Gelfand-Tsetlin $\mathfrak{gl}(n)$-modules for arbitrary characters

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Conventions

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- $\mathfrak{gl}(n)$ will denote the Lie algebra of $n \times n$ matrices over $\mathbb{C}$.

Gelfand-Tsetlin subalgebra

Let for $m \leqslant n$, $\mathfrak{gl}_m$ be the Lie subalgebra of $\mathfrak{gl}(n)$ spanned by $\{E_{ij} \mid i, j = 1, \ldots, m\}$.

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n$$

which induces a chain

$$U_1 \subset U_2 \subset \ldots \subset U_n$$

where $U_m = U(\mathfrak{gl}_m)$.

Gelfand-Tsetlin subalgebra

Definition

The Gelfand-Tsetlin subalgebra Γ of U is the subalgebra generated by $\{Z_m \mid m = 1, \ldots, n\}$.

Where the center Z_m of U_m is generated by

$$c_{mk} = \sum_{(i_1,\ldots,i_k)\in\{1,\ldots,m\}^k} E_{i_1 i_2} E_{i_2 i_3} \ldots E_{i_k i_1}. \quad (1)$$

$k = 1 \ldots, m$.

Gelfand-Tsetlin modules

Definition

A Gelfand-Tsetlin module is a U-module M such that

$$M = \bigoplus_{\chi \in \Gamma^*} M(\chi),$$

with $M(\chi)$ the set of all vectors of generalized Γ-eigenvalue χ.

$$M(\chi) = \{v \in M : \forall g \in \Gamma \text{ , } \exists k \in \mathbb{N} \text{ such that } (g - \chi(g))^k v = 0\}.$$

Theorem (Futorny, Ovsienko-2000)

Set $\chi \in \Gamma^*$ *and* $Q_n = 2! \cdots (n-1)!$. *Then*

1. *The number of isomorphism classes of irreducible* U*-modules* N *such that* $N(\chi) \neq 0$ *is always nonzero and does not exceed* Q_n.
2. *For any irreducible Gelfand-Tsetlin module* M, *such that* $M(\chi) \neq 0$ *one has*
$$\dim_{\mathbb{C}} M(\chi) \leq Q_n.$$

Problems

- Given a arbitrary Gelfand-Tsetlin character χ, construct a Gelfand-Tsetlin module M such that $M(\chi) \neq 0$.

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- Given a arbitrary Gelfand-Tsetlin character χ, construct a Gelfand-Tsetlin module M such that $M(\chi) \neq 0$.
- Given a arbitrary Gelfand-Tsetlin character χ, construct all (up to isomorphism) Gelfand-Tsetlin modules M such that $M(\chi) \neq 0$.

Irreducible Finite dimensional modules for $\mathfrak{gl}(n)$

Definition

$$\begin{array}{ccccccc}
l_{n1} & l_{n2} & & \cdots & & l_{n,n-1} & l_{nn} \\
& l_{n-1,1} & & \cdots & & l_{n-1n-1} & \\
& & \cdots & \cdots & \cdots & & \\
& & l_{21} & & l_{22} & & \\
& & & l_{11} & & &
\end{array}$$

is called a Gelfand-Tsetlin tableau. A Gelfand-Tsetlin tableau is called standard if the entries satisfied the relations

$$l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0} \quad \textit{and} \quad l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{>0}.$$

Gelfand-Tsetlin Theorem

Theorem (Gelfand-Tsetlin-1950)

If $L(\lambda)$ is a finite dimensional irreducible representation of $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$, there exists a basis of $L(\lambda)$ consisting of all standard tableaux $T(L)$'s with top row $l_{nj} = \lambda_j + j - 1$. Moreover, the action of the generators of $\mathfrak{gl}(n)$ is given by the Gelfand-Tsetlin formulas.

Gelfand-Tsetlin formulas

$$E_{k,k+1}(T(L)) = -\sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k+1}(l_{ki}-l_{k+1,j})}{\prod_{j\neq i}^{k}(l_{ki}-l_{kj})}\right)T(L+\delta^{ki}),$$

$$E_{k+1,k}(T(L)) = \sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k-1}(l_{ki}-l_{k-1,j})}{\prod_{j\neq i}^{k}(l_{ki}-l_{kj})}\right)T(L-\delta^{ki}),$$

$$E_{kk}(T(L)) = \left(k-1+\sum_{i=1}^{k}l_{ki}-\sum_{i=1}^{k-1}l_{k-1,i}\right)T(L),$$

Where $T(L\pm\delta^{ki})$ is the tableau obtained by $T(L)$ adding ± 1 to the (k,i)'s position of $T(L)$ (if a new tableau is not standard then the result of the action is zero). The formulas above are called Gelfand-Tsetlin formulas for $\mathfrak{gl}(n)$.

Action of Γ on finite dimensional modules

Theorem (Zhelobenko-1972)

If V is an irreducible finite dimensional module parameterized by tableaux as in the Gelfand-Tsetlin theorem, then the action of the generators of Γ on any tableau $T(L)$ of the basis is given by:

$$c_{mk}T(L) = \gamma_{mk}(l_{m1}, \ldots, l_{mm})T(L).$$

where γ_{mk} is an explicit symmetric polynomial of degree k in m variables.

Two approaches

1. Can we use some analogous to standard relations to obtain a well defined set of tableaux where the Gelfand-Tsetlin formulas give a module structure?

Two approaches

1. Can we use some analogous to standard relations to obtain a well defined set of tableaux where the Gelfand-Tsetlin formulas give a module structure?
2. Given an arbitrary tableau, can we construct a tableaux-type module with the action of $\mathfrak{gl}(n)$ given by some generalized Gelfand-Tsetlin formulas?

Relation modules

Relation modules

Set $\mathfrak{V} := \{(i,j) \mid 1 \leq j \leq i \leq n\}$.

$$\mathcal{R}^+ := \{((i,j);(i-1,t)) \mid 1 \leq j \leq i,\ 2 \leq i \leq n,\ 1 \leq t \leq i-1\}$$
$$\mathcal{R}^- := \{((i,j);(i+1,s)) \mid 1 \leq j \leq i \leq n-1,\ 1 \leq s \leq i+1\}$$
$$\mathcal{R}^0 := \{((n,i);(n,j)) \mid 1 \leq i \neq j \leq n\}$$

and let $\mathcal{R} := \mathcal{R}^- \cup \mathcal{R}^0 \cup \mathcal{R}^+ \subset \mathfrak{V} \times \mathfrak{V}$. From now any $\mathcal{C} \subseteq \mathcal{R}$ will be called a *set of relations*.

Associated with any $\mathcal{C} \subseteq \mathcal{R}$ we can construct a directed graph $G(\mathcal{C})$ with set of vertices $\mathfrak{V}$ and an arrow going from (i, j) to (r, s) if and only if $((i, j); (r, s)) \in \mathcal{C}$.

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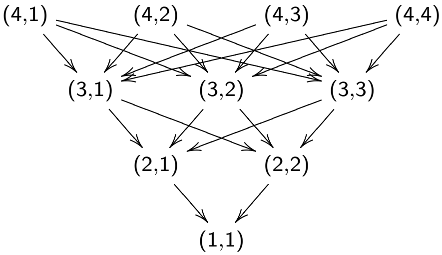
For convenience we will picture the vertex set as disposed in a triangular arrangement with n rows and k-th row given by $\{(k, 1), \ldots, (k, k)\}$.

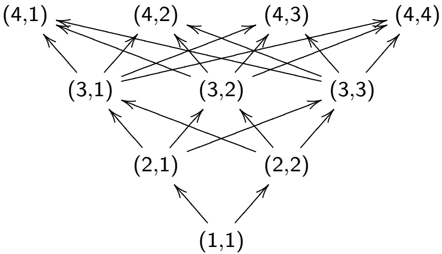
Set $\mathcal{R}^+ = \{((i,j);(i-1,t)) \mid 1 \leq j \leq i,\ 2 \leq i \leq n,\ 1 \leq t \leq i-1\}$

$G(\mathcal{R}^+)$



Set $\mathcal{R}^- = \{((i,j);(i+1,s)) \mid 1 \leq j \leq i \leq n-1,\ 1 \leq s \leq i+1\}$

$G(\mathcal{R}^-)$



Definition

Let $\mathcal{C}$ be any set of relations.

(i) $\mathcal{C}$ is called indecomposable if $G(\mathcal{C})$ is a connected graph.

(ii) Given $(i,j),\ (r,s) \in \mathfrak{V}$ we will write $(i,j) \succeq_{\mathcal{C}} (r,s)$ if there exists a path in $G(\mathcal{C})$ starting in (i,j) and finishing in (r,s).

Definition

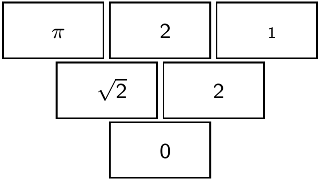
We will say that $T(L)$ is a $\mathcal{C}$-realization if:

- $l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0}$ for any $((i,j);(r,s)) \in \mathcal{C}^+ \cup \mathcal{C}^0$.
- $l_{ij} - l_{rs} \in \mathbb{Z}_{>0}$ for any $((i,j);(r,s)) \in \mathcal{C}^-$.
- For any $1 \leq k \leq n-1$ we have, $l_{ki} - l_{kj} \in \mathbb{Z}$ if and only if (k,i) and (k,j) in the same connected component of $G(\mathcal{C})$.

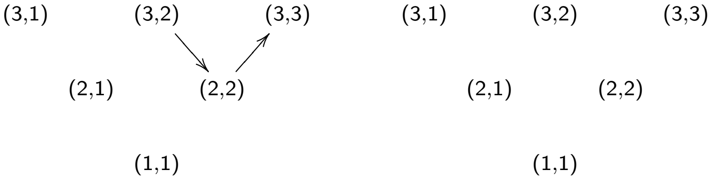
The vector space

- By $\mathcal{B}_{\mathcal{C}}(T(L))$ we denote the set of all tableaux of the form $T(L+z)$, $z \in \{z \in \mathbb{Z}^{\frac{n(n+1)}{2}} \mid z_{ni} = 0, i = 1, \ldots, n\}$ which are $\mathcal{C}$-realizations.
- By $V_{\mathcal{C}}(T(L))$ we denote the complex vector space spanned by $\mathcal{B}_{\mathcal{C}}(T(L))$.

Example



Is a $\mathcal{C}$-realization, where $G(\mathcal{C})$ is given by one of the following graphs



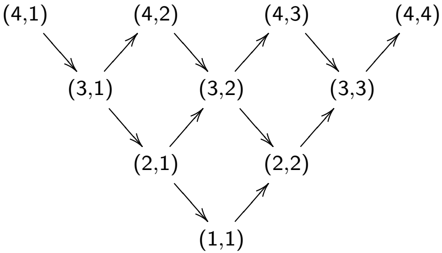
Definition $\mathcal{C} \subseteq \mathcal{R}$ is call admissible if:

- There exist a $\mathcal{C}$-realization $T(L)$.
- For any $\mathcal{C}$-realization $T(L)$, the vector space $V_{\mathcal{C}}(T(L))$ has a structure of a $\mathfrak{gl}_n$-module, endowed with the action of $\mathfrak{gl}_n$ given by the Gelfand-Tsetlin formulas.

Finite dimensional modules

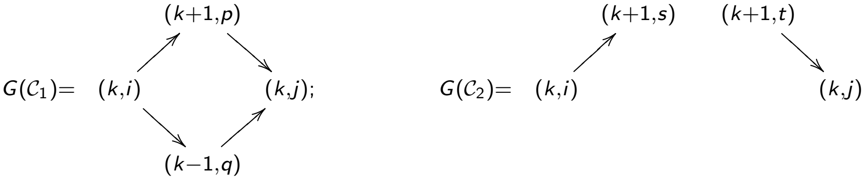
Example

$$\mathcal{S}^+ := \{(i+1,j);(i,j)) \mid 1 \leq j \leq i \leq n-1\}$$
$$\mathcal{S}^- := \{((i,j);(i+1,j+1)) \mid 1 \leq j \leq i \leq n-1\}.$$



FRZ Condition

For every adjoining pair (k,i) and (k,j), $1 \leq k \leq n-1$, there exist p, q such that $\mathcal{C}_1 \subseteq \mathcal{C}$ or, there exist $s < t$ such that $\mathcal{C}_2 \subseteq \mathcal{C}$, where the graphs associated to $\mathcal{C}_1$ and $\mathcal{C}_2$ are as follows



Theorem (Futorny, R., Zhang)

A reduced set of relations $\mathcal{C}$ without cycles and crosses is admissible if and only if $G(\mathcal{C})$ is a union of disconnected sets satisfying FRZ Condition.

Generic modules

$G(\emptyset)$.

(4,1) (4,2) (4,3) (4,4)

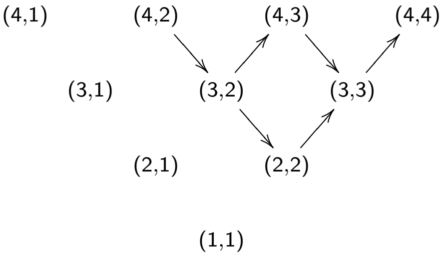
(3,1) (3,2) (3,3)

(2,1) (2,2)

(1,1)

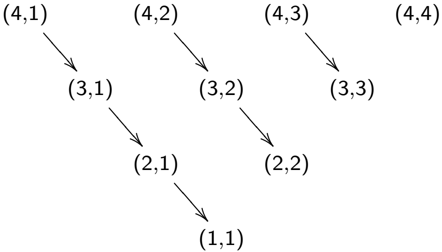
Cuspidal modules

$G(\mathcal{C}_1)$



Verma modules

$G(\mathcal{C}_2)$



Theorem (Futorny, R., Zhang)

For any admissible $\mathcal{C}$ the module $V_{\mathcal{C}}(T(L))$ is a Gelfand-Tsetlin module with diagonalizable action of the generators of the Gelfand-Tsetlin subalgebra Γ.

Approach II

Generalized GT Formulas

Universal Gelfand-Tsetlin modules

Given a Gelfand-Tsetlin character χ, and $T(L)$ a Gelfand-Tsetlin tableau associated with χ, we construct a fully supported module with basis parameterized by the set

$$\{\, T(L+\bar{z}) \mid z \in \mathbb{Z}_0^{\frac{n(n+1)}{2}} \}$$

With action of $\mathfrak{gl}(n)$ given by generalized GT-formulas.

Gelfand-Tsetlin formulas

$$E_{k,k+1}(T(L)) = -\sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k+1}(l_{ki}-l_{k+1,j})}{\prod_{j\neq i}^{k}(l_{ki}-l_{kj})}\right)T(L+\delta^{ki}),$$

$$E_{k+1,k}(T(L)) = \sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k-1}(l_{ki}-l_{k-1,j})}{\prod_{j\neq i}^{k}(l_{ki}-l_{kj})}\right)T(L-\delta^{ki}),$$

$$E_{kk}(T(L)) = \left(k-1+\sum_{i=1}^{k}l_{ki}-\sum_{i=1}^{k-1}l_{k-1,i}\right)T(L),$$

Definition

A vector $v \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ is called singular if $v_{rs} - v_{rt} \in \mathbb{Z}$ for some $1 \leq s < t \leq r \leq n-1$.

Singular Pairs

Suppose first that the singularities of v are given in pairs, which means:

$$v_{ki} - v_{kj} \in \mathbb{Z} \text{ implies } v_{ki} - v_{kt} \notin \mathbb{Z} \text{ for any } t \neq i, j.$$

Theorem (FGR16, FGR17)

There exist a labeling of the set of tableaux in $V(T(v))$ indexed by permutations in a subgroup $\tilde{G}$ of $S_{n-1} \times \cdots \times S_1$ ($\tilde{G} \simeq (S_2)^r$ where r is the number of singular pairs on v), differential operators $\{\mathcal{D}_\sigma\}_{\sigma \in \tilde{G}}$ and polynomials $\{\mathcal{P}_\sigma\}_{\sigma \in \tilde{G}}$ such that:

$$E(T_\sigma(\overline{v} + z)) = \mathcal{D}_{w_0}(P_{\sigma^{-1}} E(T(v + z)))$$

defines a $\mathfrak{gl}(n)$-module structure on $V(T(v))$.

In this case the polynomials P_σ are products of differences of singularities $\prod(v_{ki} - v_{kj})$ and the differential operators are compositions of differential operator of the form:

$$\mathcal{D}_{ij}^{\bar{v}}(f) = \frac{1}{2}\left(\frac{\partial f}{\partial v_{ki}} - \frac{\partial f}{\partial v_{kj}}\right)(\bar{v}).$$

$$\mathcal{D}_\sigma(fT(v+z)) = \sum_{\sigma' \leq \sigma} \mathcal{D}_{\sigma'}^{\bar{v}}(f)\, T_{(\sigma')^{-1}}(\bar{v}+z)$$

Universal tableaux Gelfand-Tsetlin modules

In [RZ18] a Gelfand-Tsetlin module $V(T(\bar{v}))$ is associated to any $\bar{v}$ (a similar construction using a geometric approach appears in [EMV18]). The module $V(T(\bar{v}))$ is called the *universal tableaux module associated to* $\bar{v}$. It is a module with $\mathbb{C}$-basis given by the set

$$\{D_\sigma(\bar{v}+z) \mid z \in \mathbb{D}, \sigma \in S_\pi^z\}$$

whose elements are called *derivative tableaux*. A tableau of the form $D_e(\bar{v}+z)$ is called the *classical tableau* associated to $\bar{v}+z$.

Explicit action

Given $I = [a, b]_k$ with $k < n$ we set

$$e_I = \frac{\prod_{j=1}^{k+1}(x_{k,a} - x_{k+1,j})}{\prod_{(k,j)\notin I}(x_{k,a} - x_{k,j})}; \qquad f_I = \frac{\prod_{j=1}^{k-1}(x_{k,b} - x_{k-1,j})}{\prod_{(k,j)\notin I}(x_{k,b} - x_{k,j})}.$$

. We also set

$$h_k = x_{k,1} + \cdots + x_{k,k} - (x_{k-1,1} + \cdots + x_{k-1,k-1}) + k - 1.$$

Theorem (FGRZ18)

The action of the canonical generators of $\mathfrak{gl}(n,\mathbb{C})$ *on* $V(T(\bar{v}))$ *is given by the formulas*

$$E_{k,k+1}D_\sigma(\bar{v}+z) = -\sum_{I\in\mathbb{I}(\bar{v},z)[k]}\ \sum_{\tau\leq\sigma\alpha(I)} \mathfrak{D}^{\bar{v}+z}_{\tau,\sigma\alpha(I)}(e_I)D_\tau(\bar{v}+z+\delta^{k,a(I)}),$$

$$E_{k+1,k}D_\sigma(\bar{v}+z) = \sum_{I\in\mathbb{I}(\bar{v},z)[k]}\ \sum_{\tau\leq\sigma\beta(I)} \mathfrak{D}^{\bar{v}+z}_{\tau,\sigma\beta(I)}(f_I)D_\tau(\bar{v}+z-\delta^{k,b(I)}),$$

$$E_{k,k}D_\sigma(\bar{v}+z) = h_k(\bar{v}+z)D_\sigma(\bar{v}+z),$$

where $\mathfrak{D}_{\tau,\sigma}$ *are the Postnikov-Stanley operators introduced in [FGRZ18], and elements* $D_\tau(\bar{v}+u)$ *such that* τ *is not a u-shuffle should be treated as zero.*

Conjecture

Every irreducible GT-module V with $V_\chi \neq \{0\}$ is isomorphic to a subquotient of the module constructed.

Thanks for your attention!