q-Analogue of $A_{m-1} \oplus A_{n-1} \subset A_{mn-1}$ [8]

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ICTP-SAIFR Workshop on Quantum Symmetries

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- the spin $L \rightarrow L_1 + L_2 + \ldots + L_n$,
- the iso-spin $T \rightarrow T_1 + T_2 + \ldots + T_n$.

Chains of Subgroups as Dynamical Symmetry of a System

The system Hamiltonian is usually more complicated: $H = H_{free} + H_{int}$

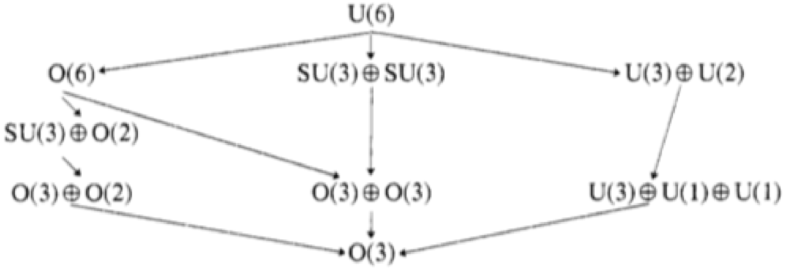

Figure: Chains of subgroups within the Interacting two-vector-boson model of collective motions in nuclei [1].

the R-matrix Quantum Matrix Group

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The Hopf-algebra relations among the regular functionals $l^{\pm}_{ij}$ [2]:

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Co - Unit : \; \varepsilon(l^{\pm}_{ij}) &= \delta_{ij}, && (2)\\
\sum_{m,p} R^{+}_{ij,mp} l^{+}_{mk} l^{-}_{pl} &= \sum_{m,p} l^{-}_{jp} l^{+}_{im} R^{+}_{mp,kl}, && (3)\\
\sum_{m,p} R^{+}_{ij,mp} l^{\pm}_{mk} l^{\pm}_{pl} &= \sum_{m,p} l^{\pm}_{jp} l^{\pm}_{im} R^{+}_{mp,kl}. && (4)
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$$\prod_{i=1}^{n} l_{ii}^{\pm} = 1 \quad ; \quad l_{ii}^{+} l_{ii}^{-} = 1 = l_{ii}^{-} l_{ii}^{+} \quad (5)$$

$$l_{ij}^{+} = 0 \quad \text{for} \quad i > j \quad \text{and} \quad l_{ij}^{-} = 0 \quad \text{for} \quad i < j \quad (6)$$

The last two relations are utilized for the Special matrix groups.

Reshetikhin's R-matrix for A^q_{n-1} and its matrix realization

For the A^q_{n-1} algebras the explicit form of the R^+-matrix is given by:

$$R^+ = q^{\frac{1}{n}} \left(q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i \neq j=1}^{n} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j=1}^{n} e_{ij} \otimes e_{ji} \right) \quad (7)$$

where e_{ij} are $n \times n$ matrixes with elements $(e_{ij})_{km} = \delta_{ik}\delta_{jm}$.

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where e_{ij} are $n \times n$ matrixes with elements $(e_{ij})_{km} = \delta_{ik}\delta_{jm}$.
This leads to the following relations for $l^{\pm}_{ij}$:

$$[l^{(\varepsilon)}_{im}, l^{(\varepsilon)}_{js}] = (1-q)(\underbrace{l^{(\varepsilon)}_{im} l^{(\varepsilon)}_{js}}_{i=j} - \underbrace{l^{(\varepsilon)}_{js} l^{(\varepsilon)}_{im}}_{m=s}) + (q - q^{-1})(\underbrace{l^{(\varepsilon)}_{jm} l^{(\varepsilon)}_{is}}_{m>s} - \underbrace{l^{(\varepsilon)}_{jm} l^{(\varepsilon)}_{is}}_{j>i}) \quad (8)$$

$$[l^{+}_{im}, l^{-}_{js}] = (1-q)(\underbrace{l^{+}_{im} l^{-}_{js}}_{i=j} - \underbrace{l^{-}_{js} l^{+}_{im}}_{m=s}) + (q - q^{-1})(\underbrace{l^{-}_{jm} l^{+}_{is}}_{m>s} - \underbrace{l^{+}_{jm} l^{-}_{is}}_{j>i}) \quad (9)$$

The q-number is defined as $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$ and used as follows[1]:

$$I^{\pm}_{ij} = \mp q^{\pm \frac{1}{2}}(q - q^{-1}) Y^{\pm}_{ij} q^{\mp \frac{1}{2}(\tilde{H}_i + \tilde{H}_j)} \text{ with } Y^{\pm}_{ii} = \mp \frac{q^{\mp \frac{1}{2}}}{q - q^{-1}} \quad (10)$$

$$[Y^{+}_{ij}, Y^{-}_{ji}] = [H_{ij}]_q \; i < j; \; H_{ij} = \tilde{H}_i - \tilde{H}_j, \; [H_{ij}, H_{km}] = 0 \quad (11)$$

[1] $Y^{\pm}_{ij}$ can be replaced by $\tilde{Y}^{\pm}_{ij} f_{ij}(q, \tilde{H})$ which will modify (11). An example of such a mapping from $su(2)$ to a deformed $su_q(2)$ has been given in [3].

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$[Y^{-}_{km}, Y^{+}_{ij}] = (q - q^{-1}) Y^{+}_{kj} Y^{-}_{im} q^{H_{ik}}$
$j > k > i > m$
$[Y^{+}_{ij}, Y^{-}_{im}] = 0 \; j > i > m$
$[Y^{+}_{ij}, Y^{-}_{ki}] = -Y^{+}_{kj} q^{H_{ik}} \; j > k > i$
$[Y^{+}_{ij}, Y^{-}_{jm}] = Y^{-}_{im} q^{H_{ij}} \; j > i > m$
$[Y^{+}_{ij}, Y^{-}_{km}] = 0$

$[Y^{+}_{ij}, Y^{-}_{km}] = (q - q^{-1}) Y^{-}_{kj} Y^{+}_{im} q^{H_{jm}}$
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$[Y^{+}_{ij}, Y^{-}_{jm}] = q^{H_{jm}} Y^{+}_{im} \; j > m > i$
$$\begin{cases} k > j > i > m; \; k > m > j > i \\ j > k > m > i; \; j > i > k > m \end{cases}$$

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Borel subalgebras $\mathcal{B}^{\pm}$

$$Borel \quad subalgebra : \mathcal{B}^{+} \qquad Borel \quad subalgebra : \mathcal{B}^{-} \tag{12}$$

$$
\begin{array}{ll}
[H_{ik}, Y_{js}^{+}] = (e_i - e_k, e_j - e_s) Y_{js}^{+} & [H_{ik}, Y_{js}^{-}] = (e_i - e_k, e_j - e_s) Y_{js}^{-} \\
[Y_{ik}^{+}, Y_{kj}^{+}]_q = Y_{ij}^{+} \quad i < k < j & [Y_{ij}^{-}, Y_{jk}^{-}]_{q^{-1}} = Y_{ik}^{-} \quad i > j > k \\
[Y_{ik}^{+}, Y_{ij}^{+}]_q = 0 \quad i < j < k & [Y_{kj}^{-}, Y_{ij}^{-}]_{q^{-1}} = 0 \quad i > k > j \\
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[Y_{ij}^{+}, Y_{km}^{+}] = 0 \quad i < j < k < m & [Y_{ij}^{-}, Y_{km}^{-}] = 0 \quad i > j > k > m \\
[Y_{ij}^{+}, Y_{km}^{+}] = 0 \quad i < k < m < j & [Y_{ij}^{-}, Y_{km}^{-}] = 0 \quad i > k > m > j \\
[Y_{km}^{+}, Y_{ij}^{+}] = (q - q^{-1}) Y_{kj}^{+} Y_{im}^{+} & [Y_{ij}^{-}, Y_{km}^{-}] = (q - q^{-1}) Y_{kj}^{-} Y_{im}^{-} \\
i < k < j < m & i > k > j > m
\end{array}
$$

where $(e_i, e_j) = \delta_{ij}$ and the q-commentator is $[A, B]_q = AB - qBA$.

the Hopf-algebra structure

From the definition of the *Co-Multiplication* $\Delta(l^{\pm}_{ij}) = \sum_{k=1}^{n} l^{\pm}_{ik} \otimes l^{\pm}_{kj}$ and the *Co-Unit* $\varepsilon(l^{\pm}_{ij}) = \delta_{ij}$ one has the following co-algebraic structure :

$$\Delta H_{ij} = H_{ij} \otimes 1 + 1 \otimes H_{ij} \; ; \; \varepsilon(H_{ij}) = 0 \; ; \; S(H_{ij}) = -H_{ij}$$

$$\varepsilon(Y^{\pm}_{ij}) = \mp\frac{q^{\mp\frac{1}{2}}}{q-q^{-1}}\delta_{ij} \; ; \quad Y^{\pm}_{ii} = \mp\frac{q^{\mp\frac{1}{2}}}{q-q^{-1}} \; ; \quad Y^{+}_{ik} = 0 \; i > k \; ; \; Y^{-}_{ik} = 0 \; i < k$$

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$$\Delta Y^{\pm}_{ij} = \mp(q - q^{-1})q^{\pm\frac{1}{2}} \sum_{i\leq k\leq j \text{ or } (j\leq k\leq i)} Y^{\pm}_{ik} q^{\pm\frac{1}{2}H_{jk}} \otimes Y^{\pm}_{kj} q^{\pm\frac{1}{2}H_{ik}} \tag{13}$$

Applying the standard definition of the antipode S:

$$m \circ (id \otimes S) \circ \Delta = m \circ (S \otimes id) \circ \Delta = i \circ \varepsilon$$

recurrent formula for the antipode of the generators $Y^{\pm}_{ij}$:

$$S(Y^{\pm}_{ij}) = -q^{\mp 1} Y^{\pm}_{ij} \pm (q - q^{-1})q^{\pm 1} \sum_{i<k<j \text{ or } (i>k>j)} Y^{\pm}_{ik} S(Y^{\pm}_{kj}) \tag{14}$$

the q-boson algebra $\mathcal{A}_q^-$(n)

q-boson creation and annihilation operators $a_i^\pm$ and their q-boson number operators N_i as in [4, 5, 6, 7].

$$a_i^- a_i^+ - q^\mp a_i^+ a_i^- = q^{\pm N_i} \text{ and } [N_i, a_j^\pm] = \pm\delta_{ij} a_j^\pm \tag{15}$$

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The irreducible Fock representations $\Gamma_q^{[m]}$ with a vacuum state $\mid 0>$ such that $a_i^- |0>= 0,\ N_i \mid 0>= 0$:

$$\Gamma_q^{[m]} := \{\mid m>=\mid m_1, ..., m_n>= \prod_{i=1}^{n} \frac{(a_i^+)^{m_i}}{\sqrt{[m_i]!}} \mid 0> ,\ \mid m = \sum_{i=1}^{n} m_i\} \tag{16}$$

with the following properties: $\dim \Gamma_q^{[m]} = \frac{(n+m-1)!}{m!(n-1)!}$

$$N \mid m>= m \mid m> \quad \text{where} \quad N = \sum_{i=1}^{n} N_i. \tag{17}$$

q-boson realization of the A^q_{n-1} generators

Cartan-Chevalley generators (Sun and Fu in [4]):

$$H_i = H_{i,i+1},\ Y_i^+ = Y_{i,i+1}^+,\ Y_i^- = Y_{i+1,i}^-$$

$$H_i = N_i - N_{i+1}\ ;\ Y_i^+ = a_i^+ a_{i+1}^-\ ;\ Y_i^- = a_{i+1}^+ a_i^- \tag{18}$$

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The operators N_i can be expressed using H_i (18) and N (17):

$$N_i = \frac{1}{n}N + \frac{1}{n}\sum_{s=2}^{n}\sum_{j=1}^{s-1} H_j - \sum_{j=1}^{i-1} H_j \tag{19}$$

q-boson realization of the A^q_{n-1} generators

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The additional generators that extend the Chevalley basis (18) to the Cartan-Weyl basis (11) can be obtained by utilizing the second set of relations in the Borel subalgebras $\mathcal{B}^{\pm}$ in (12):

$$H_{ij} = N_i - N_j\ ;\ Y_{ij}^{\pm} = a_i^+ a_j^- q^{\mp \sum_{i<k<j \text{ or } (j<k<i)} N_k} \tag{20}$$

the Δ^{n-1} homomorphism mapping

Proposition 1:

$\tilde{X}_\mu^\pm$, $\tilde{H}_\mu$ and $\tilde{Z}^{\pm s}$, $\tilde{H}^s$ satisfy the relations within $A^q_{k_1-1}$ and $A^q_{k_2-1}$.

$$A^q_{m-1} \xrightarrow{\Delta^{(n-1)}} \underbrace{A^q_{m-1} \otimes \ldots \otimes A^q_{m-1}}_{n} \quad (21)$$

$\otimes$ will be dropped and the index s (or μ) will indicate of the tensor space.

$$\tilde{H}_\mu = \sum_{s=1}^{k_2} H^s_\mu; \quad \tilde{X}^\pm_\mu = \Delta^{(k_2-1)}(X^\pm_\mu) = \sum_{s=1}^{k_2} X^{\pm s}_\mu q^{\frac{1}{2}\sum_{\sigma \neq s, \sigma=1}^{k_2} sign(\sigma - s) H^\sigma_\mu} \quad (22)$$

$$\tilde{H}^s = \sum_{\mu=1}^{k_1} H^s_\mu; \quad \tilde{Z}^{\pm s} = \Delta^{(k_1-1)}(Z^{\pm s}) = \sum_{\mu=1}^{k_1} Z^{\pm s}_\mu q^{\frac{1}{2}\sum_{\sigma \neq \mu, \sigma=1}^{k_1} sign(\sigma - \mu) H^s_\sigma} \quad (23)$$

Explicit q-boson and Δ^{k-1} realization

$$
\begin{aligned}
\tilde{X}^{+}_{\mu} &= \sum_{s=1}^{k_2} a^{+s}_{\mu} a^{-s}_{\mu+1} q^{\frac{1}{2}\sum_{\sigma\neq s,\sigma=1}^{k_2} sign(\sigma-s)(N^{\sigma}_{\mu}-N^{\sigma}_{\mu+1})} \\
\tilde{X}^{-}_{\mu} &= \sum_{s=1}^{k_2} a^{+s}_{\mu+1} a^{-s}_{\mu} q^{\frac{1}{2}\sum_{\sigma\neq s,\sigma=1}^{k_2} sign(\sigma-s)(N^{\sigma}_{\mu}-N^{\sigma}_{\mu+1})} \\
\tilde{Z}^{+s} &= \sum_{\mu=1}^{k_1} a^{+s}_{\mu} a^{-s+1}_{\mu} q^{\frac{1}{2}\sum_{\sigma\neq \mu,\sigma=1}^{k_1} sign(\sigma-\mu)(N^{s}_{\sigma}-N^{s+1}_{\sigma})} \\
\tilde{Z}^{-s} &= \sum_{\mu=1}^{k_1} a^{+s+1}_{\mu} a^{-s}_{\mu} q^{\frac{1}{2}\sum_{\sigma\neq \mu,\sigma=1}^{k_1} sign(\sigma-\mu)(N^{s}_{\sigma}-N^{s+1}_{\sigma})} \\
\tilde{H}^{s} &= \sum_{\mu=1}^{k_1} N^{s}_{\mu} - N^{s+1}_{\mu} \; ; \; \tilde{H}_{\mu} = \sum_{s=1}^{k_2} N^{s}_{\mu} - N^{s}_{\mu+1}
\end{aligned}
\tag{24}
$$

Understanding the q-bosons within $\tilde{X}_\mu$ and $\tilde{Z}^s$

$$\text{in } \tilde{X} : a_\mu^{\pm s} = \underbrace{id \otimes \ldots \otimes id \otimes \overbrace{a_\mu^\pm}^{s} \otimes id \otimes \ldots \otimes id}_{k_2}$$

$$\text{while in } \tilde{Z} : a_\mu^{\pm s} = \underbrace{id \otimes \ldots \otimes id \otimes \overbrace{a_s^\pm}^{\mu} \otimes id \otimes \ldots \otimes id}_{k_1}$$

However in both cases, they satisfy the same relations:

$$\begin{array}{ll} [a_\mu^{\pm s}, a_\nu^{\pm t}] = 0 \text{ for all } s, t, \mu, \nu & [a_\mu^{+s}, a_\nu^{-t}] = 0 \text{ for all } s \neq t; \mu \neq \nu \\ [N_\mu^s, a_\nu^{\pm t}] = \pm\delta_{\mu,\nu}\delta_{s,t}a_\nu^{\pm t} & a_\mu^{-s}a_\mu^{+s} - q^{\mp 1}a_\mu^{+s}a_\mu^{-s} = q^{\pm N_\mu^s} \end{array} \tag{25}$$

Proposition 2:

The algebras $\otimes^{k_2}\mathcal{A}_q^-(k_1)$ and $\otimes^{k_1}\mathcal{A}_q^-(k_2)$ constructed by the q-bosons $a_\mu^{\pm s}$ are isomorphic to the algebra $\mathcal{A}_q^-(k_1 k_2)$ constructed by the q-bosons $a_i^\pm$.

The Splitting correspondence

The Splitting correspondence: $i \leftrightarrow (\mu, s)$ $(k_2 \leq k_1)$ [8]:

$$\begin{array}{l} i \leftrightarrow (\mu, s) \;\; i = 1, \ldots, k_1 k_2; \;\; \mu = 1, \ldots, k_1; \;\; s = 1, \ldots, k_2 \\ \mu = 1 + \mathrm{int}[\frac{i-1}{k_2}] \;\; \text{where int} \, [x] \;\; \text{is the integer part of } x \\ s = 1 + (i-1)\mathrm{mod}(k_2) \; , \;\; i = (\mu - 1)k_2 + s. \end{array} \tag{26}$$

Proof of Proposition 2:

It follows from the introduction of (26) in equations (15) and (25).

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Lemma 1:

$A^q_{k_1-1}$ and $A^q_{k_2-1}$ have realizations within the algebra $\mathcal{A}^-_q(k_1 k_2)$ - see (24).

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Proposition 3:

The generators $\tilde{X}^{\pm}_{\mu}$, $\tilde{H}_{\mu}$ commute with the generators $\tilde{Z}^{\pm s}$, $\tilde{H}^s$. [8]

The explicit embedding $A^q_{k_1-1} \oplus A^q_{k_2-1} \subset A^q_{k_1 k_2-1}$

By using (19), (20), and the isomorphism (26) the generators of $A^q_{k_1-1}$ and $A^q_{k_2-1}$ in (24) are expressed through the generators of $A^q_{k_1 k_2-1}$ in the following way:

$$
\begin{aligned}
\tilde{Z}^{\pm s} &= \sum_{\mu=1}^{k_1} Y^{\pm}_{(\mu-1)k_2+s}\, q^{\frac{1}{2}\sum_{\sigma\neq\mu,\sigma=1}^{k_1} sign(\sigma-\mu) H_{(\sigma-1)k_2+s}} \\
\tilde{H}^{s} &= \sum_{\mu=1}^{k_1} H_{(\mu-1)k_2+s} \quad ; \quad \tilde{H}_{\mu} = \sum_{s=(\mu-1)k_2+1}^{(\mu-1)k_2+k_2} H_{s,s+k_2} \\
\tilde{X}^{+}_{\mu} &= \sum_{t=\mu k_2+1}^{(\mu+1)k_2} Y^{+}_{t-k_2,t}\, q^{\frac{1}{2}\sum_{\nu\neq t,\nu=\mu k_2+1}^{(\mu+1)k_2} sign(\nu-t) H_{\nu-k_2,\nu} + \Lambda^{+}_{t}} \\
\tilde{X}^{-}_{\mu} &= \sum_{t=\mu k_2+1}^{(\mu+1)k_2} Y^{-}_{t,t-k_2}\, q^{\frac{1}{2}\sum_{\nu\neq t,\nu=\mu k_2+1}^{(\mu+1)k_2} sign(\nu-t) H_{\nu-k_2,\nu} + \Lambda^{-}_{t}} \\
\Lambda^{\pm}_{t} &= \frac{k_2-1}{k_1 k_2}\left(N + \sum_{\sigma=2}^{k_1 k_2} H_{1,\sigma}\right) \pm \sum_{\sigma=t-k_2+1}^{t-1} H_{1,\sigma}
\end{aligned}
\tag{27}
$$

The difference $\Lambda^{\pm}_{t}$ between the expressions for $\tilde{Z}^{\pm s}$ and $\tilde{X}^{\pm}_{\mu}$ is due to the ordering of indices in (26) which leads to the appearance of different terms $q^{\pm N_k}$ in the q-boson realization (20) of the Chevalley and the additional Weyl generators. In the expression $\Lambda^{\pm}_{t}$ the operator N , in q-boson realization has the meaning of a total number of bosons operator.

Conclusion and Q&A Discussion

Proposition 4:

The elements $\tilde{X}_{\mu}^{\pm}$, $\tilde{H}_{\mu}$ of $A_{k_1-1}^{q}$ and $\tilde{Z}^{\pm s}$, $\tilde{H}^{s}$ of $A_{k_2-1}^{q}$ defined by (27) belong to the algebra $A_{k_1 k_2-1}^{q}$ and provide an explicit embedding $A_{k_1-1}^{q} \oplus A_{k_2-1}^{q} \subset A_{k_1 k_2-1}^{q}$ in the q-boson realization (20) of $A_{k_1 k_2-1}^{q}$ [8].

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The construction relays on:

- the Δ^{n-1} homomorphism mapping taking A^q_{k-1} into $\otimes^n A^q_{k-1}$,

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The construction relays on:

- the Δ^{n-1} homomorphism mapping taking A_{k-1}^{q} into $\otimes^{n} A_{k-1}^{q}$,
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- the isomorphisms $\mathcal{A}_q^{-}$ (mn)$\sim \otimes^{n}\mathcal{A}_q^{-}(m)\sim \otimes^{m}\mathcal{A}_q^{-}$(n).

Conclusion and Q&A Discussion

Proposition 4:

The elements $\tilde{X}^{\pm}_{\mu}$, $\tilde{H}_{\mu}$ of $A^q_{k_1-1}$ and $\tilde{Z}^{\pm s}$, $\tilde{H}^s$ of $A^q_{k_2-1}$ defined by (27) belong to the algebra $A^q_{k_1 k_2-1}$ and provide an explicit embedding $A^q_{k_1-1} \oplus A^q_{k_2-1} \subset A^q_{k_1 k_2-1}$ in the q-boson realization (20) of $A^q_{k_1 k_2-1}$ [8].

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THANK YOU!

FOR YOUR TIME AND ATTENTION!

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