Representations of Lestrygonian Nichols algebra

Dirceu Bagio

(In collaboration with N. Andruskiewitsch, D. Flores and S. Flora)

Federal University of Santa Maria, Brazil

Workshop on Quantum Symmetries
Sketch

◊ The Lestrygonian Nichols algebra $\mathcal{B}(L_q(1, G))$. 
Sketch

- The Lestrygonian Nichols algebra $\mathcal{B}(\mathfrak{L}_q(1, G))$.
- Finite dimensional simple modules of $\mathcal{B}(\mathfrak{L}_q(1, G))$. 
Sketch

- The Lestrygonian Nichols algebra $B(\mathcal{L}_q(1, G))$.
- Finite dimensional simple modules of $B(\mathcal{L}_q(1, G))$.
- Point modules of $B(\mathcal{L}_q(1, G))$. 
Sketch

- The Lestrygonian Nichols algebra $\mathcal{B}(\mathcal{L}_q(1, G))$.
- Finite dimensional simple modules of $\mathcal{B}(\mathcal{L}_q(1, G))$.
- Point modules of $\mathcal{B}(\mathcal{L}_q(1, G))$. 
**Notation**

- $k$ is an algebraically closed field, $\text{char } k = 0$. 
Notation

- $k$ is an algebraically closed field, $\text{char } k = 0$.

- The natural numbers are denoted by $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. 
Notation

- $k$ is an algebraically closed field, $\text{ch } k = 0$.

- The natural numbers are denoted by $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

- If $k < t \in \mathbb{N}_0$, then $\mathbb{I}_{k,t} = \{n \in \mathbb{N}_0 : k \leq n \leq t\}$, and $\mathbb{I}_t := \{1, \ldots, t\}$.
Yetter-Drinfeld modules
Let $H$ be a Hopf algebra.
Yetter-Drinfeld modules

Let $H$ be a Hopf algebra. A **Yetter-Drinfeld module** over $H$ is a vector space $V$ such that $V$ is an $H$-module and an $H$-comodule and the following compatibility is true:
Yetter-Drinfeld modules

Let $H$ be a Hopf algebra. A Yetter-Drinfeld module over $H$ is a vector space $V$ such that $V$ is an $H$-module and an $H$-comodule and the following compatibility is true:

$$\rho(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}, \quad h \in H, \ v \in V.$$
Yetter-Drinfeld modules

Let $H$ be a Hopf algebra. A **Yetter-Drinfeld module** over $H$ is a vector space $V$ such that $V$ is an $H$-module and an $H$-comodule and the following compatibility is true:

$$\rho(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}, \quad h \in H, \ v \in V.$$

If $H$ has bijective antipode then the category of Yetter-Drinfeld modules $\mathcal{H}YD^H_H$ is a braided tensor category:
Yetter-Drinfeld modules
Let $H$ be a Hopf algebra. A Yetter-Drinfeld module over $H$ is a vector space $V$ such that $V$ is an $H$-module and an $H$-comodule and the following compatibility is true:

$$\rho(h \cdot v) = h(1)v(-1)S(h(3)) \otimes h(2) \cdot v(0), \quad h \in H, \ v \in V.$$ 

If $H$ has bijective antipode then the category of Yetter-Drinfeld modules $\mathcal{YD}^H_H$ is a braided tensor category: for $V, W \in \mathcal{YD}^H_H$, 


Yetter-Drinfeld modules

Let $H$ be a Hopf algebra. A Yetter-Drinfeld module over $H$ is a vector space $V$ such that $V$ is an $H$-module and an $H$-comodule and the following compatibility is true:

$$\rho(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}, \quad h \in H, \ v \in V.$$ 

If $H$ has bijective antipode then the category of Yetter-Drinfeld modules $\mathcal{HYD}_H$ is a braided tensor category: for $V, W \in \mathcal{HYD}_H$,

$$c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}, \quad v \in V, \ w \in W.$$
**Yetter-Drinfeld modules**

Let $H$ be a Hopf algebra. A **Yetter-Drinfeld module** over $H$ is a vector space $V$ such that $V$ is an $H$-module and an $H$-comodule and the following compatibility is true:

$$\rho(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}, \quad h \in H, \, v \in V.$$ 

If $H$ has bijective antipode then the category of Yetter-Drinfeld modules $\mathcal{H} \mathcal{YD}$ is a braided tensor category: for $V, W \in \mathcal{H} \mathcal{YD},$

$$c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}, \quad v \in V, \, w \in W.$$ 

Particularly, if $V$ is an object in $\mathcal{H} \mathcal{YD}$ then $(V, c_V, V)$ is a braided vector space.
Assume that $H$ is a Hopf algebra with bijective antipode.
Assume that $H$ is a Hopf algebra with bijective antipode.

We have the following correspondence:
Assume that $H$ is a Hopf algebra with bijective antipode. We have the following correspondence:

$$V \in \mathcal{H}YD \leadsto \text{a Hopf algebra } \mathcal{B}(V) \text{ in } \mathcal{H}YD$$
Assume that $H$ is a Hopf algebra with bijective antipode. We have the following correspondence:

$$V \in \mathcal{H}_Y \mathcal{D} \leadsto \text{a Hopf algebra } \mathcal{B}(V) \text{ in } \mathcal{H}_Y \mathcal{D}$$

The **Nichols algebra** $\mathcal{B}(V)$ associates to $(V, c)$ is given by

$$\mathcal{B}(V, c) = k \oplus V \oplus_{n \geq 2} V^\otimes n / \ker \mathfrak{S}_n$$

where $\mathfrak{S}_n$ is the quantum symmetrizer.
Let $\Gamma$ an abelian group. Consider $V = V_1 \oplus V_2 \in \mathcal{YD}$, where $V_1$ has a basis $x_1, x_2$ and $V_2$ has a basis $x_3$ such that the braid is given by:
Let $\Gamma$ an abelian group. Consider $V = V_1 \oplus V_2 \in \mathbb{k}\mathbf{\Gamma}\mathcal{YD}$, where $V_1$ has a basis $x_1, x_2$ and $V_2$ has a basis $x_3$ such that the braid is given by:

$$(c(x_i \otimes x_j))_{i,j \in I_3} = \begin{pmatrix}
\varepsilon x_1 \otimes x_1 & (\varepsilon x_2 + x_1) \otimes x_1 & qx_3 \otimes x_1 \\
\varepsilon x_1 \otimes x_2 & (\varepsilon x_2 + x_1) \otimes x_2 & qx_3 \otimes x_2 \\
q^{-1} x_1 \otimes x_3 & q^{-1} (x_2 + ax_1) \otimes x_3 & q_{22} x_3 \otimes x_3
\end{pmatrix}$$

where $a, q_{22}, q, \varepsilon \in \mathbb{k}$. 
Let $\Gamma$ an abelian group. Consider $V = V_1 \oplus V_2 \in \mathbb{k}_\Gamma \mathcal{YD}$, where $V_1$ has a basis $x_1, x_2$ and $V_2$ has a basis $x_3$ such that the braid is given by:

$$(c(x_i \otimes x_j))_{i,j \in \Pi_3} = \begin{pmatrix}
\epsilon x_1 \otimes x_1 & (\epsilon x_2 + x_1) \otimes x_1 & qx_3 \otimes x_1 \\
\epsilon x_1 \otimes x_2 & (\epsilon x_2 + x_1) \otimes x_2 & qx_3 \otimes x_2 \\
q^{-1}x_1 \otimes x_3 & q^{-1}(x_2 + ax_1) \otimes x_3 & q_{22}x_3 \otimes x_3
\end{pmatrix}
$$

where $a, q_{22}, q, \epsilon \in \mathbb{k}$.

**Braided vector space $\mathcal{L}_q(1, \mathcal{G})$**

If $\mathcal{G} := -2a \in \mathbb{N}$ and $q_{22} = \epsilon = 1$ then the braided vector space above will be denoted by $\mathcal{L}_q(1, \mathcal{G})$. 
The Nichols algebra $B(\mathcal{L}_q(1, G))$ was calculated in [1].

**Generators and relations**

Let $G \in \mathbb{N}$ and $q \in \mathbb{k}^\times$. 
The Nichols algebra $\mathcal{B}(L_q(1, G))$ was calculated in [1].

**Generators and relations**

Let $G \in \mathbb{N}$ and $q \in \mathbb{k}^\times$. The Nichols algebra $\mathcal{B}(L_q(1, G))$ is presented by generators $x_1, x_2, (z_n)_{0 \leq n \leq G}$ ($z_0 = x_3$) with defining relations
The Nichols algebra $\mathcal{B}(\mathcal{L}_q(1, G))$ was calculated in [1].

Generators and relations
Let $G \in \mathbb{N}$ and $q \in k^\times$. The Nichols algebra $\mathcal{B}(\mathcal{L}_q(1, G))$ is presented by generators $x_1, x_2, (z_n)_{0 \leq n \leq G}$ ($z_0 = x_3$) with defining relations

\begin{align*}
x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, & \quad (1) \\
x_1z_0 - qz_0x_1, & \quad (2) \\
z_nz_{n+1} - q^{-1}z_{n+1}z_n, & \quad 0 \leq n < G. \quad (3) \\
x_2z_n - qz_nx_2 - z_{n+1}, & \quad 0 \leq n < G, \quad (4) \\
x_2z_G - qz_Gx_2. & \quad (5)
\end{align*}
The Nichols algebra $B(\mathcal{L}_q(1, G))$ was calculated in [1].

Generators and relations
Let $G \in \mathbb{N}$ and $q \in k^\times$. The Nichols algebra $B(\mathcal{L}_q(1, G))$ is presented by generators $x_1, x_2, (z_n)_{0 \leq n \leq G}$ ($z_0 = x_3$) with defining relations

\begin{align*}
    x_2x_1 - x_1x_2 + \frac{1}{2}x_1^2, \\
    x_1z_0 - qz_0x_1, \\
    z_nz_{n+1} - q^{-1}z_{n+1}z_n, & \quad 0 \leq n < G. \\
    x_2z_n - qz_nx_2 - z_{n+1}, & \quad 0 \leq n < G, \\
    x_2z_G - qz_Gx_2. & \quad (1, 2, 3, 4, 5)
\end{align*}

$B(\mathcal{L}_q(1, G))$ is called Lestrygonian.
Some properties of $\mathcal{B}(\mathfrak{L}_q(1, G))$:
Some properties of $B(\mathfrak{L}_q(1, G))$:

◊ $B(\mathfrak{L}_q(1, G))$ is a domain;
Some properties of $\mathcal{B}(\mathfrak{L}_q(1, G))$:

- $\mathcal{B}(\mathfrak{L}_q(1, G))$ is a domain;
- $\mathcal{B}(\mathfrak{L}_q(1, G))$ has a PBW-basis

$$B = \{x_1^{m_1} x_2^{m_2} z_{q^n}^m \ldots z_1^{n_1} z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\};$$
Some properties of $\mathcal{B}(\mathcal{L}_q(1, G))$:

- $\mathcal{B}(\mathcal{L}_q(1, G))$ is a domain;
- $\mathcal{B}(\mathcal{L}_q(1, G))$ has a PBW-basis

$$B = \{x_1^{m_1} x_2^{m_2} z_0^{n_0} \ldots z_1^{n_1} z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\};$$

- $\text{GKdim } \mathcal{B}(\mathcal{L}_q(1, G)) = 3 + G.$
Some properties of $\mathcal{B}(\mathfrak{L}_q(1, G))$:

- $\mathcal{B}(\mathfrak{L}_q(1, G))$ is a domain;
- $\mathcal{B}(\mathfrak{L}_q(1, G))$ has a PBW-basis
  \[ B = \{ x_1^{m_1} x_2^{m_2} z_1^{n_1} \cdots z_1^{n_1} z_0^{n_0} : m_i, n_j \in \mathbb{N}_0 \}; \]
- $\text{GKdim } \mathcal{B}(\mathfrak{L}_q(1, G)) = 3 + G$.
- $\mathcal{B}(\mathfrak{L}_q(1, G))$ is graded, with
  \[ \deg x_1 = \deg x_2 = 1, \quad \deg z_n = n + 1, \quad 0 \leq n \leq G. \]
Some properties of $B(\mathcal{L}_q(1, G))$:

- $B(\mathcal{L}_q(1, G))$ is a domain;
- $B(\mathcal{L}_q(1, G))$ has a PBW-basis

$$B = \{x_1^{m_1}x_2^{m_2}z_1^{n_1} \cdots z_1^{n_k}z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\};$$

- $\GKdim B(\mathcal{L}_q(1, G)) = 3 + G$.
- $B(\mathcal{L}_q(1, G))$ is graded, with

$$\deg x_1 = \deg x_2 = 1, \quad \deg z_n = n + 1, \quad 0 \leq n \leq G.$$  

- the subalgebra $A$ generated by $x_1$ and $x_2$ is isomorphic to the Jordan plane and has defining relations (1).
Some properties of $\mathcal{B}(\mathfrak{L}_q(1, G))$:

- $\mathcal{B}(\mathfrak{L}_q(1, G))$ is a domain;
- $\mathcal{B}(\mathfrak{L}_q(1, G))$ has a PBW-basis
  \[ B = \{ x_1^{m_1} x_2^{m_2} z_1^{n_1} z_2^{n_2} \cdots z_1^{n_1} z_0^{n_0} : m_i, n_j \in \mathbb{N}_0 \}; \]
- $\text{GKdim } \mathcal{B}(\mathfrak{L}_q(1, G)) = 3 + G$.
- $\mathcal{B}(\mathfrak{L}_q(1, G))$ is graded, with
  \[ \text{deg } x_1 = \text{deg } x_2 = 1, \quad \text{deg } z_n = n + 1, \quad 0 \leq n \leq G. \]
- the subalgebra $\mathcal{A}$ generated by $x_1$ and $x_2$ is isomorphic to the Jordan plane and has defining relations (1).
Quantum plane

Let $k_q[X, Y]$ be the algebra generated by $X$ and $Y$ with relation $XY - qYX$. 
Quantum plane

Let $k_q[X, Y]$ be the algebra generated by $X$ and $Y$ with relation $XY - qYX$.

◇ If $q = 1$, then this is the polynomial ring in 2 variables; its finite-dimensional simple modules are all one-dimensional and parametrized by the points of the plane.
Quantum plane

Let $k_q[X, Y]$ be the algebra generated by $X$ and $Y$ with relation $XY - qYX$.

- If $q = 1$, then this is the polynomial ring in 2 variables; its finite-dimensional simple modules are all one-dimensional and parametrized by the points of the plane.

- Assume that $q \neq 1$;
Quantum plane

Let $k_q[X, Y]$ be the algebra generated by $X$ and $Y$ with relation $XY - qYX$.

- If $q = 1$, then this is the polynomial ring in 2 variables; its finite-dimensional simple modules are all one-dimensional and parametrized by the points of the plane.
- Assume that $q \neq 1$; then $k_q[X, Y]$ is called the quantum plane of parameter $q$. 
Quantum plane

Let $k_q[X, Y]$ be the algebra generated by $X$ and $Y$ with relation $XY - qYX$.

- If $q = 1$, then this is the polynomial ring in 2 variables; its finite-dimensional simple modules are all one-dimensional and parametrized by the points of the plane.

- Assume that $q \neq 1$; then $k_q[X, Y]$ is called the quantum plane of parameter $q$.

We define for each $a \in k^\times$ the following one-dimensional modules of $k_q[X, Y]$:
Quantum plane

Let $k_q[X, Y]$ be the algebra generated by $X$ and $Y$ with relation $XY - qYX$.

- If $q = 1$, then this is the polynomial ring in 2 variables; its finite-dimensional simple modules are all one-dimensional and parametrized by the points of the plane.
- Assume that $q \neq 1$; then $k_q[X, Y]$ is called the quantum plane of parameter $q$.

We define for each $a \in k^\times$ the following one-dimensional modules of $k_q[X, Y]$:

\[
k_a^X = k : \quad X \cdot 1 = a, \quad Y \cdot 1 = 0
\]
\[
k_a^Y = k : \quad X \cdot 1 = 0, \quad Y \cdot 1 = a.
\]
Suppose that ord $q =: N < \infty$. The $N$-dimensional representation $\mathcal{U}_{a,b} = \mathbb{k}^N$ of $\mathbb{k}_q[X, Y]$ is defined by
Suppose that $\text{ord } q =: N < \infty$. The $N$-dimensional representation $\mathcal{U}_{a,b} = \mathbb{k}^N$ of $\mathbb{k}_q[X,Y]$ is defined by

$$Xe_j = aq^{j-1}e_j, \quad j \in \mathbb{I}_N,$$

$$Ye_j = e_{j+1}, \quad j \in \mathbb{I}_{N-1}, \quad Ye_n = be_1,$$

where $e_1, \ldots, e_N$ is the canonical basis of $\mathbb{k}^N$ and $a, b \in \mathbb{k}^\times$. 
Suppose that $\text{ord } q =: N < \infty$. The $N$-dimensional representation $U_{a,b} = \mathbb{k}^N$ of $\mathbb{k}_q[X,Y]$ is defined by

$$
Xe_j = aq^{j-1}e_j, \quad j \in \mathbb{Z}_N,
$$

$$
Ye_j = e_{j+1}, \quad j \in \mathbb{Z}_{N-1}, \quad Ye_n = be_1,
$$

where $e_1, \ldots, e_N$ is the canonical basis of $\mathbb{k}^N$ and $a, b \in \mathbb{k}^\times$.

**Simple modules of the quantum plane**

Assume that $q \neq 1$. Let $V$ be a finite-dimensional simple module of $\mathbb{k}_q[X,Y]$. Then
Suppose that \( \text{ord } q =: N < \infty \). The \( N \)-dimensional representation \( U_{a,b} = \mathbb{k}^N \) of \( \mathbb{k}_q[X, Y] \) is defined by

\[
Xe_j = aq^{j-1}e_j, \quad j \in I_N, \\
Ye_j = e_{j+1}, \quad j \in I_{N-1}, \quad Ye_n = be_1,
\]

where \( e_1, \ldots, e_N \) is the canonical basis of \( \mathbb{k}^N \) and \( a, b \in \mathbb{k}^\times \).

**Simple modules of the quantum plane**

Assume that \( q \neq 1 \). Let \( V \) be a finite-dimensional simple module of \( \mathbb{k}_q[X, Y] \). Then

\[
\diamond \quad \dim V = 1 \Rightarrow V \simeq \mathbb{k}_a^X \quad \text{or} \quad V \simeq \mathbb{k}_a^Y.
\]
Suppose that \( \text{ord } q =: N < \infty \). The \( N \)-dimensional representation \( \mathcal{U}_{a,b} = \mathbb{k}^N \) of \( \mathbb{k}_q[X, Y] \) is defined by

\[
Xe_j = aq^{j-1}e_j, \quad j \in \mathbb{I}_N, \\
Ye_j = e_{j+1}, \quad j \in \mathbb{I}_{N-1}, \quad Ye_n = be_1,
\]

where \( e_1, \ldots, e_N \) is the canonical basis of \( \mathbb{k}^N \) and \( a, b \in \mathbb{k}^\times \).

**Simple modules of the quantum plane**

Assume that \( q \neq 1 \). Let \( V \) be a finite-dimensional simple module of \( \mathbb{k}_q[X, Y] \). Then

\[
\diamond \text{ dim } V = 1 \Rightarrow V \simeq \mathbb{k}_a^X \text{ or } V \simeq \mathbb{k}_a^Y.
\]

\[
\diamond \text{ dim } V > 1 \Rightarrow \text{ord } q =: N < \infty \text{ and } V \simeq \mathcal{U}_{a,b}.
\]
Simple modules of the $\mathcal{B}(\mathcal{L}_q(1, G))$

The following categories are isomorphic:
Simple modules of the $\mathcal{B}(\mathfrak{L}_q(1, G))$

The following categories are isomorphic:

$$\text{Irrep } \mathcal{B}(\mathfrak{L}_q(1, G)) \simeq \text{Irrep } \mathbb{k}_q[X, Y].$$
Let $A = \bigoplus_{n \in \mathbb{N}_0} A^n$, $A_0 = k$, be a finitely graded $k$-algebra generated in degree 1.
Let $A = \bigoplus_{n \in \mathbb{N}_0} A^n$, $A_0 = \mathbb{k}$, be a finitely graded $\mathbb{k}$-algebra generated in degree 1.

**Definition**

A *point module for $A$* is a (left) graded module $V = \bigoplus_{n \in \mathbb{N}_0} V^n$ over $A$ such that
Let $A = \bigoplus_{n \in \mathbb{N}_0} A^n$, $A_0 = \mathbb{k}$, be a finitely graded $\mathbb{k}$-algebra generated in degree 1.

**Definition**

A *point module for* $A$ is a (left) graded module $V = \bigoplus_{n \in \mathbb{N}_0} V^n$ over $A$ such that

- $V$ is cyclic and generated in degree 0, i.e., $V = A \cdot V^0$, 

Let $A = \bigoplus_{n \in \mathbb{N}_0} A^n$, $A_0 = k$, be a finitely graded $k$-algebra generated in degree 1.

**Definition**

A *point module for $A$* is a (left) graded module $V = \bigoplus_{n \in \mathbb{N}_0} V^n$ over $A$ such that

- $V$ is cyclic and generated in degree 0, i.e., $V = A \cdot V^0$,
- $V$ has Hilbert series $h_V(t) = 1/(1 - t)$, in other words $\dim_k V^n = 1$, for all $n \geq 0$. 
We denote a point of the projective space $\mathbb{P}^n$ over $k$ by $(a_0 : a_1 : \cdots : a_n)$.
We denote a point of the projective space $\mathbb{P}^n$ over $k$ by $(a_0 : a_1 : \cdots : a_n)$.

**Point modules of free algebras**

Let $A = k\langle x_0, x_1, \ldots, x_n \rangle$ be the free associative algebra. The isomorphism classes of point modules over $A$ are in bijective correspondence with points in the infinite product $\mathbb{P}^n \times \mathbb{P}^n \times \cdots = \prod_{i=0}^{\infty} \mathbb{P}^n$. 
We denote a point of the projective space $\mathbb{P}^n$ over $k$ by $(a_0 : a_1 : \cdots : a_n)$.

**Point modules of free algebras**

Let $A = k\langle x_0, x_1, \ldots, x_n \rangle$ be the free associative algebra. The isomorphism classes of point modules over $A$ are in bijective correspondence with points in the infinite product $\mathbb{P}^n \times \mathbb{P}^n \times \cdots = \prod_{i=0}^{\infty} \mathbb{P}^n$. The correspondence is given by:

$$V = \bigoplus_{j \in \mathbb{N}_0} \langle v_j \rangle \mapsto (a_{0,j} : \cdots : a_{n,j}), \text{ where } x_i v_j = a_{i,j} v_{j+1}.$$
Consider a point module $V = \bigoplus_{j \in \mathbb{N}_0} \langle v_j \rangle$ of $\mathcal{B}(\mathcal{L}_q(1, G))$. 
Consider a point module $V = \bigoplus_{j \in \mathbb{N}_0} \langle v_j \rangle$ of $\mathcal{B}(\mathcal{L}_q(1, G))$. Since $x_i v_j, z_0 v_j \in \langle v_{j+1} \rangle$, $i = 1, 2$ and $j \in \mathbb{N}_0$, we assume that

$$x_1 v_j = a_j v_{j+1}, \quad x_2 v_j = b_j v_{j+1}, \quad z_0 v_j = c_j v_{j+1}, \quad a_j, b_j, c_j \in k.$$
Consider a point module $V = \bigoplus_{j \in \mathbb{N}_0} \langle v_j \rangle$ of $\mathcal{B}(\mathfrak{L}_q(1, G))$. Since $x_i v_j, z_0 v_j \in \langle v_{j+1} \rangle$, $i = 1, 2$ and $j \in \mathbb{N}_0$, we assume that

$$x_1 v_j = a_j v_{j+1}, \quad x_2 v_j = b_j v_{j+1}, \quad z_0 v_j = c_j v_{j+1}, \quad a_j, b_j, c_j \in \mathbb{k}.$$ 

Then $V$ is completely determined by $(P_0, P_1, \ldots) \in \prod_{i=0}^\infty \mathbb{P}^2$, where $P_j = (a_j : b_j : c_j)$. 
Lema 1

If $a_0 \neq 0$ then:

i) $a_j \neq 0$ for all $j \in \mathbb{N}_0$, and

ii) $V$ is determined by $P_j = (1 : b_0 / a_0 - j / 2 : 0)$. 

Lema 2

Suppose that $a_0 = 0$. 

i) If $b_j \neq 0$ for all $j \in \mathbb{N}_0$, then $V$ is determined by $P_j = (0 : 1 : q - j c_0 / b_0)$. 

ii) If $b_k = 0$ for some $k \in \mathbb{N}_0$ then $b_j = 0$ for all $j \in \mathbb{N}_0$ and $V$ is determined by $P_j = (0 : 0 : 1)$. 
**Lema 1**

If $a_0 \neq 0$ then:

i) $a_j \neq 0$ for all $j \in \mathbb{N}_0$, and
Lema 1
If $a_0 \neq 0$ then:

i) $a_j \neq 0$ for all $j \in \mathbb{N}_0$, and

ii) $V$ is determined by $P_j = (1 : b_0/a_0 - j/2 : 0)$. 
Lema 1
If $a_0 \neq 0$ then:

i) $a_j \neq 0$ for all $j \in \mathbb{N}_0$, and

ii) $V$ is determined by $P_j = (1 : b_0/a_0 - j/2 : 0)$.

Lema 2
Suppose that $a_0 = 0$. 
**Lema 1**
If $a_0 \neq 0$ then:

i) $a_j \neq 0$ for all $j \in \mathbb{N}_0$, and

ii) $V$ is determined by $P_j = (1 : b_0/a_0 - j/2 : 0)$.

**Lema 2**
Suppose that $a_0 = 0$.

i) If $b_j \neq 0$, for all $j \in \mathbb{N}_0$, then $V$ is determined by $P_j = (0 : 1 : q^{-j}c_0/b_0)$.
Lema 1
If $a_0 \neq 0$ then:
\begin{itemize}
  \item[i)] $a_j \neq 0$ for all $j \in \mathbb{N}_0$, and
  \item[ii)] $V$ is determined by $P_j = (1 : b_0/a_0 - j/2 : 0)$.
\end{itemize}

Lema 2
Suppose that $a_0 = 0$.
\begin{itemize}
  \item[i)] If $b_j \neq 0$, for all $j \in \mathbb{N}_0$, then $V$ is determined by $P_j = (0 : 1 : q^{-j}c_0/b_0)$.
  \item[ii)] If $b_k = 0$ for some $k \in \mathbb{N}_0$ then $b_j = 0$ for all $j \in \mathbb{N}_0$ and $V$ is determined by $P_j = (0 : 0 : 1)$.
\end{itemize}
Consider the following subsets of $\mathbb{P}^2$:
Consider the following subsets of $\mathbb{P}^2$:

$$Y := \{(1 : b : 0) \in \mathbb{P}^2 : b \in k\}$$

and

$$Z := \{(0 : 1 : b) \in \mathbb{P}^2 : b \in k\}$$
Consider the following subsets of $\mathbb{P}^2$:

$$Y := \{(1 : b : 0) \in \mathbb{P}^2 : b \in k\}$$

and

$$Z := \{(0 : 1 : b) \in \mathbb{P}^2 : b \in k\}$$

**Theorem 2**
The isomorphism classes of point modules over $\mathcal{B}(\mathcal{L}_q(1, G))$ are parametrized by $Y \cup Z \cup \{(0 : 0 : 1)\}$. 
References
References

