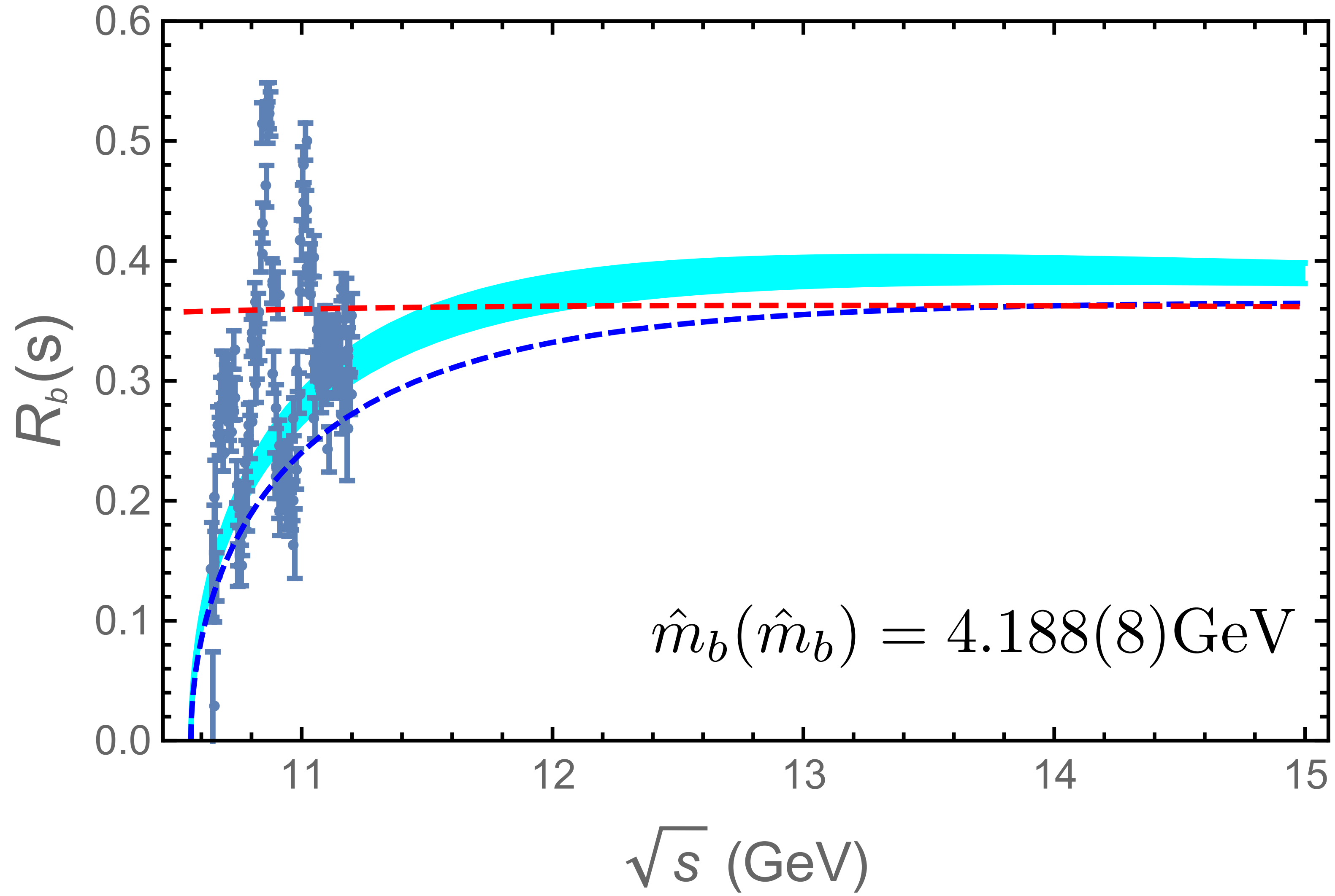
QCD Sum Rules



Consistency among moments



Validity of pQCD around threshold



Role of Belle data