Particle Decay in the Expanding Spacetime of Post-Inflationary Cosmology

Based on PRD 98, 083503 (2018) and PRD 100, 023531 (2019)

By Nathan Herring
In collaboration with Daniel Boyanovksy and Andrew Zentner
ICTP-SAIFR: Dark Universe Workshop Oct. 21-25, 2019

Overview

- Why Particle Decay?
- FRW Spacetime
- Scalar and Fermion Fields in FRW
- The Wigner-Weisskopf Method
- Example 1: Decay of Massive Scalar to Massless Scalars in Radiation Domination
- Example 2: Decay of Massive Scalar to Massless Fermions in Radiation Domination
- Potential Implications
- Conclusions

Particle Decay is a ubiquitous process in Cosmology.

Particle Decay is the simplest dynamical process in QFT.

Applications in the Early Universe

- Baryogenesis
  - Generation of Matter/Antimatter Asymmetry
  - Ex: S. Enomoto and N. Maekawa (2011)
- Leptogenesis
  - Generation Lepton/Antilepton Asymmetry
  - Ex: W. Buchmuller *et al.* (2012)
- CP Violating Decays in Early Universe
  - Matter/Antimatter asymmetry
  - Ex: L. Covi *et al.* (1996)
- Big Bang Nucleosynthesis
  - Formation of the light elements
- Particle Dark Matter
  - How was it produced? Is it stable? Can it interact with the "visible" matter?
  - Constrained to have a long lifetime

The Friedmann-Robertson-Walker Metric

- A Spatially Flat Expanding Universe is described by the FRW Metric:

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$$

- The scale factor must obey Friedmann's Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = H_0^2 \left[\frac{\Omega_M}{a^3(t)} + \frac{\Omega_R}{a^4(t)} + \Omega_\Lambda\right]$$

$$H_0 = 1.5 \times 10^{-42}\,\mathrm{GeV} \quad ; \quad \Omega_M = 0.308 \quad ; \quad \Omega_R = 5 \times 10^{-5} \quad ; \quad \Omega_\Lambda = 0.692$$

Values as determined by C.L. Bennett *et al.* (2013)

The Friedmann-Robertson-Walker Metric

- A Spatially Flat Expanding Universe is described by the FRW Metric:

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$$

- The scale factor must obey Friedmann's Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = H_0^2\left[\frac{\Omega_M}{a^3(t)} + \frac{\Omega_R}{a^4(t)} + \Omega_\Lambda\right]$$

4 Important Consequences:

1. **Homogeneous (Momentum Conservation)**
2. **Isotropic (Angular Momentum Conservation)**
3. **Time-Dependence (Energy is Not Conserved)**
4. **Conformal to Minkowski (dt = a dη)**

$$H_0 = 1.5 \times 10^{-42}\,\mathrm{GeV} \quad ; \quad \Omega_M = 0.308 \quad ; \quad \Omega_R = 5 \times 10^{-5} \quad ; \quad \Omega_\Lambda = 0.692$$

Values as determined by C.L. Bennett *et al.* (2013)

Scalar and Fermion Fields in FRW Spacetime

Conformally Rescaled Equations of Motion

- Scalar Fields:

$$\mathcal{L}_0[\chi(\eta)] = \frac{1}{2}\left[\chi'^2 - (\nabla\chi)^2 - \mathcal{M}^2(\eta)\chi^2\right]$$

$$\left[\frac{d^2}{d\eta^2} + k^2 + \mathcal{M}^2(\eta)\right]g_k(\eta) = 0$$

Scalar and Fermion Fields in FRW Spacetime

Conformally Rescaled Equations of Motion

- Scalar Fields:

$$\mathcal{L}_0[\chi(\eta)] = \frac{1}{2}\left[\chi'^2 - (\nabla\chi)^2 - \mathcal{M}^2(\eta)\chi^2\right]$$

$$\left[\frac{d^2}{d\eta^2} + k^2 + \mathcal{M}^2(\eta)\right] g_k(\eta) = 0$$

- Fermion Fields:

$$\mathcal{L}_0[\psi] = \overline{\psi}\left[i\gamma^\mu\partial_\mu - \mathcal{M}_f^2(\eta)\right]\psi$$

$$\left[\frac{d^2}{d\eta^2} + k^2 + \mathcal{M}_f^2(\eta) - i\mathcal{M}'_f(\eta)\right] f_k(\eta) = 0$$

$$\left[\frac{d^2}{d\eta^2} + k^2 + \mathcal{M}_f^2(\eta) + i\mathcal{M}'_f(\eta)\right] h_k(\eta) = 0$$

Scalar and Fermion Fields in FRW Spacetime

Conformally Rescaled Equations of Motion

- Scalar Fields:

$$\mathcal{L}_0[\chi(\eta)] = \frac{1}{2}\left[\chi'^2 - (\nabla\chi)^2 - \mathcal{M}^2(\eta)\chi^2\right]$$

$$\left[\frac{d^2}{d\eta^2} + k^2 + \mathcal{M}^2(\eta)\right] g_k(\eta) = 0$$

- Fermion Fields:

$$\mathcal{L}_0[\psi] = \overline{\psi}\left[i\gamma^\mu\partial_\mu - \mathcal{M}_f^2(\eta)\right]\psi$$

$$\left[\frac{d^2}{d\eta^2} + k^2 + \mathcal{M}_f^2(\eta) - i\mathcal{M}'_f(\eta)\right] f_k(\eta) = 0$$

$$\left[\frac{d^2}{d\eta^2} + k^2 + \mathcal{M}_f^2(\eta) + i\mathcal{M}'_f(\eta)\right] h_k(\eta) = 0$$

Adiabatic Approximation

- WKB Ansatz

$$g_k(\eta) = \frac{e^{-i\int_{\eta_i}^{\eta} W_k(\eta')\,d\eta'}}{\sqrt{2\,W_k(\eta)}}$$

- Adiabatic Expansion

$$W_k^2(\eta) = \omega_k^2(\eta)\left[1 - \frac{1}{2}\frac{\omega_k''(\eta)}{\omega_k^3(\eta)} + \frac{3}{4}\left(\frac{\omega_k'(\eta)}{\omega_k^2(\eta)}\right)^2 + \cdots\right]$$

- The Physical Character of this Expansion

$$\frac{\omega_k'(\eta)}{\omega_k^2(\eta)} = \frac{H(t)}{\gamma_k^2(t)\,E_k(t)}$$

The Wigner-Weisskopf Method

Features:

- Developed originally to calculate atomic line spectra
- Manifestly unitary and non-perturbative
- Exploits the interaction picture
- Allows for direct calculation of transition amplitudes and probabilities

$$i\frac{d}{d\eta}|\Psi(\eta)\rangle_I = H(\eta)_I|\Psi(\eta)\rangle_I$$

$$i\frac{d}{d\eta}C_A(\eta) = \sum_\kappa C_\kappa(\eta)\langle A|H_I(\eta)|\kappa\rangle$$

$$i\frac{d}{d\eta}C_\kappa(\eta) = C_A(\eta)\langle\kappa|H_I(\eta)|A\rangle \quad ; \quad C_A(\eta_i) = 1\,;\, C_\kappa(\eta_i) = 0$$

$$\Sigma_A(\eta;\eta') = \sum_\kappa \langle A|H_I(\eta)|\kappa\rangle\langle\kappa|H_I(\eta')|A\rangle$$

$$\boxed{|C_A(\eta)|^2 = e^{-\int_{\eta_i}^{\eta}\Gamma_A(\eta')d\eta'}\,|C_A(\eta_i)|^2 \quad ; \quad \Gamma_A(\eta) = 2\int_{\eta_i}^{\eta} d\eta_1\,\mathrm{Re}\left[\Sigma_A(\eta,\eta_1)\right]}$$

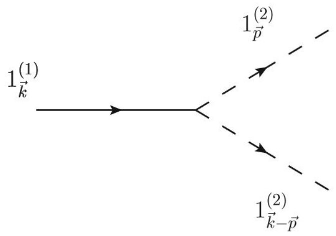
Massive Scalar -> Massless Scalars

Decay at Rest in Comoving Frame (k = 0)

$$\gamma_k(t) \to 1\,; k = 0$$

$$\mathcal{P}(t) = e^{-\frac{\lambda^2}{8\pi m}t} = e^{-\Gamma_0 t}$$

- The result exactly approaches the Minkowski, S-Matrix value.
- Valid for parent particle “born” at rest in comoving frame.



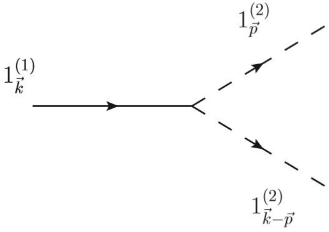
Massive Scalar -> Massless Scalars

Decay at Rest in Comoving Frame (k = 0)

$$\gamma_k(t) \to 1\,; k = 0$$

$$\mathcal{P}(t) = e^{-\frac{\lambda^2}{8\pi m}t} = e^{-\Gamma_0 t}$$

- The result exactly approaches the Minkowski, S-Matrix value.
- Valid for parent particle “born” at rest in comoving frame.



Relativistic Case (k > m)

$$\gamma_k(t) = \left[1 + \frac{t_{nr}}{t}\right]^{1/2} \quad ; \quad t_{nr} = \frac{k^2}{2m_1^2 H_R}$$

$$\mathcal{P}(t) = e^{\frac{-1}{3}\Gamma_0 t_{nr}(\frac{t}{t_{nr}})^{\frac{3}{2}}}\;;\; t \ll t_{nr}$$

$$\mathcal{P}(t) = e^{-\Gamma_0 t}\left(\frac{t}{t_{nr}}\right)^{\frac{\Gamma_0 t_{nr}}{2}}\;;\; t > t_{nr}$$

- For particles “born” with some comoving momentum
- Generically, smaller than the Minkowski rate:
  - Time Dilation
  - Cosmic Redshift

Massive Scalar -> Massless Fermions (Yukawa Coupling)

Decay at Rest in Comoving Frame (k = 0)

$$\gamma_k(t) \to 1\,;k=0$$

$$\mathcal{P}(t) = \left[\frac{t}{t_b}\right]^{\frac{-Y^2}{8\pi^2}} e^{\frac{Y^2}{4\pi^2}[\frac{t}{t_b}]^{\frac{1}{4}}}\, e^{-\Gamma_0(t-t_b)}\,\mathcal{P}(t_b)$$

- Distinctly not the Minkowski result unlike for scalar decay products
- Consequence of Renormalizability + Curved Spacetime
  - “Dressing” of the state on an associated time scale
  - Time dependent frequencies preserve the “anomalous dimension”

Massive Scalar -> Massless Fermions (Yukawa Coupling)

Decay at Rest in Comoving Frame (k = 0)

$$\gamma_k(t) \to 1\,;k=0$$

$$\mathcal{P}(t) = \left[\frac{t}{t_b}\right]^{\frac{-Y^2}{8\pi^2}} e^{\frac{Y^2}{4\pi^2}[\frac{t}{t_b}]^{\frac{1}{4}}}\, e^{-\Gamma_0(t-t_b)}\,\mathcal{P}(t_b)$$

- Distinctly not the Minkowski result unlike for scalar decay products
- Consequence of Renormalizability + Curved Spacetime
  - "Dressing" of the state on an associated time scale
  - Time dependent frequencies preserve the "anomalous dimension"

Relativistic Case (k > m)

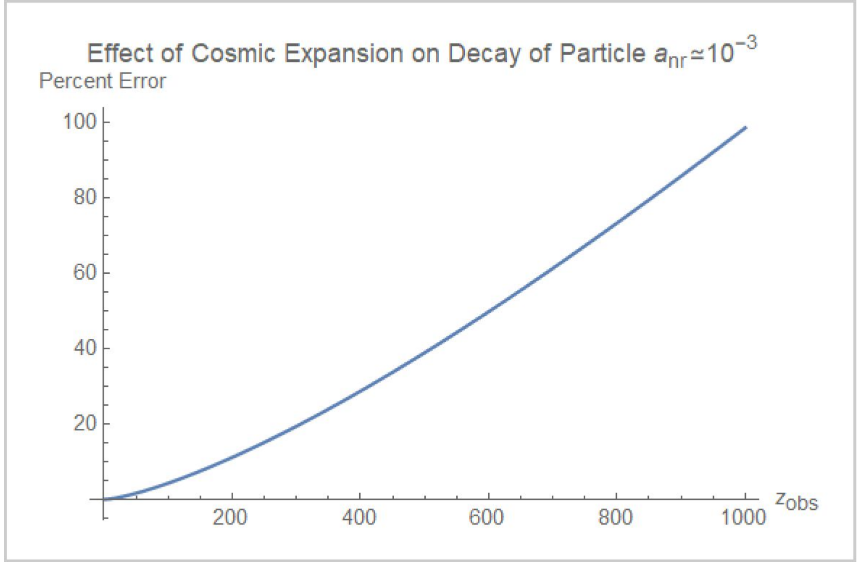
$$\gamma_k(t) = \left[1+\frac{t_{nr}}{t}\right]^{1/2} \quad ; \quad t_{nr} = \frac{k^2}{2m_1^2 H_R}$$

$$\mathcal{P}(t) = e^{-\frac{2}{3}\Gamma_0 t_{nr}(\frac{t}{t_{nr}})^{\frac{3}{2}}}\mathcal{P}(t_b)\,;\,t_b \ll t \ll t_{nr}$$

$$\mathcal{P}(t) = \left[\frac{t}{t_{nr}}\right]^{\frac{-Y^2}{8\pi^2}} e^{\frac{Y^2}{4\pi^2}[\frac{t}{t_{nr}}]^{\frac{1}{4}}} \left[\frac{t}{t_{nr}}\right]^{\frac{\Gamma_0 t_{nr}}{2}} e^{-\Gamma_0(t-t_b)}\,\mathcal{P}(t_b)\,;\,t \gg t_{nr}$$

- The UR case is nearly identical to scalar case
- In NR regime "anomalous dimension" term distinguishes fermionic from scalar case
  - Enhancement factor that preserves the short time-scale physics
  - Time dependent frequencies is the key!

Implications: Long-Lived Particles



Consider a massive scalar decaying to massless fermions with the following assumptions:

- Particle produced at $T \simeq T_{GUT} \simeq 10^{15}$ GeV
- $a_{nr} \simeq 10^{-3}$ (Matter-Radiation Equality)
- Recall $a_{nr} \stackrel{\text{def}}{=} k/m$
- Very small Yukawa coupling

1. Plotted is Minkowski Rate/FRW Rate error as a function of redshift.
2. The error is large when $z_{obs} \gtrsim 1/a_{nr}$

Implications: Early Universe Quantum Kinetics

$$\frac{dN_k}{dt} = \text{Gain} - \text{Loss}$$

$$\Gamma^k_{gain} = \frac{\pi\lambda^2}{E_k}\int \frac{d^3q}{(2\pi)^3}\frac{\delta(E_k - E_p - E_q)}{2E_p 2E_q}\, n_q n_p$$

$$\Gamma^k_{loss} = \frac{\pi\lambda^2}{E_k}\int \frac{d^3q}{(2\pi)^3}\frac{\delta(E_k - E_p - E_q)}{2E_p 2E_q}\, n_q n_p e^{\beta(E_p+E_q)}$$

$$\Gamma^k_{loss} = e^{\beta E_k}\,\Gamma^k_{gain}$$

$$\frac{dN_k}{dt} = -(\Gamma^k_{loss} - \Gamma^k_{gain})(N_k - N_k^{eq})$$

Standard Quantum Kinetic Treatment

- Quantum Kinetic Master Equation for $\chi \leftrightarrow \varphi\varphi$
- Assume φ particles are already thermalized
- Energy conserving delta functions lead to *detailed balance*.
- Solution is thermal distribution for χ particle asymptotically.

Implications: Early Universe Quantum Kinetics

$$\frac{dN_k}{dt} = \mathrm{Gain} - \mathrm{Loss}$$

$$\Gamma^k_{gain} = \frac{\pi\lambda^2}{E_k} \int \frac{d^3q}{(2\pi)^3} \frac{\delta(E_k - E_p - E_q)}{2E_p 2E_q} n_q n_p$$

$$\Gamma^k_{loss} = \frac{\pi\lambda^2}{E_k} \int \frac{d^3q}{(2\pi)^3} \frac{\delta(E_k - E_p - E_q)}{2E_p 2E_q} n_q n_p e^{\beta(E_p+E_q)}$$

$$\Gamma^k_{loss} = e^{\beta E_k}\, \Gamma^k_{gain}$$

$$\frac{dN_k}{dt} = -(\Gamma^k_{loss} - \Gamma^k_{gain})(N_k - N^{eq}_k)$$

Standard Quantum Kinetic Treatment

- Quantum Kinetic Master Equation for $\chi \leftrightarrow \varphi\varphi$
- Assume φ particles are already thermalized
- Energy conserving delta functions lead to *detailed balance*.
- Solution is thermal distribution for χ particle asymptotically.

Modifications from Cosmic Expansion

- Energy conservation is not manifest
- Gain/Loss terms are essentially decay rates which deviate from Minkowski results in FRW.

Conclusions

- We can obtain the decay law, analytically, in the expanding FRW spacetime using adiabatic expansion (zeroth order) and Wigner-Weisskopf method.

- We can obtain an *effective time-dependent decay rate* which is smaller than the analogous Minkowski result.
  - Local time dilation
  - Cosmic Redshift

- For fermions (w/ Yukawa couplings), the expanding spacetime encodes the memory of transient dynamics associated with short-scale physics into the decay function.
  - Renormalizable theory
  - Time dependent frequencies

- S-Matrix inspired results are, at best, approximations, but they miss crucial non-equilibrium dynamics and other modifications to the decay law!

Extra Slides

The S-Matrix

- The unitary time-evolution matrix constructed from the interacting Hamiltonian
- These states are asymptotically free particle states (*infinite time limit*)
- Terms in the S-Matrix correspond to Feynman Diagrams
- Transition probability is given by the square of the term integrated over phase space
- The S-matrix implicitly assumes *energy conservation* in order to extract a decay rate (Audretsch, Spangehl [1986])
- No energy conservation in FRW indicates the need for a different technique!

$$H_I(t) = \lambda \int d^3x \; \phi_1(\vec{x},t) \; : \phi_2^2(\vec{x},t) :$$

$$U_I(\infty,-\infty) = 1 - i\int_{-\infty}^{\infty} H_I(t_1)\,dt_1 + \frac{(-i)^2}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{T}[H_I(t_1)\,H_I(t_2)]\,dt_1\,dt_2 +$$

$$S_{fi} = \langle f|U_I(\infty,-\infty)|i\rangle \qquad T_{fi} = |S_{fi}^{(1)}|^2 \sim |\mathcal{M}_{fi}|^2\,\delta^{(4)}(k-P_1-P_2)V\,T$$

The Interaction Picture

The Schrodinger Picture

- Operators are time-independent
- States evolve with full Hamiltonian

$$i\frac{d}{dt}U(t,t_0) = H(t)U(t,t_0)\,, U(t_0,t_0) = 1$$

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$$

The Interaction Picture

- Operators evolve with the free field Hamiltonian
- States evolve with interacting Hamiltonian

$$\frac{d}{dt}\phi(t) = i[H_0, \phi(t)]$$

$$i\frac{d}{dt}U_I(t,t_0) = H_I(t)U_I(t,t_0)\,, U_I(t_0,t_0) = 1$$

$$|\psi(t)\rangle_I = U_I(t,t_0)|\psi(t_0)\rangle_I$$

The Heisenberg Picture

- Operators evolve with full Hamiltonian
- States are time-independent

$$\frac{d}{dt}\phi(t) = i[H, \phi(t)]$$

The Wigner-Weisskopf Method (in Detail)

Features:

- Developed originally to calculate atomic line spectra
- Manifestly unitary and non-perturbative
- Exploits the interaction picture
- Allows for direct calculation of transition amplitudes and probabilities

Implementation:

- Expand State Evolution in Fock Basis
- Set initial conditions
- Close hierarchy of equations
- Define the self-energy

$$i\frac{d}{d\eta}|\Psi(\eta)\rangle_I = H(\eta)_I|\Psi(\eta)\rangle_I$$

$$i\frac{d}{d\eta}C_n(\eta) = \sum_m C_m(\eta)\langle n|H_I(\eta)|m\rangle$$

$$i\frac{d}{d\eta}C_A(\eta) = \sum_\kappa C_\kappa(\eta)\langle A|H_I(\eta)|\kappa\rangle$$

$$i\frac{d}{d\eta}C_\kappa(\eta) = C_A(\eta)\langle \kappa|H_I(\eta)|A\rangle \quad ; \quad C_A(\eta_i) = 1\,;\, C_\kappa(\eta_i) = 0$$

$$\Sigma_A(\eta;\eta') = \sum_\kappa \langle A|H_I(\eta)|\kappa\rangle\langle \kappa|H_I(\eta')|A\rangle$$

Wigner-Weisskopf Method Cont.

- Formally solve the coupled ODEs
- Markovian Approximation
  - Integrate by parts
  - Weak Coupling Argument

$$C_\kappa(\eta) = -i \int_{\eta_i}^{\eta} \langle \kappa | H_I(\eta') | A \rangle \, C_A(\eta') \, d\eta'$$

$$\frac{d}{d\eta} C_A(\eta) = -\int_{\eta_i}^{\eta} d\eta' \, \Sigma_A(\eta, \eta') \, C_A(\eta')$$

$$\frac{d}{d\eta} C_A(\eta) = -C_A(\eta) \int_{\eta_i}^{\eta} d\eta' \; \Sigma_A(\eta; \eta') + \cdots$$

Then one defines a time-dependent decay rate:

$$\mathcal{P}_{1\to 22}(\eta, \eta_i) = \int_{\eta_i}^{\eta} d\eta_2 \int_{\eta_i}^{\eta} d\eta_1 \, \Sigma_k(\eta_2; \eta_1)$$

$$\Gamma(\eta) \equiv \frac{d}{d\eta} \mathcal{P}_{1\to 22}(\eta, \eta_i) = 2 \int_{\eta_i}^{\eta} d\eta_1 \, \mathrm{Re}[\Sigma_k(\eta; \eta_1)]$$

- The result is manifestly non-perturbative
- We formally obtain the decay law and decay rate
- Calculation of the self-energy is the key step

$$|C_A(\eta)|^2 = e^{-\int_{\eta_i}^{\eta} \Gamma_A(\eta') d\eta'} \, |C_A(\eta_i)|^2 \quad ; \quad \Gamma_A(\eta) = 2 \int_{\eta_i}^{\eta} d\eta_1 \, \mathrm{Re}\left[\Sigma_A(\eta, \eta_1)\right]$$

Scalar Field Theory in FRW Spacetime

1. Classical Action
2. Conformal Rescaling
3. Free Field Equations of Motion
4. Spatial Fourier Transform
5. Mode Function Differential Equation

$$A = \int d^4x \sqrt{|g|}\Big\{\frac{1}{2} g^{\mu\nu}\,\partial_\mu\phi_1\partial_\nu\phi_1 - \frac{1}{2}\big[m_1^2 + \xi_1 R\big]\phi_1^2 + \frac{1}{2} g^{\mu\nu}\,\partial_\mu\phi_2\partial_\nu\phi_2 - \frac{1}{2}\big[m_2^2 + \xi_2 R\big]\phi_2^2 - \lambda\phi_1 : \phi_2^2 : \Big\} \qquad R = 6\Big[\frac{\ddot a}{a} + \Big(\frac{\dot a}{a}\Big)^2\Big]$$

$$\phi_{1,2}(t) = \frac{\chi_{1,2}(\eta)}{a(\eta)} \quad ; \quad a(\eta) = a(t(\eta))$$

$$A = \int d^3x\, d\eta \sum_{j=1,2}\Big[\tfrac{1}{2}\Big(\frac{d\chi_j}{d\eta}\Big)^2 - \tfrac{1}{2}\left(\nabla\chi_j\right)^2 - \tfrac{1}{2}\chi_j^2\,\mathcal{M}_j^2(\eta)\Big] - \lambda\, a(\eta)\,\chi_1 \, : \chi_2^2 :$$

$$\mathcal{M}_j^2(\eta) = m_j^2\, a^2(\eta) - \frac{a''}{a}\,(1 - 6\xi_j) \quad ; \quad j = 1, 2\,.$$

$$\frac{d^2}{d\eta^2}\,\chi_j(\vec{x},\eta) - \nabla^2\chi_j(\vec{x},\eta) + \mathcal{M}_j^2(\eta)\,\chi_j(\vec{x},\eta) = 0 \quad ; \quad j = 1, 2$$

$$\chi(\vec{x},\eta) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} \chi_{\vec{k}}(\eta) e^{-i\vec{k}\cdot\vec{x}}$$

$$\frac{d^2}{d\eta^2}\,\chi_{\vec{k}}(\eta) + \Big[\omega_k^2(\eta) - \frac{a''}{a}\,(1 - 6\xi_j)\Big]\,\chi_{\vec{k}}(\vec{k},\eta) = 0 \quad ; \quad \omega_k^2(\eta) = k^2 + m_j^2\, a^2(\eta)$$

The WKB Solution

- In radiation domination, the mode function differential equation simplifies

$$\frac{d^2}{d\eta^2}\,\chi_{\vec{k}}(\eta) + \omega_k^2(\eta)\,\chi_{\vec{k}}(\eta) = 0$$

- The mode functions must be quantized

$$\chi = a\,g_k(\eta) + a_-^\dagger\,g_k^*(\eta)$$

- The WKB ansatz can then be implemented

$$g_k(\eta) = \frac{e^{-i\int_{\eta_i}^{\eta} W_k(\eta')\,d\eta'}}{\sqrt{2\,W_k(\eta)}} \qquad W_k^2(\eta) = \omega_k^2(\eta) - \frac{1}{2}\left[\frac{W_k''(\eta)}{W_k(\eta)} - \frac{3}{2}\left(\frac{W_k'(\eta)}{W_k(\eta)}\right)^2\right]$$

Decay of Massive Scalar to Massless Scalars

The Self-Energy

- First the self-energy must be obtained for massive particle decaying to two massless particles

$$\Sigma_k(\eta,\eta') = \frac{4\lambda^2}{2!} a(\eta)a(\eta') g_k^{(1)}(\eta')\,(g_k^{(1)}(\eta))^* \int \frac{d^3p}{(2\pi)^3}\, g_p^{(2)}(\eta)\, g_q^{(2)}(\eta)\ (g_p^{(2)}(\eta'))^*\,(g_q^{(2)}(\eta'))^*$$

- Input the mode functions: $g_k(\eta) = \dfrac{e^{-i\int_{\eta_i}^{\eta}\omega_k(\eta')d\eta'}}{\sqrt{2\omega_k(\eta)}}$ $\qquad \omega_k(\eta') = \sqrt{k^2+m^2a^2(\eta')}$

- For $m_2 = 0$ the self-energy takes the form:

$$\Sigma_k(\eta,\eta') = \frac{2\lambda^2\, a(\eta)a(\eta')\, e^{i\int_{\eta'}^{\eta}\omega_k(\eta'')d\eta''}}{\sqrt{2\omega_k^{(1)}(\eta)2\omega_k^{(1)}(\eta')}} \int \frac{d^3p}{(2\pi)^3}\frac{e^{-i(p+q)(\eta-\eta')}}{2p\,2q} \quad ; \quad q = |\vec{k}-\vec{p}|$$

Time Dependent Decay Rate

1. Integrate in small conformal time interval
2. Decay Law requires further integration
3. The result is a locally time dilated Minkowski decay law.

$$\Gamma_k(\eta) = \frac{\lambda^2\, a^2(\eta)}{8\pi\, \omega_k^{(1)}(\eta)}\, \frac{1}{2}\Big[1+\mathcal{S}(\eta)\Big]\,, \quad \mathcal{S}(\eta) = \frac{2}{\pi}\int_0^{\eta} P[\eta,\eta']\, \frac{\sin\big[A(\eta,\eta')\big]}{\eta-\eta'}\, d\eta'$$

$$\Gamma_k(\eta) = \frac{\lambda^2\, a^2(\eta)}{8\pi\, \omega_k^{(1)}(\eta)}\, \frac{1}{2}\left[1+\frac{2}{\pi}\, Si[A_0(z(\eta);\eta)]\right]$$

$$A_0(z(\eta);\eta) = z(\eta)\left[1-\left(1-\frac{1}{\gamma_k^2(\eta)}\right)^{1/2}\right]$$

$$\int_0^{\eta} \Gamma_k(\eta)\, d\eta \equiv \Gamma_0 \int_0^{t} \frac{\mathcal{F}(t')}{\gamma_k(t')}\, dt'$$

$$\Gamma_0 = \frac{\lambda^2}{8\pi m_1} \quad ; \quad \mathcal{F}(t') = \frac{1}{2}\left[1+\frac{2}{\pi}\, Si[A_0(t')]\right]$$

$$\gamma_k(t) = \left[1+\frac{t_{nr}}{t}\right]^{1/2} \quad ; \quad t_{nr} = \frac{k^2}{2m_1^2 H_R}$$

Limiting Cases

Simple Non-relativistic Case

$$\gamma_k(t) \to 1\,;k = 0$$

$$\int_0^{\eta} \Gamma_{k=0}(\eta')\; d\eta' = \frac{\lambda^2}{8\pi m_1}$$

- The result exactly approaches the Minkowski, S-Matrix value.
- However, few particles are “born” with k=0 in early cosmology.

Ultra-relativistic Case

$$\gamma_k(t) \to \infty\,;m_1 = 0$$

$$\int_0^{\eta} \Gamma_k(\eta)\; d\eta \equiv \frac{\lambda^2}{16\pi} \int_0^{t} \frac{1}{k_p(t')}\; dt'$$

$$\left|C^{(1)}_{\vec{k}}(t)\right|^2 = e^{-(t/t^*)^{3/2}} \quad ; \quad t^* = \left[\frac{\lambda^2\;(2H_R)^{1/2}}{24\pi\, k}\right]^{-2/3}$$

- The UR Result is markedly distinct from Minkowski result.
- Time-dilation
- Cosmic Redshift

General Non-Relativistic Case

- For particles "born" with some comoving momentum
- Time Dilation
- Cosmic Redshift
- Always smaller than the Minkowski rate
- The "G" function interpolates between the UR and NR regimes
- For General NR case take t >> t_{nr}

$$\mathcal{F} \simeq 1 \qquad \int_0^t \Gamma_k(t')\,dt' = \Gamma_0\, t_{nr}\, G_k(t)$$

$$G_k(t) = \left[\frac{t}{t_{nr}}\left(1 + \frac{t}{t_{nr}}\right)\right]^{1/2} - \ln\left[\sqrt{1 + \frac{t}{t_{nr}}} + \sqrt{\frac{t}{t_{nr}}}\right]$$

$$\left|C^{(1)}_{\vec{k}}(t)\right|^2 = e^{-\Gamma_0\, t}\left(\frac{t}{t_{nr}}\right)^{\Gamma_0 t_{nr}/2}$$

Decay of Massive Scalar to Massive Scalars

Time Dependent Decay Rate

1. Much harder calculation (momentum integral is difficult)
2. Integrate in conformal time first
3. Use spectral density of states
4. Discover Cosmological Fermi's Golden Rule
5. Cosmic Expansion introduces a new "uncertainty timescale"

$$\Gamma_k(\eta) = \frac{2\,\lambda^2\,a^2(\eta)}{\omega_k^{(1)}(\eta)} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\,\omega_p^{(2)}(\eta)\,2\,\omega_q^{(2)}(\eta)} \frac{\sin\left[\left(\omega_k^{(1)}(\eta) - \omega_p^{(2)}(\eta) - \omega_q^{(2)}(\eta)\right)\eta\right]}{\left(\omega_k^{(1)}(\eta) - \omega_p^{(2)}(\eta) - \omega_q^{(2)}(\eta)\right)}$$

$$q = |\vec{k} - \vec{p}|$$

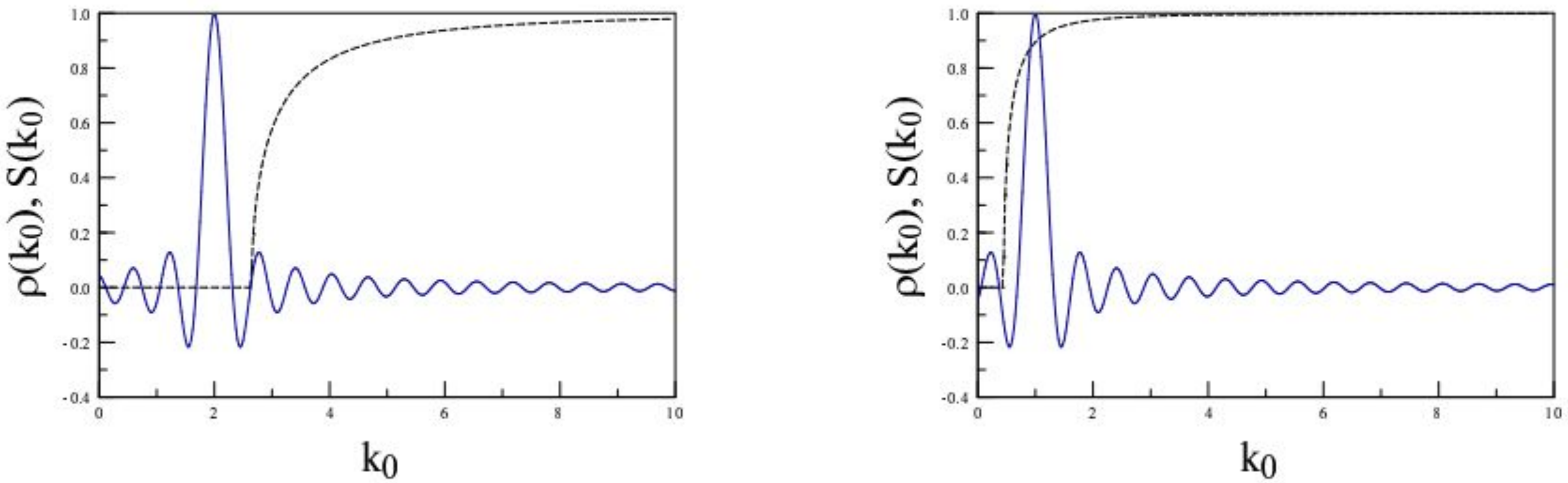
$$\Gamma_k(\eta) = \int_{-\infty}^{\infty} dk_0\;\rho(k_0,k)\,\frac{\sin\left[\left(k_0 - E_k^{(1)}(\eta)\right)\widetilde{T}\right]}{\pi\left(k_0 - E_k^{(1)}(\eta)\right)} \qquad \widetilde{T} = \frac{1}{H}$$

$$\rho(k_0,k;\eta) = \frac{\lambda^2\,a(\eta)}{E_k^{(1)}(\eta)} \int \frac{d^3p}{(2\pi)^3} \frac{(2\pi)\,\delta\left[k_0 - E_p^{(2)}(\eta) - E_q^{(2)}(\eta)\right]}{2\,E_p^{(2)}(\eta)\,2\,E_q^{(2)}(\eta)}$$

$$\frac{\sin\left[\left(k_0 - E_k^{(1)}(\eta)\right)\widetilde{T}\right]}{\pi\left(k_0 - E_k^{(1)}(\eta)\right)} \longrightarrow \delta\left(k_0 - E_k^{(1)}(\eta)\right)$$

$$\Gamma(\eta) = \frac{\lambda^2\,a(\eta)}{8\pi\,E_k^{(1)}(\eta)}\left[1 - \frac{4\,m_2^2}{m_1^2}\right]^{1/2}\Theta(m_1^2 - 4\,m_2^2)$$

Threshold Relaxation



$$\left(E_k^{(1)}(\eta)+\pi H\right)^2 \gg (E_k^{(1)}(\eta))^2+(4m_2^2-m_1^2) \simeq 2\pi E_k^{(1)}(\eta)\, H(\eta) \gg 4m_2^2-m_1^2$$