Heavy quark masses from heavy quarkonium

Antonio Pineda

Universitat Autònoma de Barcelona & IFAE

Workshop on determinations of fundamental QCD parameter, September 30- October 4, ICTP-SAIFR, Sao Paulo, Brazil, 2019

Motivation

m_Q. Fundamental parameter of the Standard Model.
Heavy Quarkonium (near threshold) physics very sensitive to its value.

Usual definitions:

- $m_{\overline{\rm MS}} \rightarrow$ short distance mass.
- $m_{\rm OS} \rightarrow$ natural definition for heavy quark physics near threshold.

Heavy Quarkonium (non-perturbative = Coulomb resummation) dynamics.

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Nonrelativistic systems near threshold

$$v=\sqrt{1-\frac{4m_Q^2}{s}}\ll 1$$

$$m_Q \gg m_Q v \gg m_Q v^2 \quad (EFT's)$$

$$\frac{\alpha_s}{v}\sim 1 \rightarrow \text{Coulomb resummation} \rightarrow \text{Schroedinger equation (pNRQCD)}$$

$$\text{LO} \sim \sum_{n=0}^{\infty} c_n \frac{\alpha_s^n}{v^n}$$

$$\text{NLO} \sim \sum_{n=0}^{\infty} c_n \frac{\alpha_s^n}{v^n} \times (\alpha_s, v)$$

$$\text{NLL} \sim \sum_{n=0}^{\infty} c_n \frac{\alpha_s^n}{v^n} \times (\alpha_s, v) \sum_{k=0}^{\infty} d_k \alpha_s^k \ln^k(\alpha, v)$$

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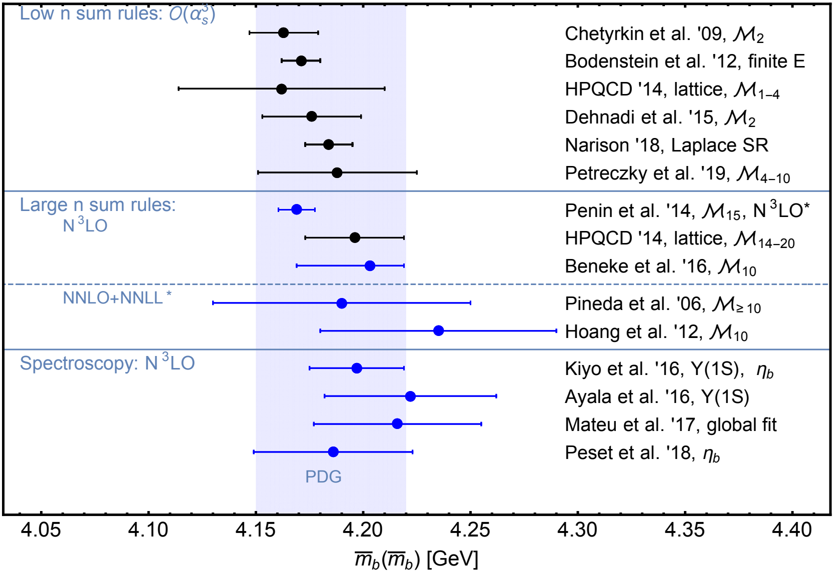
Heavy Quarkonium (non-perturbative = Coulomb resummation) dynamics:

- Heavy Quarkonium mass
- Non-relativistic sum rules
- t-$\bar{t}$ production near threshold

Highest sensitive to the heavy quark pole mass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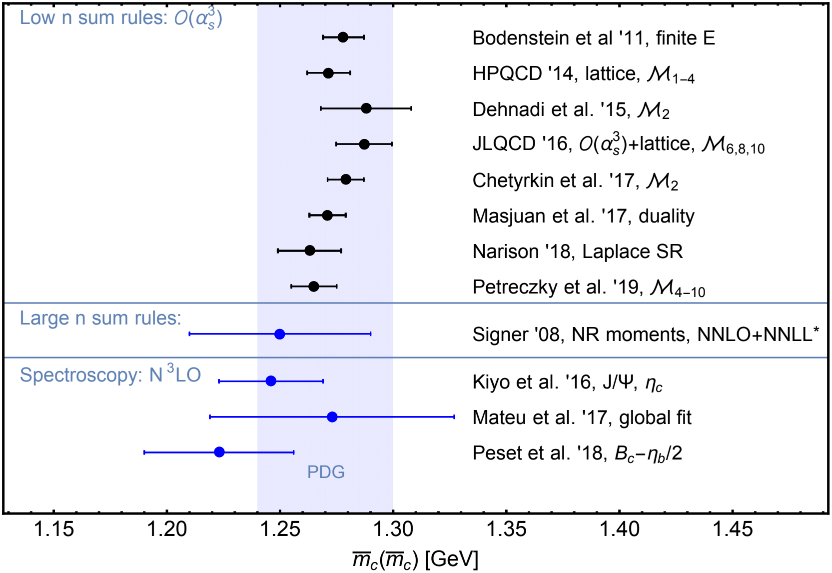

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$\overline{m}_c(\overline{m}_c)$ [GeV]

Spectrum

$$M_n = 2m - \frac{m\alpha^2}{4n^2}\left(1 + \frac{\alpha}{\pi}P_1(L_\nu) + \frac{\alpha^2}{\pi^2}P_2(L_\nu) + \frac{\alpha^3}{\pi^3}P_3(L_\nu)\right)$$

- $\mathcal{O}(m\alpha^5 \ln \frac{1}{\alpha})$ $m_1 = m_2$; Brambilla et al. (1999)
- $\mathcal{O}(m\alpha^5)$ $m_1 = m_2$ n=1; Penin, Steinhauser (2002)
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Fixed order computation $\rightarrow$ NNNLO
Resummation of logarithms $\rightarrow$ NNNLL (not yet complete)

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Pole mass and renormalons

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1} \,,$$

Renormalon (OPE) analysis predicts $r_n \sim n!$. Therefore $m_{\rm OS}$ suffers from renormalon ambiguities:

$$\delta^{\rm (pert.)}_{np} m_{\rm OS} = \delta^{\rm (pert.)}_{np} (m_{\overline{\rm MS}} + r_0\alpha_s + r_1\alpha_s^2 + \cdots) \sim \Lambda_{\rm QCD}!$$

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$$m_{\rm OS} = m_{\overline{\rm MS}} + \int_0^\infty {\rm d}t\, e^{-t/\alpha_s} B[m_{\rm OS}](t)\,, \qquad B[m_{\rm OS}](t) \equiv \sum_{n=0}^\infty r_n \frac{t^n}{n!}.$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$B[m_{\rm OS}](t) = N_m \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + \cdots\right) + (\text{analytic term}),$$

$$r_n \stackrel{n\to\infty}{=} N_m\, \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \cdots\right).$$

$$b = \frac{\beta_1}{2\beta_0^2}\,, \qquad c_1 = \frac{1}{4\, b \beta_0^3}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right)\,, \qquad \cdots$$

Beneke, Pineda, ...

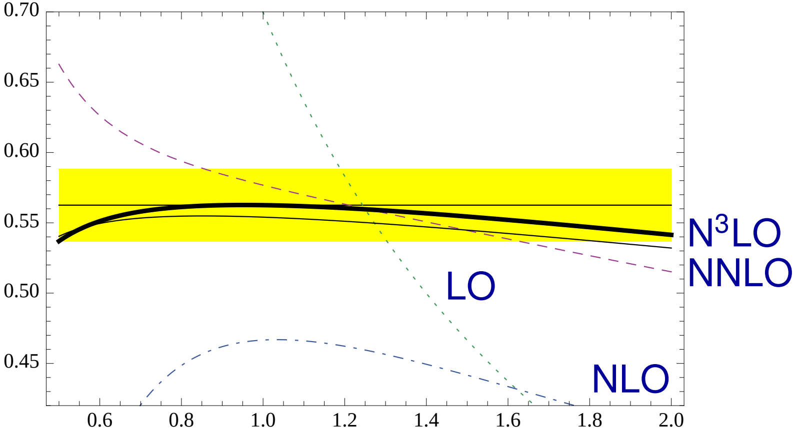

Figure: $2m + V$ renormalon free. $-N_V/2 = N_m$ for $n_l = 3$, as a function of $x \equiv \nu r$, obtained from $-(N_V/2)v_n/v_n^{asym}$. v_n^{asym} is truncated at $\mathcal{O}(1/n^3)$.

$$\boxed{N_m(n_l = 0) = 0.600(29)}, \qquad \boxed{N_m(n_l = 3) = 0.563(26)}.$$

$$M_{\Upsilon(nS)} = 2m_{\rm OS} + E_n^{\rm OS} = 2m_{\rm OS} + A_n^{1,{\rm OS}}\alpha^2 + A_n^{2,{\rm OS}}\alpha^3 + \cdots$$

$M_{\Upsilon(nS)}$ is renormalon free $\rightarrow E_n^{\rm OS}$ has a renormalon.

Renormalon free definitions:

$$m_{\rm RS}(\nu_f) = m_{\rm OS} - \sum_{n=0(1)}^{\infty} N_m\, \nu_f \left(\frac{\beta_0}{2\pi}\right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}\,.$$

Expansion in $\alpha_s(\nu)$

$$m_{\rm RS}(\nu_f) = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n^{\rm RS}\alpha_s^{n+1}\,,$$

where $r_n^{\rm RS} = r_n^{\rm RS}(m_{\overline{\rm MS}}, \nu, \nu_f)$ are of natural size. We now do not loose accuracy if we first obtain $m_{\rm RS}$ and later on $m_{\overline{\rm MS}}$.

$$M_{\Upsilon(nS)} = 2m_{\rm RS} + E_n^{\rm RS} = 2m_{\rm RS} + \sum_{m=1}^{\infty} A_n^{m,{\rm RS}}\alpha_s^{m+1}\,.$$

Alternative: kinetic, PS, MSR, MRS, PV, ...

Issues

Bare data set: $\bar{b}b$, $\bar{c}c$, $\bar{b}c$, ...
Options:
Ground state only
Ground state +excitations
Pseudoscalar/vector

Bottom mass determinations: finite mass charm quark effects.

NP vs ultrasoft effects

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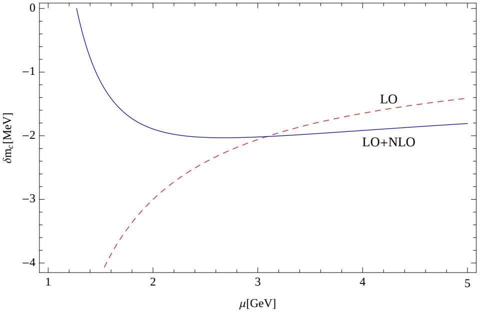
$n_f = 3$ light fermions in the $\overline{\rm MS}$-pole mass relation

Light fermions: $n_f = 4 \to n_f = 3$
For soft scale Brambilla et al.:

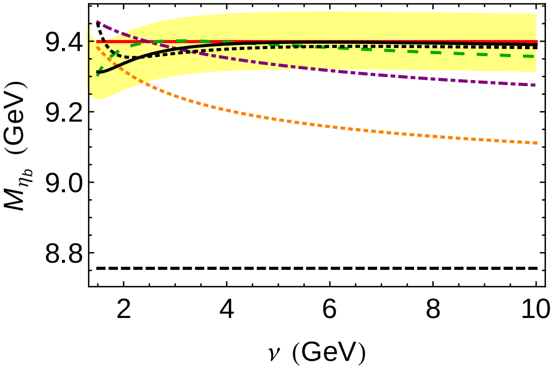
$$m_b \alpha_s < m_c$$

For hard scale Ayala et al.:

$$m_b e^{-n} < m_c$$

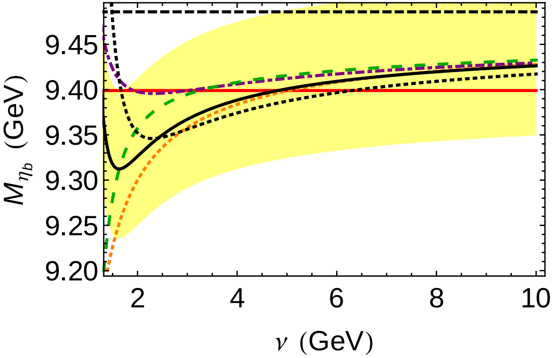


$\overline{m}_b$ from M_{η_b}



RS scheme
Good convergence

$\overline{m}_b$ from M_{η_b}



RS' determination (RS determination virtually identical); $\nu = 5$ GeV

$$\overline{m}_b(\overline{m}_b) = 4186(37)\ MeV$$

Error dominated by ν dependence.
Ultrasoft effects are not important.

Ultrasoft and NP effects

$$\delta E\Big|_{\mathcal{O}(\alpha_s^5)} = Am\alpha_s^5(\ln\frac{m\alpha_s}{\nu_{us}} + K)\Big|_{\rm soft} + \delta E_{us}(\nu_{us})$$

$$\delta E_{us}(\nu_{us}) \simeq \frac{T_F}{3N_c}\int_0^\infty dt \langle n|\mathbf{r}e^{-t(H_o-E_n)}\mathbf{r}|n\rangle\langle g\mathbf{E}^a(t)\phi(t,0)^{\rm adj}_{ab}g\mathbf{E}^b(0)\rangle(\nu_{us}),$$

where $H_o \equiv \frac{\mathbf{p}^2}{m} + \frac{1}{2N_c}\frac{\alpha_s}{r}$ and $E_n \equiv -mC_f^2\alpha_s^2/(4n^2)$.

If $\Lambda_{\rm QCD} \ll m\alpha_s^2$

$$\delta E_{us}(\nu_{us}) = \delta E_{us,\,pert.} + \delta E_{us,\,nopert.}\,,$$

$$\delta E_{us,\,pert.} = Am\alpha_s^5(\ln\frac{\nu_{us}}{m\alpha_s^2} + K')\,, \quad \delta E_{us,\,nopert.} = \sum_{n=0}^{\infty} C_n O_n\,,$$

where $C_n \sim 1/(m^{3+2n}\alpha_s^{4+4n})$ and $O_n \sim \Lambda_{\rm QCD}^{4+2n}$.

Voloshin, Leutwyler, Pineda, Rauh

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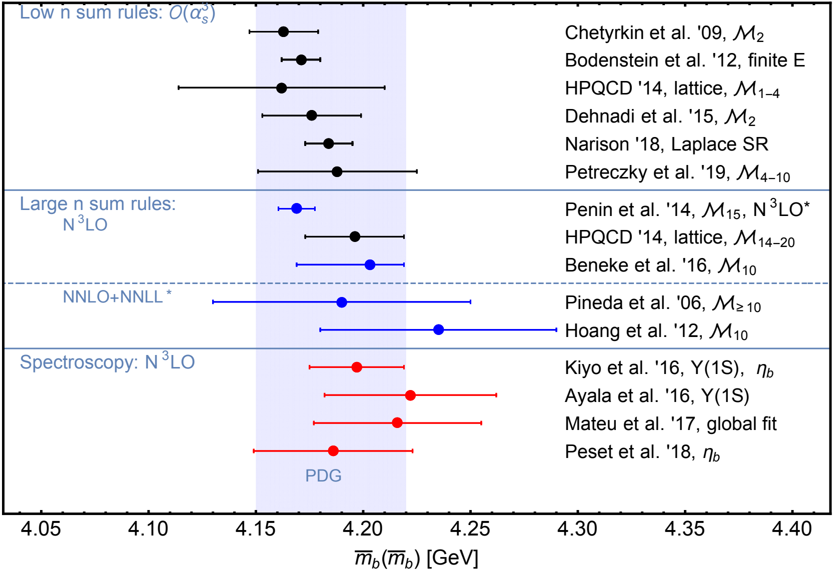
If $\Lambda_{\rm QCD} \gg m\alpha_s^2$

$$\delta E_{us}(\nu_{us}) \sim \Lambda^3_{\rm QCD}\langle r^2\rangle_n$$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$\overline{m}_c$ from $M_{B_c} - M_{\eta_b}/2$ and M_{η_c}

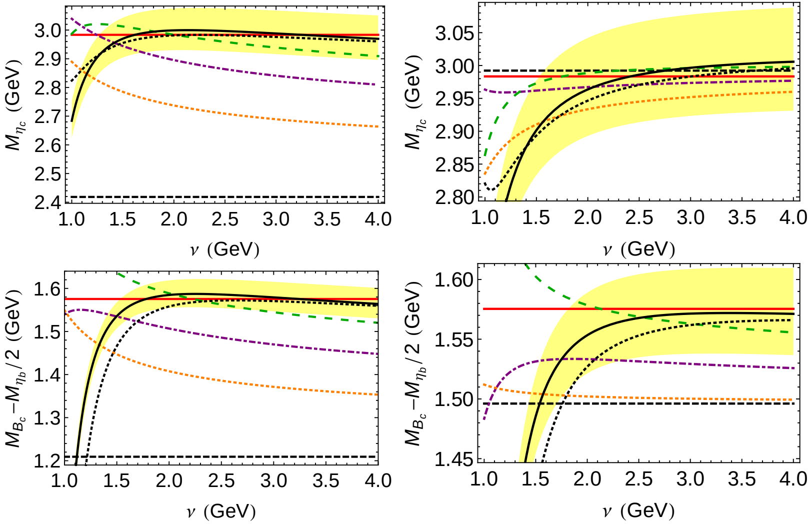


Figure: **Left panels:** RS scheme. **Right panels:** RS' scheme.

Determinations from M_{η_c}

$$
\begin{aligned}
m_{c,\mathrm{RS}}(1\mathrm{GeV}) &= 1\,202^{+15}_{+16}(\nu)^{-15}_{+11}(\nu_f)^{-10}_{+10}(\alpha_s)^{-34}_{+34}(N_m);\\
\Rightarrow \quad \overline{m}_c &= 1\,217^{+12}_{+13}(\nu)^{-1}_{-6}(\nu_f)^{-8}_{+8}(\alpha_s)^{+10}_{-10}(N_m).\\
m_{c,\mathrm{RS}'}(1\ \mathrm{GeV}) &= 1\,495^{-11}_{+50}(\nu)^{-9}_{+20}(\nu_f)^{+4}_{-4}(\alpha_s)^{-20}_{+20}(N_m);\\
\Rightarrow \quad \overline{m}_c &= 1\,222^{-9}_{+40}(\nu)^{-7}_{+16}(\nu_f)^{-7}_{+7}(\alpha_s)^{+11}_{-11}(N_m),
\end{aligned}
$$

Determinations from $M_{B_c} - M_{\eta_c}/2$

$$
\begin{aligned}
m_{c,\mathrm{RS}}(1\mathrm{GeV}) &= 1\,204^{+27}_{-8}(\nu)^{-26}_{+18}(\nu_f)^{-17}_{+16}(\alpha_s)^{-33}_{+33}(N_m)^{-1}_{+1}(\overline{m}_b);\\
\Rightarrow \quad \overline{m}_c &= 1\,220^{+21}_{-6}(\nu)^{-7}_{-4}(\nu_f)^{-14}_{+13}(\alpha_s)^{+11}_{-11}(N_m)^{-1}_{+1}(\overline{m}_b).\\
m_{c,\mathrm{RS}'}(1\ \mathrm{GeV}) &= 1\,501^{+1}_{+23}(\nu)^{-14}_{-27}(\nu_f)^{-2}_{+2}(\alpha_s)^{-18}_{+18}(N_m)^{-1}_{+1}(\overline{m}_b);\\
\Rightarrow \quad \overline{m}_c &= 1\,227^{-1}_{+18}(\nu)^{+11}_{-22}(\nu_f)^{-12}_{+12}(\alpha_s)^{+13}_{-13}(N_m)^{-1}_{+1}(\overline{m}_b)
\end{aligned}
$$

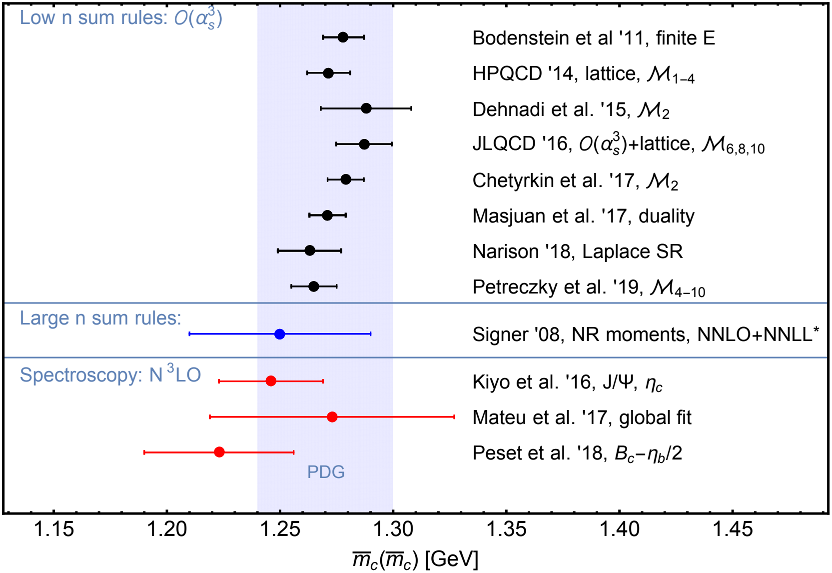
$$\overline{m}_c(\overline{m}_c) = 1223(33)\ MeV$$

NP effects vs ultrasoft effects

$$\delta E \sim \Lambda_{\rm QCD}^3 \left(\langle r^2 \rangle_{Bc} - \frac{1}{2} \langle r^2 \rangle_{\eta_b} \right)$$

and for the latter

$$\delta E \sim \Lambda_{\rm QCD}^3 \langle r^2 \rangle_{\eta_c} \ .$$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Extra comments:

- Vector fits appear to give larger values for the masses than pseudoscalars
  Possible explanation $\rightarrow$ hyperfine splitting sensitive to the resummation of logarithms.
- Resummation of logarithms will increase sensitivity to smaller scales.
- Experimental precision infinity $\rightarrow$ no need for excitations that may introduce systematics.
  No problem if the theoretical errors are properly introduced.
  They can be used to test assumptions.

Nonrelativistic Sum rules (b-$\bar{b}$, c-$\bar{c}$)

$$J^\mu = \bar{Q}\gamma^\mu Q = B_1 \psi^\dagger \boldsymbol{\sigma} \chi + \cdots ,$$

$$B_1 = 1 + a_1\alpha_s + a_2\alpha_s^2 + a_3\alpha_s^3 + \cdots$$

B_1 at NNLO: Hoang(QED); Beneke, Signer, Smirnov; Czarnecki, Melnikov

B_1 at NNNLO: Marquard, Piclum, Seidel, Steinhauser

B_1, B_0 at NLL: Pineda; Hoang, Stewart

B_1/B_0 at NNLL: Penin, Pineda, Smirnov, Steinhauser

B_1, B_0 at NNLL (partial): Pineda, Signer; Hoang, Stahlhofen

$$(q_\mu q_\nu - g_{\mu\nu})\Pi(q^2) = i\int d^4x e^{iqx} \langle \mathrm{vac}|J_\mu(x)J_\nu(0)|\mathrm{vac}\rangle$$

$$\Pi(q^2) \sim B_1^2 \langle \mathbf{r}=\mathbf{0}|\frac{1}{E-H}|\mathbf{r}=\mathbf{0}\rangle$$

$$G(0,0,E) = \sum_{m=0}^{\infty} \frac{|\phi_{0m}(0)|^2}{E_{0m} - E + i\epsilon - i\Gamma_t} + \frac{1}{\pi}\int_0^\infty dE' \frac{|\phi_{0E'}(0)|^2}{E_{0E'} - E + i\epsilon - i\Gamma_t}$$

Relation of the vacuum polarization with $\sigma_{t\bar{t}}$, non-relativistic sum rules and $\Gamma(V_Q(nS) \to e^+e^-)$

$$\Gamma(V \to e^+e^-) \sim \frac{1}{m^2} B_1^2 |\phi(\mathbf{0})|^2$$

$$\sigma_{t-\bar{t}} \sim B_1(\nu)^2 \mathrm{Im} G(0,0,\sqrt{s}) + \cdots$$

$$M_n \equiv \frac{12\pi^2 e_b^2}{n!} \left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q^2=0} = \int_0^\infty \frac{ds}{s^{n+1}} R_{b\bar{b}}(s),$$

$$M_n = 48\pi e_b^2 N_c \int_{-\infty}^{\infty} \frac{dE}{(E+2m_b)^{2n+3}} \left(B_1^2 - B_1 d_1 \frac{E}{3m_b}\right) \mathrm{Im}\, G(0,0,E)$$

Non-relativistic Sum rules: bottomonium

Pineda-Signer

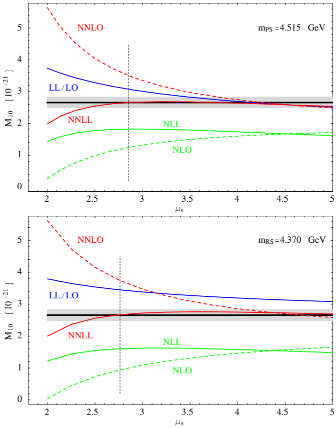


Figure: *The moment M_{10} as a function of μ_s at LO/LL, NLO, NLL, NNLO and NNLL for $m_{b\rm PS}$(2 GeV) = 4.515 GeV in the PS scheme (upper figure), and for $m_{b\rm RS}$(2 GeV) = 4.370 GeV in the RS scheme (lower figure).*

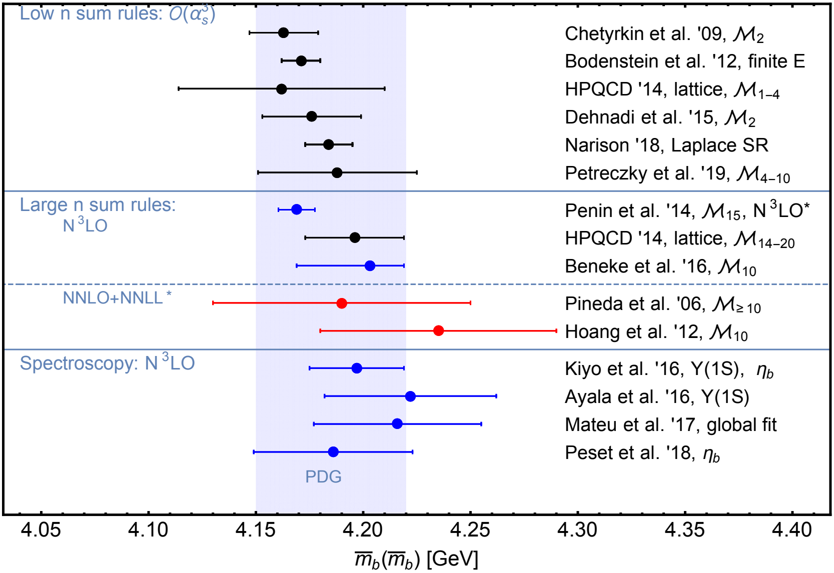
$$m_{b,\mathrm{PS}}(2\mathrm{GeV}) = 4.52 \pm 0.06 \ \mathrm{GeV},$$

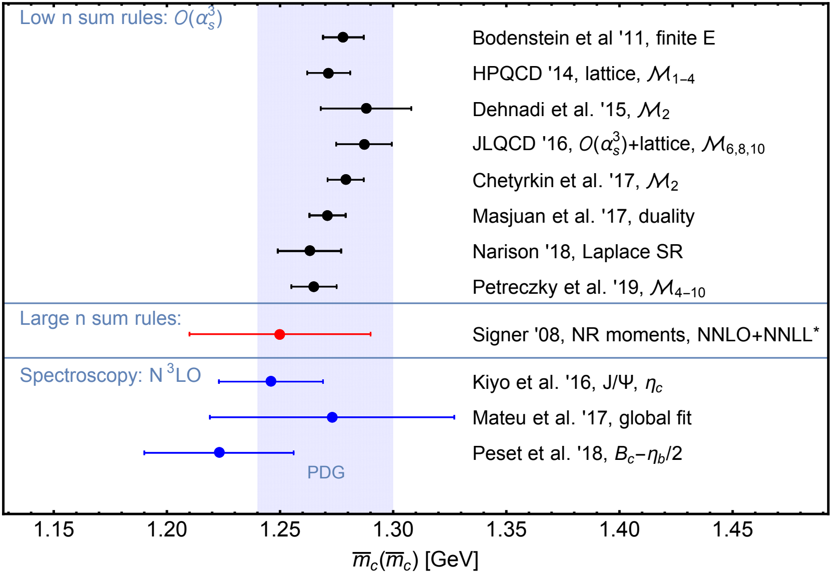
$$m_{b,\mathrm{RS}}(2\mathrm{GeV}) = 4.37 \pm 0.07 \ \mathrm{GeV}.$$

$$\overline{m}_b(\overline{m}_b) = 4.19 \pm 0.06 \, \mathrm{GeV}.$$

The perturbative series is sign-alternating. This is the opposite than for electromagnetic decays. The convergence of the perturbative series in sum rules is also better than for electromagnetic decays.

Low n sum rules very sensitive to the high energy behavior of R_{bb}. $n = 6$ Error 50 MeV, and bigger for smaller values of n.


Low n sum rules: $O(\alpha_s^3)$
Chetyrkin et al. '09, $\mathcal{M}_2$
Bodenstein et al. '12, finite E
HPQCD '14, lattice, $\mathcal{M}_{1-4}$
Dehnadi et al. '15, $\mathcal{M}_2$
Narison '18, Laplace SR
Petreczky et al. '19, $\mathcal{M}_{4-10}$
Large n sum rules: N^3LO
Penin et al. '14, $\mathcal{M}_{15}$, N^3LO*
HPQCD '14, lattice, $\mathcal{M}_{14-20}$
Beneke et al. '16, $\mathcal{M}_{10}$
NNLO+NNLL *
Pineda et al. '06, $\mathcal{M}_{\geq 10}$
Hoang et al. '12, $\mathcal{M}_{10}$
Spectroscopy: N^3LO
Kiyo et al. '16, Υ(1S), η_b
Ayala et al. '16, Υ(1S)
Mateu et al. '17, global fit
Peset et al. '18, η_b
PDG
4.05
4.10
4.15
4.20
4.25
4.30
4.35
4.40
$\overline{m}_b(\overline{m}_b)$ [GeV]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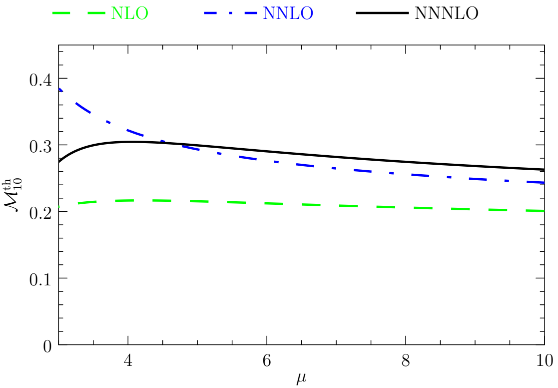
Dehnadi et al. '15, $\mathcal{M}_2$
JLQCD '16, $O(\alpha_s^3)$+lattice, $\mathcal{M}_{6,8,10}$
Chetyrkin et al. '17, $\mathcal{M}_2$
Masjuan et al. '17, duality
Narison '18, Laplace SR
Petreczky et al. '19, $\mathcal{M}_{4-10}$
Large n sum rules:
Signer '08, NR moments, NNLO+NNLL*
Spectroscopy: N ^{3}LO
Kiyo et al. '16, J/Ψ, η_c
Mateu et al. '17, global fit
Peset et al. '18, $B_c-\eta_b/2$
PDG
1.15
1.20
1.25
1.30
1.35
1.40
1.45
$\overline{m}_c(\overline{m}_c)$ [GeV]

Martin Beneke, Andreas Maier, Jan Piclum, Thomas Rauh (NNNLO)



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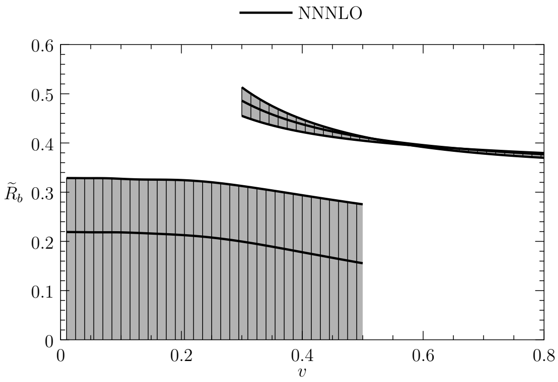

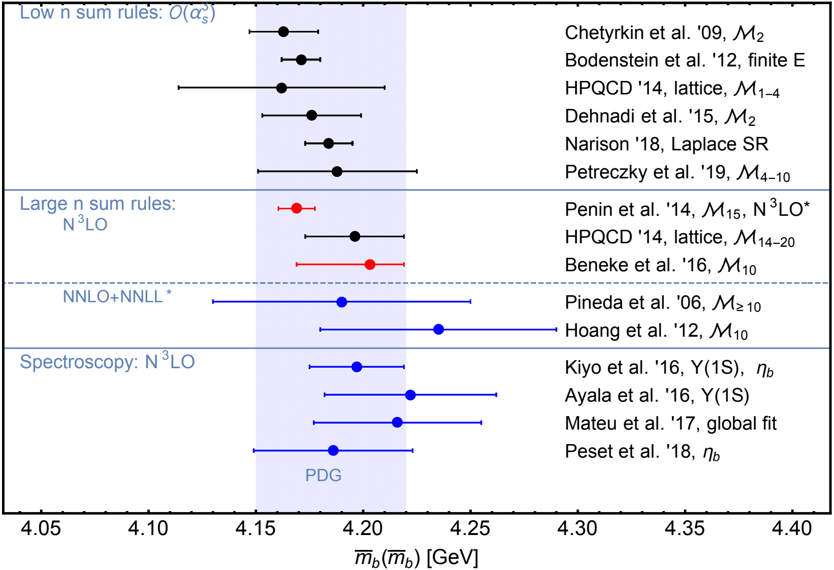

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CONCLUSIONS

Heavy quarkonium (near threshold) observables $\rightarrow$ ideal place to determine the heavy quark mass.
They show highest sensitivity to the heavy quark mass.

- Clean theoretical observables
- Different systematics than alternative determinations
- Similar numbers than alternative determinations
- Spectrum: Scale dependence increases from $NNLO \rightarrow N^3LO$

Possible issues:
mv^2 versus $\Lambda_{\rm QCD}$
New effects show up at N^{3}LO
Perturbative ultrasoft effects small (with fixed order computations)
Renormalization group improvement?
No clear signal of the existence of $\mathcal{O}(\Lambda^3_{\rm QCD} r^2)$ effects.
Lattice simulations of the static potential also seem to support this.
A dedicated lattice simulation of δE_{us} needed.

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If $mv^2 \gg \Lambda_{\rm QCD}$
Nonperturbative effects are parametrically equal in spectroscopy and nonrelativistic sum rules. Overall coefficient significantly smaller in sum rules.

Non-relativistic sum rules

- 1) for which n is non-relativistic
  2) for which n the perturbative expansion breaks down (smoking gun signal?)
  3) charm effects?
- Relativistic sum rules:
  1) Dependence with the continuum: Large experimental error
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