Examples of finite-dimensional pointed Hopf algebras in positive characteristic

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$\Bbbk = \overline{\Bbbk}$ algebraically closed field.

Plan:

I. Braided vector spaces and Nichols algebras.

II. Nichols algebras of decompositions.

III. Nichols algebras of blocks.

IV. Nichols algebras of blocks + points.

I. Braided vector spaces and Nichols algebras.

Γ (finitely generated) abelian group, $\Bbbk\Gamma$ its group algebra.

${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} =$ category of Yetter-Drinfeld modules over $\Bbbk\Gamma$:

- $V = \oplus_{g\in\Gamma} V_g$ is a Γ-graded vector space;
- V is a left Γ-module such that $g \cdot V_h = V_h$ (compatibility).

Definition. (V, c) braided vector space:
V vector space $+ c \in GL(V \otimes V)$ satisfies the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

$V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} \implies V$ braided vector space:

$$c(v \otimes w) = g \cdot w \otimes v, \qquad v \in V_g, w \in V.$$

If $\mathcal{B}$ is a Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, then $\mathcal{B}\#\Bbbk\Gamma$ is a usual Hopf algebra (here $\#$ is the Radford-Majid bosonization).

Definition. $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} \implies \exists$ unique (up to isomorphism) $\mathcal{B}(V) = \oplus_{n\in\mathbb{N}_0}\mathcal{B}^n(V)$ (graded) Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that

$$\mathcal{B}^0(V) \simeq \Bbbk, \qquad \mathcal{B}^1(V) \simeq V,$$
$$\mathcal{B}(V) = \Bbbk\langle V\rangle, \qquad \mathrm{Prim}(\mathcal{B}(V)) = \mathcal{B}^1(V).$$

$\mathcal{B}(V)$ is called the **Nichols algebra** of V.

Remark. $\exists\Omega : T(V) \to T^c(V)$ (quantum symmetrizer) such that $\mathcal{B}(V) =$ image of $\Omega \implies$ depends just on c.

Problem. Classify V in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that the Nichols algebra $\mathcal{B}(V)$ has finite dimension (or finite GKdim).

Note that $\mathcal{B}(V) = T(V)/\mathcal{I}(V)$ where $\mathcal{I}(V)$ is the maximal homogeneous Hopf ideal intersecting trivially $\Bbbk \oplus V$.

But $\mathcal{I}(V)$ is difficult to determine explicitly in general. One needs a variety of indirect techniques to deal with $\mathcal{B}(V)$.

II. Nichols algebras of decompositions.

Let $n \in \mathbb{N}_{\geq 2}$, $\mathbb{I}_n = \{1, 2, \dots, n\}$ and W a braided vector space.

Definition. (Graña) A **decomposition** of W is a family of subspaces $(W_i)_{i \in \mathbb{I}_n}$, all $\neq 0$, such that

$$W = \oplus_{i \in \mathbb{I}_n} W_i, \qquad c(W_i \otimes W_j) = W_j \otimes W_i, \qquad i, j \in \mathbb{I}_n.$$

Remark. (Graña) $V_1, V_2 \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, $V = V_1 \oplus V_2$.

If $c_{V_2,V_1} c_{V_1,V_2} = c^2_{V_1,V_2} = \mathrm{id}_{V_1 \otimes V_2} \implies \mathcal{B}(V) \simeq \mathcal{B}(V_1) \underline{\otimes} \mathcal{B}(V_2)$.

One goal is to study Nichols algebras of decomposable BVS (assuming the components are known).

Braided vector space (V, c) is of diagonal type if admits a decomposition whose components have dimension 1 **(points)**. That is, $\exists$ a basis $(x_i)_{i \in \mathbb{I}_\theta}$ of V and $\mathfrak{q} = (q_{ij})_{i,j \in \mathbb{I}_\theta}$ such that

$$c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i, \qquad i, j \in \mathbb{I}_\theta.$$

We attach to $\mathfrak{q}$ its Dynkin diagram (generalized): $\cdots \overset{q_i}{\circ} \overset{\widetilde{q}_{ij}}{\text{———}} \overset{q_j}{\circ} \cdots$

Here $\widetilde{q}_{ij} := q_{ij} q_{ji}$. We will assume that the diagram is connected.

Remark. If $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is semisimple $\implies$ is of diagonal type.

Theorem (char $\Bbbk = 0$). [Heckenberger]
The classification of the $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, G a finite abelian group, such that $\dim \mathcal{B}(V) < \infty$ is known.

Remark. The proof uses the technology of Weyl groupoids and (generalized) root systems. Given a decomposition

$$W = \oplus_{i \in \mathbb{I}_n} W_i, \qquad c(W_i \otimes W_j) = W_j \otimes W_i, \qquad i, j \in \mathbb{I}_n,$$

such that all W_i are **simple** Yetter-Drinfeld modules over a Hopf algebra H, then it has also a Weyl groupoid [A-Heckenberger-Schneider, Heckenberger-Schneider].

Theorem (char $\Bbbk = 0$). [Heckenberger-Vendramin]
The classification of the $W \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, G a finite group, with $n > 1$ such that $\dim \mathcal{B}(W) < \infty$ is known.

III. Nichols algebras of blocks.

Blocks: Indecomposable but not simple braided vector spaces (the easiest ones): $\mathcal{V}(\epsilon,\ell) \in {}^{\Bbbk\mathbb{Z}}_{\Bbbk\mathbb{Z}}\mathcal{YD}$, $\dim \mathcal{V}(\epsilon,\ell) = \ell \geq 2$; in a basis $(x_i)_{i\in\mathbb{I}_\ell}$ the braiding is

$$c(x_i \otimes x_j) = \begin{cases} \epsilon x_1 \otimes x_i, & j = 1 \\ (\epsilon x_j + x_{j-1}) \otimes x_i, & j \geq 2, \end{cases} \qquad i \in \mathbb{I}_\ell.$$

Theorem (char $\Bbbk = 0$). [AAH]

$$\operatorname{GKdim} \mathcal{B}(\mathcal{V}(\epsilon,\ell)) < \infty \iff \ell = 2 \text{ and } \epsilon \in \{\pm 1\}.$$

$\mathcal{V}(\epsilon, 2)$ is called an ϵ-block.

Open (char $\Bbbk > 0$): $\operatorname{GKdim} \mathcal{B}(\mathcal{V}(\epsilon,\ell)) < \infty \iff$ **???**

Proposition (char $\Bbbk = 0$). [AAH]
$\mathcal{B}(\mathcal{V}(1,2)) = \Bbbk\langle x_1, x_2 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2 \rangle$ (Jordan plane).

$\{x_1^a x_2^b : a, b \in \mathbb{N}_0\}$ is a basis of $\mathcal{B}(\mathcal{V}(1,2))$.

GKdim $\mathcal{B}(\mathcal{V}(1,2)) = 2$; $\mathcal{B}(\mathcal{V}(1,2))$ is a domain.

Proposition (char $\Bbbk > 2$). [Cibils-Lauda-Witherspoon]
$\mathcal{B}(\mathcal{V}(1,2)) = \Bbbk\langle x_1, x_2 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, x_1^p, \ x_2^p \rangle$
(Jordan plane in char $p > 2$).

$\{x_1^a x_2^b : 0 \leq a, b < p\}$ basis of $\mathcal{B}(\mathcal{V}(1,2)) \implies \dim \mathcal{B}(\mathcal{V}(1,2)) = p^2$.

Now $\mathcal{V}(-1,2)$. Let $x_{21} = \operatorname{ad}_c x_2\, x_1 = x_2x_1 + x_1x_2$.

Proposition (char $\Bbbk = 0$). [AAH]
$\mathcal{B}(\mathcal{V}(-1,2)) = \Bbbk\langle x_1, x_2 | x_1^2,\ x_2x_{21} - x_{21}x_2 - x_1x_{21}\rangle$
(super Jordan plane).
$\{x_1^a x_{21}^b x_2^c : a \in \{0,1\}, b, c \in \mathbb{N}_0\}$ is a basis of $\mathcal{B}(\mathcal{V}(-1,2))$.
$\operatorname{GKdim} \mathcal{B}(\mathcal{V}(-1,2)) = 2$.

Proposition (char $\Bbbk > 2$). [AAH2]
$\mathcal{B}(\mathcal{V}(-1,2)) = \Bbbk\langle x_1, x_2 | x_1^2,\ x_2x_{21} - x_{21}x_2 - x_1x_{21}, x_{21}^p,\ x_2^{2p}\rangle$
(super Jordan plane in char $p > 2$).

$\{x_1^a x_{21}^b x_2^c : a \in \mathbb{I}_{0,1}, b \in \mathbb{I}_{0,p-1}, c \in \mathbb{I}_{0,2p-1}\}$ basis of $\mathcal{B}(\mathcal{V}(-1,2))$
$\implies \dim \mathcal{B}(\mathcal{V}(-1,2)) = 4p^2$.

IV. Nichols algebras of blocks + points.

Let $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with a decomposition

$$V = V_1 \oplus \cdots \oplus V_\theta$$

such that each V_i is either an ϵ-block ($\epsilon^2 = 1$) or a point.

We assume at least one block.

Theorem (char $\Bbbk = 0$). [AAH]

The Nichols algebras $\operatorname{GKdim} \mathcal{B}(V) < \infty \iff V$ belongs to a list.

One block and one point.

Let V_1 be an ϵ-block with basis $\{x_1, x_2\}$, $V_2 = \Bbbk x_3$ a point with label q_{22} and $V = V_1 \oplus V_2$. Let $q_{11} := \epsilon \in \{\pm 1\}$. For some $q_{12}, q_{21} \in \Bbbk^\times$, $a \in \Bbbk$, the braiding is of the form $(c(x_i \otimes x_j))_{i,j \in \mathbb{I}_3} =$

$$= \begin{pmatrix} \epsilon x_1 \otimes x_1 & (\epsilon x_2 + x_1) \otimes x_1 & q_{12} x_3 \otimes x_1 \\ \epsilon x_1 \otimes x_2 & (\epsilon x_2 + x_1) \otimes x_2 & q_{12} x_3 \otimes x_2 \\ q_{21} x_1 \otimes x_3 & q_{21}(x_2 + a x_1) \otimes x_3 & q_{22} x_3 \otimes x_3 \end{pmatrix}.$$

Remark: $c^2_{|V_1 \otimes V_2} = \mathrm{id} \iff q_{21} q_{21} = 1$ and $a = 0$.

$\widetilde{q}_{12} = q_{21} q_{21} =$ *interaction* between the block V_1 and V_2:

weak if $\widetilde{q}_{12} = 1$, **mild** if $\widetilde{q}_{12} = -1$, **strong** if $\widetilde{q}_{12} \notin \{\pm 1\}$.

(char $\Bbbk = 0$). The *ghost* is a normalized version of a:

$$\mathscr{G} = \begin{cases} -2a, & \epsilon = 1, \\ a, & \epsilon = -1. \end{cases} \qquad \mathscr{G} \in \mathbb{N} \iff_{\text{def.}} \text{ the ghost is } \textit{discrete}.$$

Lemma. If the interaction is **strong**, then $\operatorname{GKdim} \mathcal{B}(V) = \infty$.
If it is **mild** and $\epsilon = 1$, then $\operatorname{GKdim} \mathcal{B}(V) = \infty$.

Graphical description. **Weak interaction:**

$\epsilon = 1$: $\boxplus \overset{\mathscr{G}}{\rule[0.5ex]{3em}{0.4pt}} \overset{q_{22}}{\bullet}$, $\quad \boxplus \qquad \overset{q_{22}}{\bullet}$ when $\mathscr{G} = 0$;

$\epsilon = -1$: $\boxminus \overset{\mathscr{G}}{\rule[0.5ex]{3em}{0.4pt}} \overset{q_{22}}{\bullet}$, $\quad \boxminus \qquad \overset{q_{22}}{\bullet}$ when $\mathscr{G} = 0$;

Mild interaction: $\epsilon = -1$: $\boxminus \overset{(-,\mathscr{G})}{\rule[0.5ex]{3em}{0.4pt}} \overset{q_{22}}{\bullet}$

Theorem (char $\Bbbk = 0$). [AAH]
Let V be a braided vector space **a block + a point**. Then $\dim \mathcal{B}(V) < \infty \iff V$ is as in the Table.

Nichols algebras of a block & point with finite GKdim**,** char 0

V		GKdim	V		GKdim
$\mathfrak{L}(1, \mathscr{G})$	$\boxplus \overset{\mathscr{G}}{\text{———}} \overset{1}{\bullet}$	$\mathscr{G} + 3$	$\mathfrak{L}(-1, \mathscr{G})$	$\boxplus \overset{\mathscr{G}}{\text{———}} \overset{-1}{\bullet}$	2
$\mathfrak{L}_-(1, \mathscr{G})$	$\boxminus \overset{\mathscr{G}}{\text{———}} \overset{1}{\bullet}$	$\mathscr{G} + 3$	$\mathfrak{L}_-(-1, \mathscr{G})$	$\boxminus \overset{\mathscr{G}}{\text{———}} \overset{-1}{\bullet}$	$\mathscr{G} + 2$
$\mathfrak{L}(\omega, 1)$	$\boxplus \overset{1}{\text{———}} \overset{\omega}{\bullet}$	2	$\mathfrak{C}_1$	$\boxminus \overset{(-1,1)}{\text{———}} \overset{-1}{\bullet}$	2

(char $\Bbbk = p > 2$). Again the braiding is $(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_3} =$

$$= \begin{pmatrix} \epsilon x_1 \otimes x_1 & (\epsilon x_2 + x_1) \otimes x_1 & q_{12}x_3 \otimes x_1 \\ \epsilon x_1 \otimes x_2 & (\epsilon x_2 + x_1) \otimes x_2 & q_{12}x_3 \otimes x_2 \\ q_{21}x_1 \otimes x_3 & q_{21}(x_2 + a x_1) \otimes x_3 & q_{22}x_3 \otimes x_3 \end{pmatrix}.$$

V has **discrete ghost** if $a \in \mathbb{F}_p^\times$.

In this case, the *ghost* is a normalized version of a: we pick a representative $\mathtt{r} \in \mathbb{Z}$ of $2a$ by imposing

$$\mathtt{r} \in \begin{cases} \{1-p, \dots, -1\}, & \epsilon = 1, \\ \{1, \dots, 2p-1\} \cap 2\mathbb{Z}, & \epsilon = -1; \end{cases} \quad \text{set } \mathscr{G} := \begin{cases} -\mathtt{r}, & \epsilon = 1, \\ \mathtt{r}, & \epsilon = -1. \end{cases}$$

Then $\mathscr{G}$ is called the *ghost*.

Theorem (char $\Bbbk = p > 2$). [AAH2]
Let V be a braided vector space **a block + a point**. If V is as in the Table, then $\dim \mathcal{B}(V) < \infty$.

Finite-dimensional Nichols algebras of a block and a point

V	diagram	$\mathscr{G}$	$\dim K$	$\dim \mathcal{B}(V)$
$\mathfrak{L}(1, \mathscr{G})$	$\boxplus \overset{\mathscr{G}}{\text{———}} \overset{1}{\bullet}$	discrete	$p^{\mathrm{r}+1}$	$p^{\mathrm{r}+3}$
$\mathfrak{L}(-1, \mathscr{G})$	$\boxplus \overset{\mathscr{G}}{\text{———}} \overset{-1}{\bullet}$	discrete	$2^{\mathrm{r}+1}$	$2^{\mathrm{r}+1}p^2$
$\mathfrak{L}(\omega, 1)$	$\boxplus \overset{1}{\text{———}} \overset{\omega}{\bullet}$	1	3^3	3^3p^2
$\mathfrak{L}_-(1, \mathscr{G})$	$\boxminus \overset{\mathscr{G}}{\text{———}} \overset{1}{\bullet}$	discrete	$2^{\frac{\mathrm{r}}{2}}p^{\frac{\mathrm{r}}{2}+1}$	$2^{\frac{\mathrm{r}}{2}+2}p^{\frac{\mathrm{r}}{2}+3}$
$\mathfrak{L}_-(-1, \mathscr{G})$	$\boxminus \overset{\mathscr{G}}{\text{———}} \overset{-1}{\bullet}$	discrete	$2^{\frac{\mathrm{r}}{2}+1}p^{\frac{\mathrm{r}}{2}}$	$2^{\frac{\mathrm{r}}{2}+3}p^{\frac{\mathrm{r}}{2}+2}$
$\mathfrak{C}_1$	$\boxminus \overset{(-1,1)}{\text{———}} \overset{-1}{\bullet}$	1	16	$64p^2$

To prove the Theorem, observe that

$$V = V_1 \oplus V_2 \quad \Longrightarrow \quad \mathcal{B}(V) \rightleftarrows \mathcal{B}(V_1) \quad \Longrightarrow \quad \mathcal{B}(V) \simeq K\#\mathcal{B}(V_1),$$

where $K = \mathcal{B}(V)^{\mathrm{co}\,\mathcal{B}(V_1)}$. By [HS, Prop. 8.6], $K \simeq \mathcal{B}(K^1)$ where

$$K^1 = \mathrm{ad}_c\,\mathcal{B}(V_1)(V_2) \in {}^{\mathcal{B}(V_1)\#\Bbbk\Gamma}_{\mathcal{B}(V_1)\#\Bbbk\Gamma}\mathcal{YD}.$$

To describe K^1, we set

$$z_n := (\mathrm{ad}_c\, x_2)^n x_3, \qquad f_n := (\mathrm{ad}_c\, x_1)^n x_3, \qquad n \in \mathbb{N}_0.$$

By explicit computing the braiding, we see that this is mostly a symmetric or super symmetric algebra (up to a twist).

$\mathfrak{L}(1,\mathscr{G})$: Let $\mathscr{G} \in \mathbb{N}$ and $V \leftrightsquigarrow \boxplus \overset{\mathscr{G}}{\text{———}} \overset{1}{\bullet}$. Let $z_n := (\operatorname{ad}_c x_2)^n x_3$.

Proposition (char $\Bbbk = 0$). [AAH].

$$\begin{aligned} \mathcal{B}(V) = \Bbbk\langle x_1, x_2, x_3 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, \\ x_1x_3 - q_{12}x_3x_1 \\ z_t z_{t+1} - q_{12}^{-1} z_{t+1} z_t, \quad 0 \leq t < \mathscr{G}, \\ z_{1+\mathscr{G}} \rangle; \end{aligned}$$

$$B = \{x_1^{m_1} x_2^{m_2} z_{\mathscr{G}}^{n_{\mathscr{G}}} \dots z_1^{n_1} z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\}$$

is a PBW-basis, hence

$\operatorname{GKdim} \mathcal{B}(V) = \mathscr{G} + 3$

$\mathfrak{L}(1, \mathscr{G})$: Let $\mathscr{G} \in \mathbb{N}$ and $V \rightsquigarrow \boxplus \overset{\mathscr{G}}{\rule{4em}{0.4pt}} \overset{1}{\bullet}$. Let $z_n := (\mathrm{ad}_c\, x_2)^n x_3$.

Proposition (char $\Bbbk > 2$). [AAH2]

$$\begin{aligned}\mathcal{B}(V) = \Bbbk\langle x_1, x_2, x_3 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2,\\ x_1x_3 - q_{12}x_3x_1\\ z_tz_{t+1} - q_{12}^{-1}z_{t+1}z_t, \quad 0 \le t < \mathscr{G},\\ z_{1+\mathscr{G}},\\ x_1^p, x_2^p, z_t^p, \quad 0 \le t < \mathscr{G}\rangle;\end{aligned}$$

$$B = \{x_1^{m_1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}} \dots z_1^{n_1}z_0^{n_0} : m_i, n_j \in \mathbb{I}_{0,p-1}\}$$

is a PBW-basis, hence

$\dim \mathcal{B}(V) = p^{\mathscr{G}+3}$.

$\mathfrak{L}(-1, \mathscr{G})$: Let $\mathscr{G} \in \mathbb{N}$ and $V \rightsquigarrow \boxplus \overset{\mathscr{G}}{\rule[0.5ex]{5em}{0.4pt}} \overset{-1}{\bullet}$. Let $z_n := (\operatorname{ad}_c x_2)^n x_3$.

Proposition (char $\Bbbk = 0$). [AAH].

$$\begin{aligned}\mathcal{B}(V) = \Bbbk\langle x_1, x_2, x_3 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2,\\ x_1x_3 - q_{12}x_3x_1\\ z_tz_{t+1} - q_{12}^{-1}z_{t+1}z_t, \quad 0 \leq t < \mathscr{G},\\ z_t^2, \quad 0 \leq t < \mathscr{G},\\ z_{1+\mathscr{G}}\rangle;\end{aligned}$$

$B = \{x_1^{m_1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}} \dots z_1^{n_1}z_0^{n_0} : n_i \in \{0, 1\}, m_j \in \mathbb{N}_0\}$
is a PBW-basis, hence
$\operatorname{GKdim} \mathcal{B}(V) = 2$.

$\mathfrak{L}(-1, \mathscr{G})$: Let $\mathscr{G} \in \mathbb{N}$ and $V \rightsquigarrow \boxplus \overset{\mathscr{G}}{\text{———}} \overset{-1}{\bullet}$. Let $z_n := (\mathrm{ad}_c\, x_2)^n x_3$.

Proposition (char $\Bbbk > 2$). [AAH2]

$$
\begin{aligned}
\mathcal{B}(V) = \Bbbk\langle x_1, x_2, x_3 | x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, & \\
& x_1x_3 - q_{12}x_3x_1 \\
& z_tz_{t+1} - q_{12}^{-1}z_{t+1}z_t, \quad 0 \le t < \mathscr{G}, \\
& z_t^2, \quad 0 \le t < \mathscr{G}, \\
& z_{1+\mathscr{G}}, \\
& x_1^p, x_2^p;
\end{aligned}
$$

$$B = \{x_1^{m_1} x_2^{m_2} z_{\mathscr{G}}^{n_{\mathscr{G}}} \dots z_1^{n_1} z_0^{n_0} : n_i \in \{0,1\}, m_j \in \mathbb{I}_{0,p-1}\}$$

is a PBW-basis, hence

$\dim \mathcal{B}(V) = 2^{\mathscr{G}+1}p^2$.

One block and several points.

Theorem (char $\Bbbk = 0$). [AAH] Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\theta$ be a decomposable braided vector space, V_1 a 1-block, $V_2, \dots, V_\theta$ points, braidings as above. Then
$\operatorname{GKdim} \mathcal{B}(V) < \infty \iff$ for every connected component J of $V_2 \oplus \cdots \oplus V_\theta$, either $\mathscr{G}_J = 0$, or else V_J is as in the Table. Also,

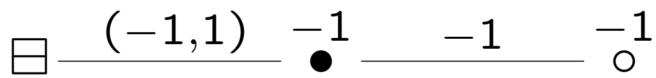
$$\operatorname{GKdim} \mathcal{B}(V) = 2 + \sum_{J \in \mathbb{X}} \operatorname{GKdim} \mathcal{B}(K_J).$$

Let $\omega \in \mathbb{G}'_3$, $\mathfrak{d}_J = \operatorname{GKdim} \mathcal{B}(K_J)$.

A block and several points, finite GKdim.

V_J	type	$\mathscr{G}_J$	K_J	$\mathfrak{d}_J$
$\overset{1}{\circ}$	A_1	discrete	$(A_1)^{\mathscr{G}_J+1}$	$\mathscr{G}_J+1$
$\overset{-1}{\circ}$	A_1	discrete	$(A_1)^{\mathscr{G}_J+1}$	0
$\overset{\omega}{\circ}$	A_1	1	A_2	0
$\overset{-1}{\circ}\overset{-1}{\text{——}}\overset{-1}{\circ}\ \dots\ \overset{-1}{\circ}\overset{-1}{\text{——}}\overset{-1}{\circ}$	$A_{\theta-1}$	$(1,0,\dots,0)$	A_3, $\theta=3$	0
			D_θ, $\theta>3$	
$\overset{-1}{\circ}\overset{-1}{\text{——}}\overset{-1}{\circ}$	A_2	$(2,0)$	D_4	0
$\overset{-1}{\circ}\overset{\omega}{\text{——}}\overset{-1}{\circ}$	$\mathfrak{sl}(2\vert 1)$	$(1,0)$	$\mathfrak{g}(2,3)$	0
$\overset{-1}{\circ}\overset{\omega^2}{\text{——}}\overset{\omega}{\circ}$	$\mathfrak{sl}(2\vert 1)$	$(1,0)$	$\mathfrak{sl}(2\vert 2)$	0
		$(0,1)$	$\mathfrak{g}(2,3)$	0
$\overset{-1}{\circ}\overset{\omega}{\text{——}}\overset{\omega^2}{\circ}\overset{\omega}{\text{——}}\overset{\omega^2}{\circ}$	$\mathfrak{sl}(1\vert 3)$	$(1,0,0)$	$\mathfrak{g}(3,3)$	0
$\overset{-1}{\circ}\overset{\omega}{\text{——}}\overset{\omega^2}{\circ}\overset{\omega^2}{\text{——}}\overset{\omega}{\circ}$	$\mathfrak{osp}(2,4)$	$(1,0,0)$	$\mathfrak{g}(3,3)$	0
$\overset{-1}{\circ}\overset{r^{-1}}{\text{——}}\overset{r}{\circ}$, $r\notin\mathbb{G}_\infty$	$\mathfrak{sl}(2\vert 1)$	$(1,0)$	$\mathfrak{sl}(2\vert 2)$	2
$\overset{-1}{\circ}\overset{r^{-1}}{\text{——}}\overset{r}{\circ}$, $r\in\mathbb{G}'_N, N>3$	$\mathfrak{sl}(2\vert 1)$	$(1,0)$	$\mathfrak{sl}(2\vert 2)$	0

Theorem (char $\Bbbk = 0$). [AAH] Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\theta$ be a decomposable braided vector space as above, but now V_1 a -1-block. Then $\operatorname{GKdim} \mathcal{B}(V) < \infty \iff$
essentially one more example called $\mathfrak{C}_2$ with diagram



Theorem (char $\Bbbk = p > 2$). [AAH2] The following Table gives examples of finite-dimensional Nichols algebras:

A block and several points, finite dim.

V	diagram	$dim\mathcal{B}(V)$
$\mathfrak{L}(A_{\theta-1})$, $\theta > 2$	$\boxplus \overset{1}{\text{———}} \overset{-1}{\bullet} \overset{-1}{\text{———}} \overset{-1}{\circ} \dots \overset{-1}{\circ} \overset{-1}{\text{———}} \overset{-1}{\circ}$ $\theta - 1$ vertices	$p^2 2^6$ $p^2 2^{(\theta-1)(\theta-2)}$
$\mathfrak{L}(A_2, 2)$	$\boxplus \overset{2}{\text{———}} \overset{-1}{\bullet} \overset{-1}{\text{———}} \overset{-1}{\circ}$	$p^2 2^{12}$
$\mathfrak{L}(A(1\vert 0)_2; \omega)$	$\boxplus \overset{1}{\text{———}} \overset{-1}{\bullet} \overset{\omega}{\text{———}} \overset{-1}{\circ}$	$p^2 2^7 3^4$
$\mathfrak{L}(A(1\vert 0)_1; \omega)$	$\boxplus \overset{1}{\text{———}} \overset{-1}{\bullet} \overset{\omega^2}{\text{———}} \overset{\omega}{\circ}$	$p^2 2^4 3^2$
$\mathfrak{L}(A(1\vert 0)_3; \omega)$	$\boxplus \overset{1}{\text{———}} \overset{\omega}{\bullet} \overset{\omega^2}{\text{———}} \overset{-1}{\circ}$	$p^2 2^7 3^4$
$\mathfrak{L}(A(1\vert 0)_1; r)$	$\boxplus \overset{1}{\text{———}} \overset{-1}{\bullet} \overset{r^{-1}}{\text{———}} \overset{r}{\circ}$, $r \in \mathbb{G}'_N, N > 3$	$p^2 2^4 N^2$
$\mathfrak{L}(A(2\vert 0)_1; \omega)$	$\boxplus \overset{1}{\text{———}} \overset{-1}{\bullet} \overset{\omega}{\text{———}} \overset{\omega^2}{\circ} \overset{\omega}{\text{———}} \overset{\omega^2}{\circ}$	$p^2 2^8 3^9$
$\mathfrak{L}(D(2\vert 1); \omega)$	$\boxplus \overset{1}{\text{———}} \overset{-1}{\bullet} \overset{\omega}{\text{———}} \overset{\omega^2}{\circ} \overset{\omega^2}{\text{———}} \overset{\omega}{\circ}$	$p^2 2^8 3^9$

Final remarks. (i). The Nichols algebras in char $p > 2$ presented above can be realized in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with G finite. Thus we obtain new examples of pointed Hopf algebras in char $p > 2$ not of diagonal type.

(ii). There are abelian groups G and $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ of the form

$$V = \oplus_{1 \le i \le \theta} V_i, \qquad c(V_i \otimes V_j) = V_j \otimes V_i, \quad 1 \le i, j \le \theta,$$

such that $\operatorname{GKdim} \mathcal{B}(V) < \infty$ but some of the V_i's are neither blocks nor points. More precisely the V_i's could be semisimple but indecomposable in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$. These are called **pale blocks**. See the Appendix of [AAH] and [A-A-Moya], in preparation.

[AAH] N. Andruskiewitsch, I. Angiono and I. Heckenberger. *On finite GK-dimensional Nichols algebras over abelian groups*. Mem. Amer. Math. Soc., to appear.

[AAH2] N. Andruskiewitsch, I. Angiono and I. Heckenberger. *Examples of finite-dimensional Nichols algebras over abelian groups in positive characteristic*, to appear.

[CLW] C. Cibils, A. Lauve, S. Witherspoon, *Hopf quivers and Nichols algebras in positive characteristic*, Proc. Amer. Math. Soc. 137(12) (2009) 40294041.

[HS] I. Heckenberger and H.-J Schneider, *YetterDrinfeld modules over bosonizations of dually paired Hopf algebras*, Adv. Math. **244** (2013), 354–394.