The Top Quark Mass at Lepton Colliders
and the Inclusive Top Quark Pair Production Cross Section from Threshold to Continuum

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Workshop on Determination of Fundamental QCD Parameters
02/10/2019
Outline

● Top Quark Mass Determination at Lepton Colliders
  ○ Overview
  ○ Threshold Scan
  ○ Radiative Events [Boronat, Fullana, Fuster, Gomis, Hoang, Mateu, AW, Marcel Vos]

● Top Quark Pair Production Cross Section from Threshold to Continuum
  [Dehnadi, Hoang, Mateu, Stahlhofen, AW]
  ○ Threshold Region
  ○ Continuum Region
  ○ Mass Schemes at and above Threshold
  ○ Matching at NNLL_{threshold} + NNNLO_{continuum}

● Conclusions
Top Quark Mass Determination at Lepton Colliders
Overview

lepton colliders

<table>
<thead>
<tr>
<th>Facility</th>
<th>Time Period</th>
<th>Energy (GeV)</th>
<th>Notes</th>
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</thead>
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<tr>
<td>LEP (CERN)</td>
<td>1989 – 2000</td>
<td>209</td>
<td></td>
</tr>
<tr>
<td>ILC (Japan)</td>
<td>proposed</td>
<td>250 – 1000</td>
<td>[Baer et al. 2013]</td>
</tr>
<tr>
<td>CLIC (CERN)</td>
<td>proposed</td>
<td>350 – 3000</td>
<td>[CLIC collaboration 2016]</td>
</tr>
<tr>
<td>CEPC (China)</td>
<td>proposed</td>
<td>91 – 240</td>
<td>[CEPC study group 2018]</td>
</tr>
<tr>
<td>FCC-ee (CERN)</td>
<td>proposed</td>
<td>90 – 365</td>
<td>[FCC collaboration 2019]</td>
</tr>
</tbody>
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Overview

top quark mass measurements (main methods)

- threshold scan 350 GeV $\sigma(e^+e^- \rightarrow t\bar{t})$ < 75 MeV precision [Simon 2019]
- radiative events 380 GeV, 500 GeV $\sigma(e^+e^- \rightarrow t\bar{t}\gamma)$ ~ 110 - 150 MeV precision [Boronat et al. 2019 - in preparation]
- direct reconstruction 380 GeV, 500 GeV invariant mass ~ 50 - 100 MeV precision [Abramowicz et al. 2019], [Seidel et al. 2013]
top quark mass measurements (main methods)

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- radiative events 380 GeV, 500 GeV \( \sigma(e^+e^- \rightarrow t\bar{t}\gamma) \) \( \sim \) 110 - 150 MeV precision [Boronat et al. 2019 - in preparation]
- direct reconstruction 380 GeV, 500 GeV invariant mass \( \sim \) 50 - 100 MeV precision

well-defined mass scheme (low-scale short-distance mass schemes)

Monte Carlo top quark mass ➞ talk from Daniel Samitz
Threshold Scan (340 GeV - 350 GeV)

- top mass: precision < 75 MeV [Simon 2019]
  (now: $m_{t}^{\text{MC}} = 172.9 \pm 0.4$ GeV [PDG])

- top width: precision < 100 MeV [Simon 2019]
  (now: $\Gamma_t = 1.42^{+0.19}_{-0.15}$ GeV [PDG])

- threshold also sensitive to top Yukawa coupling, strong coupling constant
Threshold Scan (340 GeV - 350 GeV)

- top mass: precision < 75 MeV [Simon 2019]
  
  \( m_t^{\text{MC}} = 172.9 \pm 0.4 \text{ GeV} \quad \text{[PDG]} \)

- top width: precision < 100 MeV [Simon 2019]
  
  \( \Gamma_t = 1.42^{+0.19}_{-0.15} \text{ GeV} \quad \text{[PDG]} \)

- threshold also sensitive to top Yukawa coupling, strong coupling constant

---

uncertainties for top quark mass determination

<table>
<thead>
<tr>
<th>Source</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>QCD scale variation</td>
<td>~ 40 MeV</td>
</tr>
<tr>
<td>parametric ( \alpha_s )</td>
<td>~ 30 MeV</td>
</tr>
<tr>
<td>(for ( \Delta \alpha_s = 0.001 ))</td>
<td></td>
</tr>
<tr>
<td>statistical</td>
<td>~ 20 MeV</td>
</tr>
<tr>
<td>systematic (experimental)</td>
<td>~ 25 - 50 MeV</td>
</tr>
</tbody>
</table>

[Abramowicz et al. 2019]
Radiative Events (380 GeV, 500 GeV)
[Boronat, Fullana, Fuster, Gomis, Hoang, Mateu, Vos, AW 2019 - to appear soon]

invariant mass of top quark pair:

$$ (q')^2 = s' = s \left( 1 - \frac{2E_{\gamma}}{\sqrt{s}} \right) $$
Radiative Events (380 GeV, 500 GeV)
[Boronat, Fullana, Fuster, Gomis, Hoang, Mateu, Vos, AW 2019 - to appear soon]

invariant mass of top quark pair:

\[(q')^2 = s' = s \left(1 - \frac{2E_\gamma}{\sqrt{s}} \right)\]

cross section factorizes (in ISR approximation):

\[
\frac{d\sigma_{t\bar{t}\gamma}}{d \cos \theta d \sqrt{s'}} = \frac{\alpha_{\text{em}}}{\pi \sqrt{s}} g(x, \theta) \sigma_{t\bar{t}}(s') + O(\alpha_{\text{em}}^2), \quad g(x, \theta) = \frac{2\sqrt{1-2x}}{x \sin^2 \theta} \left[1 - 2x + (1 + \cos^2 \theta)x^2\right], \quad x = \frac{E_\gamma}{\sqrt{s}}
\]

- large photon energy \(E_\gamma > 5\) GeV
- \(\theta\) integrated from 8° to 172°
- highest mass sensitivity for collinear top quarks
  - \(s' \sim 4m_t^2\)
  - radiative return to threshold
Radiative Events (380 GeV, 500 GeV)

[Boronat, Fullana, Fuster, Gomis, Hoang, Mateu, Vos, AW 2019 - to appear soon]

invariant mass of top quark pair:

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Radiative Events (380 GeV, 500 GeV)

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\]

\[
\begin{array}{|l|l|l|}
\hline
\text{cms energy} & \text{CLIC, } \sqrt{s} = 380 \text{ GeV} & \text{ILC, } \sqrt{s} = 500 \text{ GeV} \\
\text{luminosity [fb}^{-1}] & 500 & 1000 & 500 & 4000 \\
\text{statistical} & 140 \text{ MeV} & 90 \text{ MeV} & 350 \text{ MeV} & 110 \text{ MeV} \\
\text{theory} & 46 \text{ MeV} & 55 \text{ MeV} & \text{20 MeV} & \text{20 MeV} \\
\text{lum. spectrum} & 20 \text{ MeV} & 85 \text{ MeV} & \text{20 MeV} & \text{20 MeV} \\
\text{photon response} & 16 \text{ MeV} & 85 \text{ MeV} & \text{20 MeV} & \text{20 MeV} \\
\text{total} & 150 \text{ MeV} & 110 \text{ MeV} & 360 \text{ MeV} & 150 \text{ MeV} \\
\hline
\end{array}
\]
Inclusive Top Quark Pair Production Cross Section from Threshold to Continuum
Inclusive Cross Section - Theory Overview

non-relativistic QCD (NRQCD)

**NNLL**\_threshold
[Hoang, Stahlhofen 2013]
- QCD + LO electroweak (double-resonant)
- vNRQCD
- 1S mass

**NNNLO**\_threshold
[Beneke, Kiyo, Marquard, Penin, Piclum, Steinhauser 2015]
[Beneke, Kiyo, Maier, Piclumn]
- QCD + EW + Higgs
- pNRQCD
- PS mass
Inclusive Cross Section - Theory Overview

**non-relativistic QCD (NRQCD)**

- **NNLL**\textsuperscript{threshold}: [Hoang, Stahlhofen 2013]
  - QCD + LO electroweak (double-resonant)
  - vNRQCD
  - 1S mass

- **NNNLO**\textsuperscript{threshold}: [Beneke, Kiyo, Marquard, Penin, Piclum, Steinhauser 2015]
  - QCD + EW + Higgs
  - pNRQCD
  - PS mass

**full QCD**

- **NNNLO**\textsuperscript{continuum}: [Hoang, Mateu, Zebarjad 2009]
- **NNLL**\textsuperscript{continuum}: [Kiyo, Maier, Maierhofer, Marquard 2009]
  - QCD corrections

---

**Inclusive QCD Cross Section** $\sigma(e^+e^- \rightarrow \gamma^* \rightarrow t\bar{t})$

$
\sigma_{\text{matched}}^{\text{NNLL}} \sim \alpha_s
$

---

\[\sqrt{s} \text{ [GeV]}\]
Inclusive Cross Section - Theory Overview

**non-relativistic QCD (NRQCD)**

- NNLL$^\text{threshold}$
  - QCD + LO electroweak (double-resonant)
  - vNRQCD
  - 1S mass

- NNNLO$^\text{threshold}$
  - QCD + EW + Higgs
  - pNRQCD
  - PS mass

**full QCD**

NNNLO$^\text{continuum}$

- QCD + LO electroweak (double-resonant)
- QCD + LO electroweak (double-resonant)
- vNRQCD
- 1S + MSR mass scheme

---

**Matched NRQCD + QCD**

- photon-induced cross section
- QCD + LO EW at threshold
- 1S + MSR mass scheme

---

**Inclusive QCD Cross Section $\sigma(e^+e^- \rightarrow \gamma^* \rightarrow t\bar{t})$**

$v \sim \alpha_s$
Inclusive Cross Section -

Threshold
Threshold - Coulomb Resummation

At threshold: \( v \sim \alpha_s, \; \alpha_s \log(v) \sim 1 \)
Threshold - Coulomb Resummation

At threshold: $v \sim \alpha_s, \alpha_s \log(v) \sim 1$

→ ladder diagrams are enhanced

\[ e^- + e^+ \rightarrow t + \bar{t} \] + \[ \frac{v}{\alpha_s} \] + \[ \left( \frac{v}{\alpha_s} \right)^2 \] + ... 

→ resummation of ladder diagrams with Schrödinger equation

→ numerical solution with Toppik [Hoang, Teubner 1999]
Threshold - Coulomb Resummation

At threshold: $v \sim \alpha_s, \alpha_s \log(v) \sim 1$

$\rightarrow$ ladder diagrams are enhanced

$e^- \quad t$  
\hspace{1cm} +  
\hspace{1cm} $\frac{v}{\alpha_s}$  
\hspace{1cm} +  
\hspace{1cm} $\left(\frac{v}{\alpha_s}\right)^2$  
\hspace{1cm} +  
\hspace{1cm} $\ldots$

$e^+ \quad \bar{t}$

$1$

$\rightarrow$ resummation of ladder diagrams with Schrödinger equation

$\rightarrow$ numerical solution with Toppik [Hoang, Teubner 1999]

$\rightarrow$ upgraded version of Toppik

- precision now $10^{-4}$
- 10 - 50 times faster than original version
Threshold - Coulomb Resummation

At threshold: \( v \sim \alpha_s, \alpha_s \log(v) \sim 1 \)

- resummation of ladder diagrams gives toponium resonances
- large top quark width smears out the top quark resonances

\[ \Gamma_t = 10^{-4} \text{ GeV} \]

\[ \Gamma_t = 1.5 \text{ GeV} \]

\[ \sqrt{s} + i\epsilon \rightarrow \sqrt{s} + i\Gamma_t \] (gives LO electroweak contributions at threshold) [Fadin, Khoze 1987]
Threshold - Large Logarithms

At threshold: \( v \sim \alpha_s, \alpha_s \log(v) \sim 1 \)

\[ \Rightarrow \text{resummation with vNRQCD (velocity non-relativistic QCD) [Hoang, Stahlhofen 2013]} \]

Contributions to the cross section at threshold:

\[
\sigma_{\text{NRQCD}}^{\text{NNLL}} = v \sum_{n,m} \left( \frac{\alpha_s}{v} \right)^n (\alpha_s \log v)^m \quad \text{LL}
\]

\[ + v^2 \sum_{n,m} \left( \frac{\alpha_s}{v} \right)^n (\alpha_s \log v)^m \quad \text{NLL} \]

\[ + v^3 \sum_{n,m} \left( \frac{\alpha_s}{v} \right)^n (\alpha_s \log v)^m \quad \text{NNLL} \]
Threshold - Large Logarithms

At threshold: \( v \sim \alpha_s, \alpha_s \log(v) \sim 1 \)

\( \rightarrow \) resummation with vNRQCD (velocity non-relativistic QCD) [Hoang, Stahlhofen 2013]

Contributions to the cross section at threshold:

\[
\text{Threshold Cross Section}
\]

\( \sigma_{\text{NRQCD}} \) [pb]

\( \sqrt{s} \) [GeV]

\( m_t = 171.6 \text{ GeV} \)

(error bands from variation of renormalization scales)
Inclusive Cross Section - Continuum
Continuum Cross Section

The inclusive cross section is related to the vacuum polarization by the optical theorem:

\[
\sigma_{t\bar{t}} = \frac{(4\pi\alpha)^2}{s} Q_t^2 \text{Im} \left[ \Pi(\sqrt{s} + i\Gamma_t) \right]
\]

In the continuum:

\[
\sigma_{QCD}^{N^3\text{LO}} = \frac{(4\pi\alpha)^2}{s} Q_t^2 \text{Im} \left[ \Pi^{(0)} + \alpha_s \Pi^{(1)} + \alpha_s^2 \Pi^{(2)} + \alpha_s^3 \Pi^{(3)} \right]
\]

\(\Pi^{(0)},\ \Pi^{(1)}\) \(\ldots\) known analytically

\(\Pi^{(2)},\ \Pi^{(3)}\) \(\ldots\) reconstructed with Padé approximations

(validity of Padé approximations for \(\Pi^{(2)}\) shown by comparison to exact numerical result in [Maier, Marquard 2017])
The inclusive cross section is related to the vacuum polarization by the optical theorem:

\[
\sigma_{t\bar{t}} = \frac{(4\pi\alpha)^2}{s} Q_t^2 \text{Im} \left[ \sum \pi \right] = \frac{(4\pi\alpha)^2}{s} Q_t^2 \text{Im} \left[ \Pi(\sqrt{s} + i\Gamma_t) \right]
\]

In the continuum:
Inclusive Cross Section -

Mass Schemes

- pole mass scheme
- 1S mass scheme
- MS mass scheme
- MSR mass scheme

- renormalon
- for the threshold
- for the continuum
- for all regions
Mass Schemes - Pole Mass

Full propagator:

\[ S_F^0 = \frac{i}{\not{p} - m_0 + \Sigma(\not{p}, m_0)} \]

Pole mass:

\[ \not{p} - m_0 + \Sigma(\not{p}, m_0)|_{p^2=m_{pole}^2} = 0 \]

\[ \Sigma(\not{p}, m_0) = \cdots + \ldots \]

→ pole mass renormalon leads to bad convergence of the cross section already at lower orders

Renormalon at threshold:
Mass Schemes - MS Mass

Full propagator: 

\[ S_F = \frac{i}{\phi - \overline{m}(\mu) + \sum_{\text{finite}}(\phi, \overline{m}(\mu))} \, \Sigma(\phi, m_0) = \] 

Conversion:

\[ m_{\text{pole}} = \overline{m} + \sum_{n=1}^{\infty} a_n(n_l, n_h) \alpha_s(\overline{m})^n \]

\[ = \overline{m} + \overline{m} \alpha_s a_1 + \ldots \quad (\overline{m} = \overline{m}^{(n_l+1)}(\overline{m}(n_l+1)) \big) \]

\[ \sim m v \]

→ works in the continuum, but not at threshold

Breaking of non-relativistic power counting at threshold in the MS scheme:

\[ v_{\text{pole}} = \sqrt{\frac{\sqrt{s} - 2 m_{\text{pole}}}{m_{\text{pole}}}} \]

\[ = \sqrt{\frac{\sqrt{s} - 2 (\overline{m} + \overline{m} a_1 \alpha_s/4\pi)}{\overline{m} + \overline{m} a_1 \alpha_s/4\pi}} \]

\[ = v_{\text{MS}} - a_1 \left( \frac{\alpha_s}{4\pi} \right) \left( \frac{v_{\text{MS}}}{2} + \frac{1}{v_{\text{MS}}} \right) + a_1^2 \left( \frac{\alpha_s}{4\pi} \right)^2 \left( \frac{3 v_{\text{MS}}}{8} + \frac{1}{2 v_{\text{MS}}} - \frac{1}{2 v_{\text{MS}}^3} \right) + \mathcal{O}(\alpha_s^3) \]

\[ \sim \alpha_s \]

\[ \sim \alpha_s^0 \]

\[ \sim \alpha_s^{-1} \]
Mass Schemes - MS Mass

Full propagator:

\[
S_F = \frac{i}{\not\!p - \bar{m}(\mu) + \sum_{\text{finite}}(\not\!p, \bar{m}(\mu))}, \quad \Sigma(\not\!p, m_0) = \quad + \ldots
\]

Conversion:

\[
m_{\text{pole}} = \bar{m} + \bar{m} \sum_{n=1}^{\infty} a_n(n_l, n_h) \alpha_s(\bar{m})^n = \bar{m} + \bar{m} \alpha_s a_1 + \ldots \quad (\bar{m} = m_l^{(n_l+1)}(\bar{m}(n_l+1)))
\]

\[\sim m v\]

→ works in the continuum, but not at threshold

Breaking of non-relativistic power counting at threshold in the MS scheme:
Mass Schemes - 1S Mass

[Hoang, Ligeti, Manohar 1998]

Mass of 1S resonance: \[ M_{1S}^{3S1} = E_{\text{bin}} + 2m_{\text{pole}} \]

1S mass: \[ m_{1S} = \frac{1}{2} M_{tt}^{3S1} = m_{\text{pole}} + \frac{1}{2} E_{\text{bin}} \]

\[ \Gamma_t = 10^{-4} \text{ GeV} \]

other low-scale short-distance mass schemes:
PS mass [Beneke 1998], RS mass [Pineda 2001], kinetic mass [Czarnecki, Melnikov, Uraltsev 1998]
Mass Schemes - 1S Mass

Mass of 1S resonance: \( M_{tt}^{3S_1} = E_{\text{bin}} + 2m_{\text{pole}} \)

1S mass:
\[
m_{1S} = \frac{1}{2} M_{tt}^{3S_1} = m_{\text{pole}} + \frac{1}{2} E_{\text{bin}}
\]

Conversion:
\[
m_{1S} = m_{\text{pole}} + (C_F \alpha_s(\mu)m_{\text{pole}}) \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} c_{n,k} \alpha_s(\mu)^n \log \left( \frac{\mu}{C_F \alpha_s(\mu)m_{\text{pole}}} \right)
\]
\[
= m_{\text{pole}} - \frac{2}{9} \alpha_s^2 m_{\text{pole}} + \ldots
\]
\[
\sim m v^2
\]

no breaking of the non-relativistic power counting at threshold:

\[
v_{\text{pole}} = \sqrt{\frac{\sqrt{s} - 2 m_{\text{pole}}}{m_{\text{pole}}}}
\]
\[
= \sqrt{\frac{\sqrt{s} - 2 (m_{1S} + m_{1S} a_1 \alpha_s/4\pi)}{m_{1S} + m_{1S} a_1 \alpha_s/4\pi}}
\]
\[
= v_{1S} - a_1 \left( \frac{\alpha_s}{4\pi} \right)^2 \left( \frac{v_{1S}}{2} + \frac{1}{v_{1S}} \right) + a_1^2 \left( \frac{\alpha_s}{4\pi} \right)^4 \left( \frac{3 v_{1S}}{8} + \frac{1}{2 v_{1S}} - \frac{1}{2 v_{1S}^3} \right) + \mathcal{O}(\alpha_s^6)
\]
\[
\sim \alpha_s
\]
\[
\sim \alpha_s
\]
\[
\sim \alpha_s
\]
Mass Schemes - MSR Mass

[Hoang, Jain, Scimemi, Stewart 2008], [Hoang, Jain, Lepenik, Mateu, Preisser, Scimemi, Stewart 2017]

Conversion:

\[ m_{\text{pole}} = \overline{m} + \overline{m} \sum_{n=1}^{\infty} a_n \alpha_s(\overline{m})^n = \overline{m} + \overline{m} \alpha_s a_1 + \ldots \]

\[ m_{\text{pole}} = m_{\text{MSR}}(R) + R \sum_{n=1}^{\infty} a_n \alpha_s(R)^n = m_{\text{MSR}}(R) + R \alpha_s a_1 + \ldots \]

→ no breakdown of the non-relativistic power counting at threshold
→ improves convergence of the continuum cross section in the intermediate region:

MSR mass scheme

\[ \Gamma_t = 1.5 \text{ GeV}, \overline{m}_t = 163 \text{ GeV} \]

R profile

\[ \sqrt{s} \text{ [GeV]} \]
Mass Schemes - MSR Mass

[Hoang, Jain, Scimemi, Stewart 2008], [Hoang, Jain, Lepenik, Mateu, Preisser, Scimemi, Stewart 2017]

Conversion:

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→ no breakdown of the non-relativistic power counting at threshold
→ improves convergence of the continuum cross section in the intermediate region:
Inclusive Cross Section - Matching

Inclusive QCD Cross Section $\sigma(e^+e^- \rightarrow \gamma^* \rightarrow t\bar{t})$

intermediate region

$\sigma[^{\text{NNLL}}\text{matched}][\text{pb}]$

$\sqrt{s} \text{ [GeV]}$

0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4

340 350 360 370 380 390 400
Matching

\[ \sigma_{\text{matched}} = \sigma_{\text{QCD}} + (\sigma_{\text{vNRQCD}} - \sigma_{\text{double-counted}}) \cdot f_s \]
Matching

\[ \sigma_{\text{matched}} = \sigma_{\text{QCD}} + (\sigma_{\text{vNRQCD}} - \sigma_{\text{double-counted}}) \cdot \hat{f}_s \]

\[
\sigma_{\text{NNLO}}^{\text{vNRQCD}} = \nu + \alpha_s + \frac{\alpha_s^2}{\nu} + \frac{\alpha_s^3}{\nu^2} + \frac{\alpha_s^4}{\nu^3} + \ldots \\
+ \nu^2 + \alpha_s \nu + \frac{\alpha_s^2}{\nu} + \frac{\alpha_s^3}{\nu^2} + \frac{\alpha_s^4}{\nu^2} + \ldots \\
+ \nu^3 + \alpha_s \nu^2 + \frac{\alpha_s^2}{\nu} + \frac{\alpha_s^3}{\nu^2} + \frac{\alpha_s^4}{\nu} + \ldots
\]

\[
\sigma_{\text{N^3LO}}^{\text{QCD}} = \nu + \nu^2 + \nu^3 + \nu^4 + \ldots \\
+ \alpha_s + \alpha_s \nu + \alpha_s \nu^2 + \alpha_s \nu^3 + \ldots \\
+ \frac{\alpha_s^2}{\nu} + \frac{\alpha_s^2}{\nu} + \frac{\alpha_s^2}{\nu} + \frac{\alpha_s^2}{\nu^2} + \ldots \\
+ \frac{\alpha_s^3}{\nu^2} + \frac{\alpha_s^3}{\nu} + \frac{\alpha_s^3}{\nu^2} + \frac{\alpha_s^3}{\nu^3} + \ldots
\]
Matching

\[ \sigma_{\text{matched}} = \sigma_{\text{QCD}} + (\sigma_{\text{vNRQCD}} - \sigma_{\text{double-counted}}) \cdot f_s \]

\[ \sigma_{\text{NNLO}}^{\text{vNRQCD}} = \begin{align*}
&v + \alpha_s + \frac{\alpha_s^2}{v} + \frac{\alpha_s^3}{v^2} + \frac{\alpha_s^4}{v^3} + \ldots \\
&+ v^2 + \alpha_s v + \alpha_s^2 + \frac{\alpha_s^3}{v} + \frac{\alpha_s^4}{v^2} + \ldots \\
&+ v^3 + \alpha_s v^2 + \alpha_s^2 v + \alpha_s^3 + \frac{\alpha_s^4}{v} + \ldots
\end{align*} \]

\[ \sigma_{\text{N}^3\text{LO}}^{\text{QCD}} = \begin{align*}
&v + v^2 + v^3 + v^4 + \ldots \\
&+ \alpha_s + \alpha_s v + \alpha_s v^2 + \alpha_s v^3 + \ldots \\
&+ \frac{\alpha_s^2}{v} + \alpha_s^2 + \alpha_s^2 v + \alpha_s^2 v^2 + \ldots \\
&+ \frac{\alpha_s^3}{v^2} + \alpha_s^3 v + \alpha_s^3 v^2 + \ldots \\
&+ \frac{\alpha_s^4}{v^3} + \alpha_s^4 v + \alpha_s^4 v^2 + \ldots
\end{align*} \]

\[ \sigma_{\text{double-counted}} \]
Matching

\[
\sigma_{\text{matched}} = \sigma_{\text{QCD}} + (\sigma_{\text{vNRQCD}} - \sigma_{\text{double-counted}}) \cdot f_s
\]

switch-off function:

- variation gives an error estimate of the matching
- introduces scheme dependence
- do we get convergence when going to higher orders?

mass schemes:

- \(\sigma_{\text{vNRQCD}}\)
- \(\sigma_{\text{QCD}}\)
- \(\sigma_{\text{double-counted}}\) 1S mass scheme
- MSR mass scheme
- MSR mass scheme
Matching

matched cross section from lowest to highest order:

- error from variation of renormalization scales and the switch off function
- matching smoothly connects threshold with continuum
- overall error reduces from order to order
Matching

- good convergence from order to order
- matching error smaller than variation of renormalization scales
- matching error reduces from order to order
Matching

threshold cross section vs. matched cross section

→ matched cross section starts to differ from the threshold cross section immediately above the peak region

→ higher order corrections from continuum cross section give small shift at threshold

continuum cross section vs. matched cross section

→ matched cross section and continuum MSR cross section overlap above 365 GeV

→ MSR mass scheme valid down to smaller center-of-mass energies than pole mass scheme and MS mass scheme
Theory Error for NNNLO\textsubscript{continuum} the cross section at NNNLO\textsubscript{continuum} shows a difference between the pole scheme and the MSR scheme:

- cross section in the pole scheme and the MSR scheme are incompatible (error bands do not overlap)
- scale variation seems to underestimate the error
- difference corresponds to 1 GeV difference in the top quark mass

- higher order mass corrections seem to favor MSR mass scheme
Conclusions

- The top quark pair production cross section at lepton colliders will provide high precision measurements for the top mass and width from a threshold scan and radiative events.

- We constructed a consistent matched cross section at QCD NNLL\textsubscript{threshold} + NNNLO\textsubscript{continuum} with LO electroweak corrections at threshold.

- The MSR mass provides a consistent mass scheme in all regions from threshold to the continuum.

- The cross section at NNNLO\textsubscript{continuum} in the pole and MSR scheme show a large difference.

- **Outlook:**
  - study on the difference of the continuum cross section in different mass schemes
  - differential matched cross section at NLL\textsubscript{threshold} + NLO\textsubscript{continuum}
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Thank you for your attention!