

Towards charm- and bottom-quark masses with five-loop accuracy

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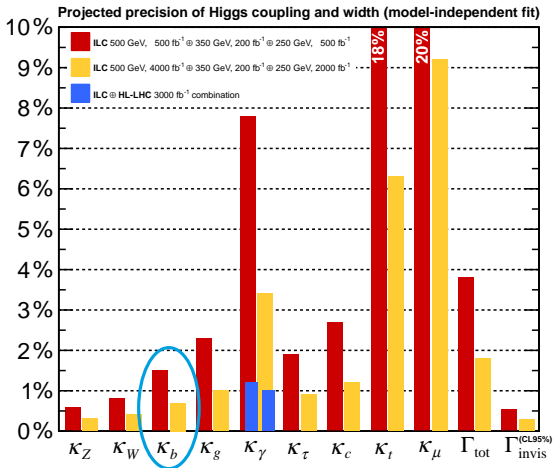
York Schröder



São Paulo, 30 September 2019

How much precision do we need?

[arXiv:1506.05992]



Higgs coupling proportional to mass \Rightarrow need $\Delta m_b < 0.5\%$

How much precision do we have?

[Chetyrkin, Kühn, Maier, Maierhöfer, Marquard, Steinhauser, Sturm 2009]

$$\begin{aligned}m_b(10 \text{ GeV}) &= (3610 \pm 10(\text{exp}) \pm 12(\alpha_s) \pm 3(\mu)) \text{ MeV} \\ &= (3610 \pm 16) \text{ MeV} \\ m_b(m_b) &= (4163 \pm 16) \text{ MeV}\end{aligned}$$

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[Dehnadi, Hoang, Mateu 2015]

$$\begin{aligned}m_b(m_b) &= (4176 \pm 4(\text{stat}) \pm 19(\text{sys}) \pm 7(\alpha_s) \pm 10(\mu)) \text{ MeV} \\ &= (4176 \pm 23) \text{ MeV}\end{aligned}$$

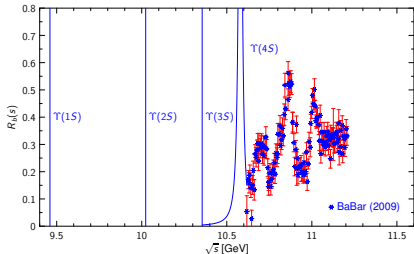
Goal: eliminate theory error

How do we determine heavy-quark masses?

Total inclusive cross section for $e^+e^- \rightarrow Q\bar{Q}$ near threshold

- Highly sensitive to heavy-quark mass m_Q
- Experimentally clean
- Perturbative predictions to high orders
- Non-perturbative effects
 - ▶ negligible for top quarks
 - ▶ highly suppressed in *sum rules* for charm and bottom quarks

Sum rules



Consider *moments* \mathcal{M}_n of $R_Q(s) = \frac{\sigma(e^+e^- \rightarrow Q\bar{Q})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$

$$\mathcal{M}_n = \int_{s_0}^{\infty} ds \frac{R_Q(s)}{s^{n+1}} = \frac{12\pi^2}{n!} \underbrace{\left[\left(\frac{d}{dq^2} \right)^n \Pi_Q(q^2) \right]_{q^2=0}}_{\text{perturbative for } n \lesssim 10}$$

Extract quark mass from $\mathcal{M}_n^{\text{theory}} = \mathcal{M}_n^{\text{experiment or lattice}}$

Sum rules

Use fixed-order perturbation theory for $n \sim 1$:

$$\mathcal{M}_n^{\text{theory}} = \frac{12\pi^2}{n!} \left(\frac{d}{dq^2} \right)^n \left[\frac{1}{\bar{q}} \text{circle} + \text{circle with vertical line} + \text{circle with horizontal line} + \text{circle with diagonal lines} + \dots \right]_{q^2=0}$$

- Exact three-loop results up to $n = 30$

[Chetyrkin, Kühn, Steinhauser 1995]

[Boghezal, Czakon, Schutzmeier 2006; Maier, Maierhöfer, Marquard 2007]

- Exact four-loop results up to $n = 4$

[Chetyrkin, Kühn, Sturm 2006; Boghezal, Czakon, Schutzmeier 2006]

[Maier, Maierhöfer, Marquard, Smirnov 2008–2009; Maier, Marquard 2017]

- Approximate four-loop results up to $n = 10$

[Hoang, Mateu, Zebarjad 2008; Kiyoyama, Maier, Maierhöfer, Marquard 2009; Greynat, Masjuan, Peris 2011]

Next: $n = 1$ at five loops

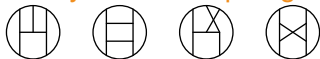
The first moment at five loops

- 1 Generate diagrams (qgraf [Nogueira 1991])

$$\mathcal{M}_1^{5 \text{ loop}} = 12\pi^2 \frac{d}{dq^2} \left[\text{Diagram} + (18378 \text{ others}) \right]_{q^2=0}$$

The first moment at five loops

- 1 Generate diagrams (`qgraf` [Nogueira 1991])
- 2 Identify vacuum topologies



34 combinations of massive and massless lines

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- 3 Insert Feynman rules, Simplify expressions, ...

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Integration-by-parts (IBP) identities:

[Chetyrkin, Tkachov 1981]

$$\begin{aligned} T(a_1) &= \text{Diagram} = \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - m_Q^2)^{a_1}} \\ 0 &= \int \frac{d^d l}{(2\pi)^d} \frac{\partial}{\partial l_\mu} l_\mu \frac{1}{(l^2 - m_Q^2)^{a_1}} \\ &= \int \frac{d^d l}{(2\pi)^d} \frac{d}{(l^2 - m_Q^2)^{a_1}} - a_1 \frac{2l^2}{(l^2 - m_Q^2)^{a_1+1}} \\ &= dT(a_1) - 2a_1[T(a_1) + m_Q^2 T(a_1 + 1)] \end{aligned}$$

- ▶ Insert numerical values for a_1
- ▶ Solve linear system of equations $\Rightarrow T(1)$ as master integral

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- 5 Calculate & insert master integrals

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$$I_i = \int \frac{1}{D_1^{\alpha_{i,1}} \dots D_{15}^{\alpha_{i,15}}}, \quad \alpha_{i,j} \in \{0, 1, \dots, ?\}$$

Preferred method: difference equations (SPADES [Luthe, Schröder])

[Laporta 2000]

Alternative: sector decomposition (FIESTA [Smirnov et al. 2008–2015])

[Hepp 1966; Speer 1968; Speer 1977; Binoth, Heinrich 2000]

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- 5 Calculate & insert master integrals
- 6 Renormalise

Calculation of master integrals

Difference equations

- Raise one denominator to symbolic power x :

$$I_i(x) = \int \frac{1}{D_1^x \cdots D_{15}^{\alpha_P}}$$

- IBP reduction \Rightarrow Coupled first-order difference equations

$$I_i(x \pm 1) = \sum_j p_{ij}^{\pm}(d, x) I_j(x)$$

- Recurrence relations from factorial series ansatz

$$I_i(x) = \sum_{s=0}^{\infty} a_{i,s} \frac{\Gamma(x+1)}{\Gamma(x+s-d/2+1)}$$

- Boundary conditions from $x \rightarrow \infty$

\Rightarrow high-precision numerical solution

Calculation of master integrals

Sector decomposition

Goal: numerical integrations over hypercube

- 1 Introduce Feynman parameters ($N_\alpha = \sum_j \alpha_j$)

$$I(x) = \int \frac{1}{D_1^{\alpha_1} \cdots D_N^{\alpha_N}}$$
$$\propto \prod_{j=1}^N \left(\int_0^\infty dx_j x_j^{\alpha_j-1} \right) \delta\left(1 - \sum_j x_j\right) \frac{\mathcal{U}^{N_\alpha-3d}}{\mathcal{F}^{N_\alpha-5d/2}}$$

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- 2 Transform to sector integrals over hypercube
 - ▶ Split into *primary sectors* $s = 1, \dots, N$ such that $x_s \geq x_j$
 - ▶ Rescale integration variables $x_j = t_j x_s$ with $t_j \in [0, 1]$

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- 2 Transform to sector integrals over hypercube

$$I(x) \propto \sum_{s=1}^N \prod_{j=1}^{N-1} \left(\int_0^1 dt_j t_j^{\alpha_j-1} \right) \frac{\mathcal{U}_s^{N_\alpha-3d}}{\mathcal{F}_s^{N_\alpha-5d/2}}$$

- 3 Iteratively decompose until $\mathcal{F}_s, \mathcal{U}_s \neq 0$ for all $t_j = 0$

Example: zero for $t_1 = t_2 = 0$

$$t_2 \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \begin{array}{|c|} \hline \text{I} \\ \hline \end{array} = \tilde{t}_2 t_1 \begin{array}{|c|} \hline \text{I} \\ \hline \end{array} + t_2 \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \begin{array}{|c|} \hline \tilde{t}_1 t_2 \\ \hline \end{array}$$

Calculation of master integrals

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- 4 Expand around $d = 4$

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- 3 Iteratively decompose until $\mathcal{F}_s, \mathcal{U}_s \neq 0$ for all $t_j = 0$
- 4 Expand around $d = 4$
- 5 Integrate \Rightarrow low-precision numerical result

The first moment at five loops

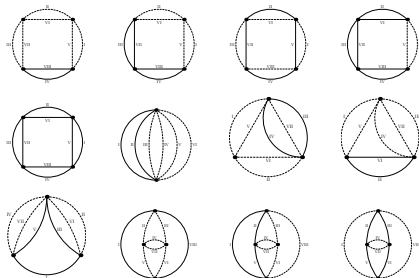
First results

n_h : Closed heavy (massive) quark loops

n_l : Closed light (massless) quark loops

	n_l^0	n_l^1	n_l^2	n_l^3
n_h^1	✗	✗	✓	✓
n_h^2	✗	✗	✓	
n_h^3	✗	✓		
n_h^4	✓			

Master integrals:



The first moment at five loops

Preliminary results

$$\mathcal{M}_1^{5 \text{ loop}} = \frac{3\pi^2}{m_Q^2} \left(\frac{\alpha_s}{\pi} \right)^4 n_h C_F \left[0.6 T_F^3 n_l^3 + 1.2 T_F^3 n_l^2 n_h + 0.9 T_F^3 n_l n_h^2 \right. \\ \left. + 0.2 T_F^3 n_h^3 + (C_F - 5C_A) T_F^2 n_l^2 + \dots \right]$$

Conclusions

- Sum rules allow high-precision determinations of charm- and bottom-quark masses
- Goal: eliminate theory uncertainty
- First results at five loops