Gelfand-Tsetlin $\mathfrak{gl}(n)$-modules for arbitrary characters

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Conventions

- We fix $n \geq 2$. 
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- $\mathfrak{gl}(n)$ will denote the Lie algebra of $n \times n$ matrices over $\mathbb{C}$. 
Gelfand-Tsetlin subalgebra

Let for $m \leq n$, $\mathfrak{gl}_m$ be the Lie subalgebra of $\mathfrak{gl}(n)$ spanned by
\[ \{ E_{ij} \mid i, j = 1, \ldots, m \} . \]

\[ \mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n \]

which induces a chain

\[ U_1 \subset U_2 \subset \ldots \subset U_n \]

where $U_m = U(\mathfrak{gl}_m)$. 
Gelfand-Tsetlin subalgebra

Definition

The Gelfand-Tsetlin subalgebra $\Gamma$ of $U$ is the subalgebra generated by $\{Z_m \mid m = 1, \ldots, n\}$. Where the center $Z_m$ of $U_m$ is generated by

$$c_{mk} = \sum_{(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k} E_{i_1i_2} E_{i_2i_3} \cdots E_{i_ki_1}. \quad (1)$$

$k = 1 \ldots, m$. 
Gelfand-Tsetlin modules

Definition

A Gelfand-Tsetlin module is a $U$-module $M$ such that

$$M = \bigoplus_{\chi \in \Gamma^*} M(\chi),$$

with $M(\chi)$ the set of all vectors of generalized $\Gamma$-eigenvalue $\chi$.

$$M(\chi) = \{ v \in M : \forall g \in \Gamma, \exists k \in \mathbb{N} \text{ such that } (g - \chi(g))^k v = 0 \}.$$
Theorem (Futorny, Ovsienko-2000)

Set \( \chi \in \Gamma^* \) and \( Q_n = 2! \cdots (n - 1)! \). Then

1. The number of isomorphism classes of irreducible \( U \)-modules \( N \) such that \( N(\chi) \neq 0 \) is always nonzero and does not exceed \( Q_n \).

2. For any irreducible Gelfand-Tsetlin module \( M \), such that \( M(\chi) \neq 0 \) one has

\[
\dim_{\mathbb{C}} M(\chi) \leq Q_n.
\]
Given a arbitrary Gelfand-Tsetlin character $\chi$, construct a Gelfand-Tsetlin module $M$ such that $M(\chi) \neq 0$. 
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Given a arbitrary Gelfand-Tsetlin character $\chi$, construct all (up to isomorphism) Gelfand-Tsetlin modules $M$ such that $M(\chi) \neq 0$. 
Definition

\[
\begin{array}{cccc}
  l_{n1} & l_{n2} & \cdots & l_{n,n-1} \\
  l_{n-1,1} & \cdots & \cdots & l_{n-1,n-1} \\
  l_{21} & l_{22} & \cdots & \cdots \\
  l_{11} & \cdots & \cdots & \cdots \\
\end{array}
\]

is called a **Gelfand-Tsetlin tableau**. A Gelfand-Tsetlin tableau is called **standard** if the entries satisfied the relations

\[
l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{>0}.
\]
Gelfand-Tsetlin Theorem

Theorem (Gelfand-Tsetlin-1950)

If $L(\lambda)$ is a finite dimensional irreducible representation of $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, there exists a basis of $L(\lambda)$ consisting of all standard tableaux $T(L)$'s with top row $l_{nj} = \lambda_j + j - 1$. Moreover, the action of the generators of $\mathfrak{gl}(n)$ is given by the Gelfand-Tsetlin formulas.
Gelfand-Tsetlin formulas

\[
E_{k,k+1}(T(L)) = - \sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k+1} (l_{ki} - l_{k+1,j})}{\prod_{j \neq i}^{k} (l_{ki} - l_{kj})} \right) T(L + \delta^{ki}),
\]

\[
E_{k+1,k}(T(L)) = \sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k-1} (l_{ki} - l_{k-1,j})}{\prod_{j \neq i}^{k} (l_{ki} - l_{kj})} \right) T(L - \delta^{ki}),
\]

\[
E_{kk}(T(L)) = \left( k - 1 + \sum_{i=1}^{k} l_{ki} - \sum_{i=1}^{k-1} l_{k-1,i} \right) T(L),
\]

Where \( T(L \pm \delta^{ki}) \) is the tableau obtained by \( T(L) \) adding \( \pm 1 \) to the \((k, i)\)'s position of \( T(L) \) (if a new tableau is not standard then the result of the action is zero). The formulas above are called Gelfand-Tsetlin formulas for \( \mathfrak{gl}(n) \).
Theorem (Zhelobenko-1972)

If $V$ is an irreducible finite dimensional module parameterized by tableaux as in the Gelfand-Tsetlin theorem, then the action of the generators of $\Gamma$ on any tableau $T(L)$ of the basis is given by:

$$c_{mk} T(L) = \gamma_{mk}(l_{m1}, \ldots, l_{mm}) T(L).$$

where $\gamma_{mk}$ is an explicit symmetric polynomial of degree $k$ in $m$ variables.
Two approaches

Can we use some analogous to standard relations to obtain a well defined set of tableaux where the Gelfand-Tsetlin formulas give a module structure?
Two approaches

Can we use some analogous to standard relations to obtain a well defined set of tableaux where the Gelfand-Tsetlin formulas give a module structure?

Given an arbitrary tableau, can we construct a tableaux-type module with the action of $\mathfrak{gl}(n)$ given by some generalized Gelfand-Tsetlin formulas?
Approach I

Relation modules
Set $\mathcal{V} := \{(i, j) \mid 1 \leq j \leq i \leq n\}$.

$\mathcal{R}^+ := \{((i, j); (i - 1, t)) \mid 1 \leq j \leq i, \ 2 \leq i \leq n, \ 1 \leq t \leq i - 1\}$

$\mathcal{R}^- := \{((i, j); (i + 1, s)) \mid 1 \leq j \leq i \leq n - 1, \ 1 \leq s \leq i + 1\}$

$\mathcal{R}^0 := \{((n, i); (n, j)) \mid 1 \leq i \neq j \leq n\}$

and let $\mathcal{R} := \mathcal{R}^- \cup \mathcal{R}^0 \cup \mathcal{R}^+ \subset \mathcal{V} \times \mathcal{V}$. From now any $\mathcal{C} \subseteq \mathcal{R}$ will be called a set of relations.
Associated with any $\mathcal{C} \subseteq \mathcal{R}$ we can construct a directed graph $G(\mathcal{C})$ with set of vertices $\mathcal{V}$ and an arrow going from $(i, j)$ to $(r, s)$ if and only if $((i, j); (r, s)) \in \mathcal{C}$. 
Associated with any $C \subseteq \mathcal{R}$ we can construct a directed graph $G(C)$ with set of vertices $\mathcal{V}$ and an arrow going from $(i, j)$ to $(r, s)$ if and only if $((i, j); (r, s)) \in C$.

For convenience we will picture the vertex set as disposed in a triangular arrangement with $n$ rows and $k$-th row given by $\{(k, 1), \ldots, (k, k)\}$. 
Set $\mathcal{R}^+ = \{((i, j); (i - 1, t)) \mid 1 \leq j \leq i, \ 2 \leq i \leq n, \ 1 \leq t \leq i - 1\}$

$G(\mathcal{R}^+)$
Set $\mathcal{R}^- = \{((i,j); (i+1, s)) \mid 1 \leq j \leq i \leq n - 1, \ 1 \leq s \leq i + 1\}$
Definition

Let $\mathcal{C}$ be any set of relations.

(i) $\mathcal{C}$ is called indecomposable if $G(\mathcal{C})$ is a connected graph.

(ii) Given $(i, j), (r, s) \in \mathcal{C}$ we will write $(i, j) \succeq_{\mathcal{C}} (r, s)$ if there exists a path in $G(\mathcal{C})$ starting in $(i, j)$ and finishing in $(r, s)$. 
Definition

We will say that $T(L)$ is a $C$-realization if:

- $l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0}$ for any $((i, j); (r, s)) \in C^+ \cup C^0$.
- $l_{ij} - l_{rs} \in \mathbb{Z}_{> 0}$ for any $((i, j); (r, s)) \in C^-$.
- For any $1 \leq k \leq n - 1$ we have, $l_{ki} - l_{kj} \in \mathbb{Z}$ if and only if $(k, i)$ and $(k, j)$ in the same connected component of $G(C)$. 
The vector space

- By $\mathcal{B}_\mathcal{C}(T(L))$ we denote the set of all tableaux of the form $T(L + z)$, $z \in \mathbb{Z}^{n(n+1)/2}$ such that $z_{ni} = 0$, $i = 1, \ldots, n$ which are $\mathcal{C}$-realizations.
- By $V_{\mathcal{C}}(T(L))$ we denote the complex vector space spanned by $\mathcal{B}_\mathcal{C}(T(L))$. 
Example

Is a $C$-realization, where $G(C)$ is given by one of the following graphs:

\[
\begin{array}{ccc}
(3,1) & (3,2) & (3,3) \\
(2,1) & (2,2) & \\
(1,1) & & (1,1)
\end{array}
\]
Definition $\mathcal{C} \subseteq \mathcal{R}$ is call admissible if:

- There exist a $\mathcal{C}$-realization $T(L)$.
- For any $\mathcal{C}$-realization $T(L)$, the vector space $V_{\mathcal{C}}(T(L))$ has a structure of a $\mathfrak{gl}_n$-module, endowed with the action of $\mathfrak{gl}_n$ given by the Gelfand-Tsetlin formulas.
Finite dimensional modules

Example

\[ S^+ := \{(i + 1, j); (i, j)\} \mid 1 \leq j \leq i \leq n - 1 \]  
\[ S^- := \{((i, j); (i + 1, j + 1)) \mid 1 \leq j \leq i \leq n - 1 \}. \]
FRZ Condition

For every adjoining pair \((k, i)\) and \((k, j)\), \(1 \leq k \leq n - 1\), there exist \(p, q\) such that \(C_1 \subseteq C\) or, there exist \(s < t\) such that \(C_2 \subseteq C\), where the graphs associated to \(C_1\) and \(C_2\) are as follows:

\[
G(C_1) = (k, i) \quad \quad \quad \quad G(C_2) = (k, i)
\]

\[
\begin{array}{ccc}
(k+1, p) & \rightleftharpoons & (k, j) \\
\downarrow & & \downarrow \\
(k-1, q) & \rightleftharpoons & (k+1, s) \\
\end{array}
\]

\[
\begin{array}{ccc}
(k+1, t) & \rightleftharpoons & (k, j) \\
\end{array}
\]
Theorem (Futorny, R., Zhang)

A reduced set of relations $\mathcal{C}$ without cycles and crosses is admissible if and only if $\text{G}(\mathcal{C})$ is a union of disconnected sets satisfying FRZ Condition.
### Generic modules

$$G(\emptyset).$$

<table>
<thead>
<tr>
<th>(4,1)</th>
<th>(4,2)</th>
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<tr>
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<td>(2,1)</td>
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<td>(1,1)</td>
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Cuspidal modules

\[ G(C_1) \]

\[
\begin{array}{cccc}
(4,1) & (4,2) & (4,3) & (4,4) \\
(3,1) & (3,2) & (3,3) & \\
(2,1) & (2,2) & \\
(1,1) & \\
\end{array}
\]
Verma modules

\[ G(C_2) \]

\[
\begin{array}{c}
(4,1) \\
\downarrow \\
(3,1) \\
\downarrow \\
(2,1) \\
\downarrow \\
(1,1)
\end{array}
\quad
\begin{array}{c}
(4,2) \\
\downarrow \\
(3,2) \\
\downarrow \\
(2,2)
\end{array}
\quad
\begin{array}{c}
(4,3) \\
\downarrow \\
(3,3)
\end{array}
\quad
\begin{array}{c}
(4,4)
\end{array}
\]
Theorem (Futorny, R., Zhang)

For any admissible $C$ the module $V_C(T(L))$ is a Gelfand-Tsetlin module with diagonalizable action of the generators of the Gelfand-Tsetlin subalgebra $\Gamma$. 
Approach II

Generalized GT Formulas
Universal Gelfand-Tsetlin modules

Given a Gelfand-Tsetlin character $\chi$, and $T(L)$ a Gelfand-Tsetlin tableau associated with $\chi$, we construct a fully supported module with basis parameterized by the set

$$\{ T(L + \bar{z}) \mid z \in \mathbb{Z}_0^{n(n+1)/2} \}$$

With action of $\mathfrak{gl}(n)$ given by generalized GT-formulas.
\[ E_{k,k+1}(T(L)) = - \sum_{i=1}^{k} \left( \prod_{j=1}^{k+1} \left( l_{ki} - l_{k+1,j} \right) \right) \frac{\prod_{j \neq i}^{k} \left( l_{ki} - l_{kj} \right)}{\prod_{j \neq i}^{k} \left( l_{ki} - l_{kj} \right)} T(L + \delta^{ki}), \]

\[ E_{k+1,k}(T(L)) = \sum_{i=1}^{k} \left( \prod_{j=1}^{k-1} \left( l_{ki} - l_{k-1,j} \right) \right) \frac{\prod_{j \neq i}^{k} \left( l_{ki} - l_{kj} \right)}{\prod_{j \neq i}^{k} \left( l_{ki} - l_{kj} \right)} T(L - \delta^{ki}), \]

\[ E_{kk}(T(L)) = \left( k - 1 + \sum_{i=1}^{k} l_{ki} - \sum_{i=1}^{k-1} l_{k-1,i} \right) T(L), \]
Definition

A vector \( \mathbf{v} \in \mathbb{Z}^{n(n+1)/2} \) is called **singular** if \( v_{rs} - v_{rt} \in \mathbb{Z} \) for some \( 1 \leq s < t \leq r \leq n - 1 \).
Singular Pairs

Suppose first that the singularities of \( \nu \) are given in pairs, which means:

\[
\nu_{ki} - \nu_{kj} \in \mathbb{Z} \text{ implies } \nu_{ki} - \nu_{kt} \notin \mathbb{Z} \text{ for any } t \neq i, j.
\]

Theorem (FGR16, FGR17)

There exist a labeling of the set of tableaux in \( V(T(\nu)) \) indexed by permutations in a subgroup \( \tilde{G} \) of \( S_{n-1} \times \cdots \times S_1 \) (\( \tilde{G} \cong (S_2)^r \)) where \( r \) is the number of singular pairs on \( \nu \)), differential operators \( \{D_\sigma\}_{\sigma \in \tilde{G}} \) and polynomials \( \{P_\sigma\}_{\sigma \in \tilde{G}} \) such that:

\[
E(T_\sigma(\nu + z)) = D_{w_0}(P_{\sigma^{-1}}E(T(\nu + z)))
\]

defines a \( \mathfrak{gl}(n) \)-module structure on \( V(T(\nu)) \).
In this case the polynomials $P_\sigma$ are products of differences of singularities $\prod (\nu_{ki} - \nu_{kj})$ and the differential operators are compositions of differential operator of the form:

$$D_{ij}^{\bar{v}}(f) = \frac{1}{2} \left( \frac{\partial f}{\partial \nu_{ki}} - \frac{\partial f}{\partial \nu_{kj}} \right) (\bar{v}).$$

$$D_\sigma(fT(\nu + z)) = \sum_{\sigma' \leq \sigma} D_{\sigma'}^{\bar{v}}(f) T_{(\sigma')^{-1}}(\bar{v} + z)$$
In [RZ18] a Gelfand-Tsetlin module \( V(T(\bar{v})) \) is associated to any \( \bar{v} \) (a similar construction using a geometric approach appears in [EMV18]). The module \( V(T(\bar{v})) \) is called the \textit{universal tableaux module associated to} \( \bar{v} \). It is a module with \( \mathbb{C} \)-basis given by the set

\[
\left\{ D_\sigma(\bar{v} + z) \mid z \in \mathbb{D}, \sigma \in S^{\mathbb{Z}}_{\pi} \right\}
\]

whose elements are called \textit{derivative tableaux}. A tableau of the form \( D_e(\bar{v} + z) \) is called the \textit{classical tableau} associated to \( \bar{v} + z \).
Generalized GT-formulas

Explicit action

Given \( l = [a, b]_k \) with \( k < n \) we set

\[
e_l = \frac{\prod_{j=1}^{k+1} (x_{k,a} - x_{k+1,j})}{\prod_{(k,j) \notin l} (x_{k,a} - x_{k,j})};
\]

\[
f_l = \frac{\prod_{j=1}^{k-1} (x_{k,b} - x_{k-1,j})}{\prod_{(k,j) \notin l} (x_{k,b} - x_{k,j})}.
\]

We also set

\[
h_k = x_{k,1} + \cdots + x_{k,k} - (x_{k-1,1} + \cdots + x_{k-1,k-1}) + k - 1.
\]
Theorem (FGRZ18)

The action of the canonical generators of $\mathfrak{gl}(n, \mathbb{C})$ on $V(T(\bar{v}))$ is given by the formulas

$$E_{k,k+1}D_\sigma(\bar{v} + z) = - \sum_{l \in \Pi(\bar{v}, z)[k]} \sum_{\tau \leq \sigma \alpha(l)} \mathcal{D}_{\tau,\sigma \alpha(l)}(e_I)D_\tau(\bar{v} + z + \delta^{k,a(l)})$$,

$$E_{k+1,k}D_\sigma(\bar{v} + z) = \sum_{l \in \Pi(\bar{v}, z)[k]} \sum_{\tau \leq \sigma \beta(l)} \mathcal{D}_{\tau,\sigma \beta(l)}(f_I)D_\tau(\bar{v} + z - \delta^{k,b(l)})$$,

$$E_{k,k}D_\sigma(\bar{v} + z) = h_k(\bar{v} + z)D_\sigma(\bar{v} + z)$$,

where $\mathcal{D}_{\tau,\sigma}$ are the Postnikov-Stanley operators introduced in [FGRZ18], and elements $D_\tau(\bar{v} + u)$ such that $\tau$ is not a $u$-shuffle should be treated as zero.
Conjecture

Every irreducible GT-module $V$ with $V_{\chi} \neq \{0\}$ is isomorphic to a subquotient of the module constructed.
Thanks for your attention!