The Mandelbrot set and its copies

Luna Lomonaco

USP

October 6, 2019

- Dynamical systems is the subject that deals with change.
- Dynamical systems is concerned with the prediction of long-time behaviour of processes which evolve over time.
- More formally, a dynamical system is a triple (M, T, f), where M is a (topological) space, T is a time set and f is the rule which describes how the system evolves over time.
- There are two main types of dynamical systems:
  1. continuous in time,
  2. discrete in time.
- Differential equations describe the evolution of systems which are continuous in time.

- In discrete dynamics, the dynamical system (often) arises by iterating a map $f : \Omega \to \Omega$.
- Iterating a map f means composing this map with itself: $f \circ f \circ ... \circ f$, where $f \circ f \circ f(z) = f(f(f(z)))$.
- The orbit z_n of a point z_0 under f is the sequence

$$z_n = \{f^n(z_0),\, n \in \mathbb{N}\} = \{z_0, f(x_0), f^2(z_0), ...\}$$

We are interested in knowing the following:

1. For what points $z \in \Omega$ the sequence converges?
2. If $z \approx w$, will we have $f^n(z) \approx f^n(w)$?

- A holomorphic map is a complex-valued map of one or more complex variables that is complex-differentiable in a neighborhood of every point in its domain
- **Definition:** Given a complex-valued map f of a single complex variable, the derivative of f at a point z_0 in its domain is defined by the limit

  $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

  If the limit exists, we say that f is $\mathbb{C}$-differentiable at the point z_0. If f is complex differentiable at every point $z_0 \in U$, we say that f is holomorphic on U.
- The existence of a complex derivative is a very strong condition, for it implies that any holomorphic map is actually infinitely differentiable and equal to its own Taylor series

- The Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$ is the compactification of the complex plane by adding a point, namely infinity.

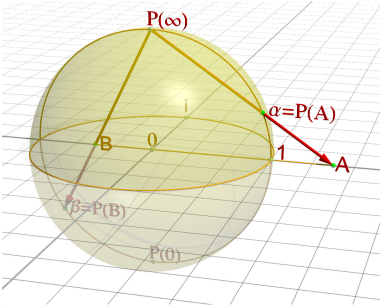
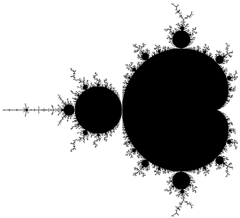
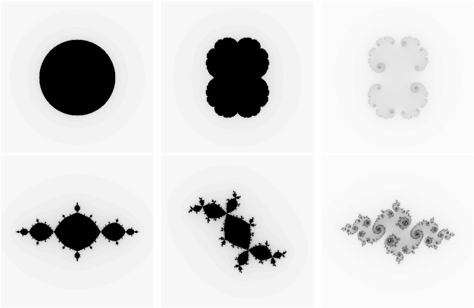


Figure: Stereographic projection of a complex number A onto a point α of the Riemann sphere.

- To study $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ around infinity, look at $g(w) = \frac{1}{f(\frac{1}{w})}$.
- The holomorphic maps in $\widehat{\mathbb{C}}$ are the rational maps.

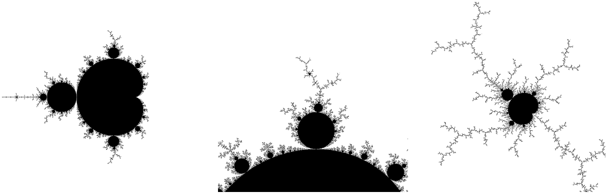
- $P_c(z) = z^2 + c$, ∞ (super)attracting fixed point, with basin $\mathcal{A}_c(\infty)$.
- Filled Julia set $K_c = K_{P_c} = \widehat{\mathbb{C}} \setminus \mathcal{A}_c(\infty)$, $J_P = \partial K_P = \partial \mathcal{A}_c(\infty)$
- Mandelbrot set: set of parameters for which K_{P_c} is connected (**connectedness locus** for the family P_c).





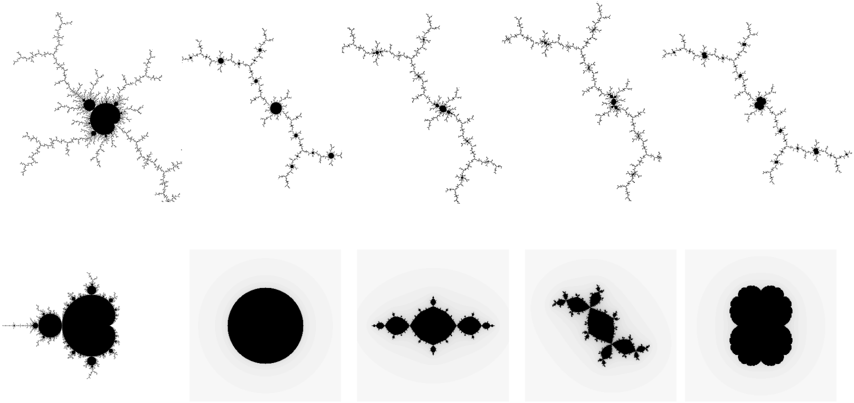
Little copies of M inside M

- Striking: little copies of M in M.
- Satellite copies of M (attached to some hyperbolic component of M, without a cusp).
- Primitive copies (non satellite, with a cusp)

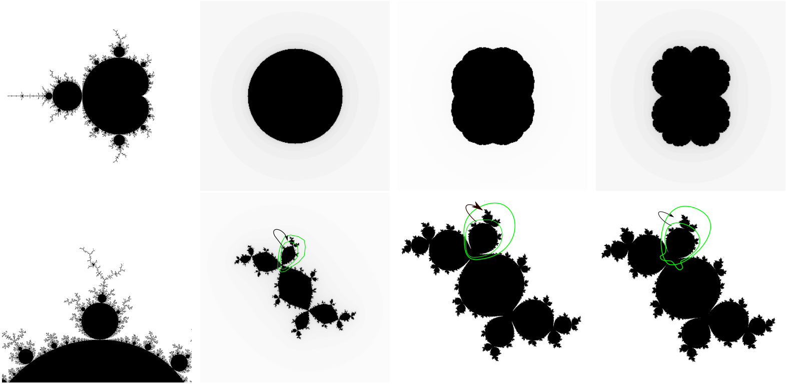


- Their presence explained by theory of polynomial-like mappings (Douady-Hubbard, '85)

- D-H,('85): Primitive copies are homeomorphic to the whole Mandelbrot set.
- Lyubich, ('99): This homeomorphism is quasiconformal.



- D-H,('85): Satellite copies are homeomorphic to the set except at the root.
- Lyubich, ('99): This homeomorphism is quasiconformal.
- Haissinsky ('00): χ homeomorphism at the root in the satellite case.



Are the satellite copies mutually qc homeomorphic?

Theorem (L-Petersen, Invent. Math. '17) For p/q and P/Q irreducible rationals with $q \neq Q$,

$$\xi_{\frac{p}{q},\frac{P}{Q}} := \chi_{P/Q}^{-1} \circ \chi_{p/q} : M_{p/q} \to M_{P/Q}$$

is not quasi-conformal, *i.e.* it does not admit a quasi-conformal extension to any neighborhood of the root.

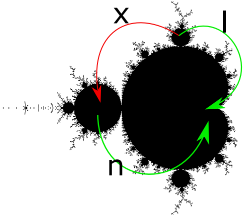


Figure: The map $\xi_{\frac{1}{3},\frac{1}{2}} = \chi_{1/2}^{-1} \circ \chi_{1/3} : M_{1/3} \to M_{1/2}$.

Thank you for your attention!